Two-fermion bound states in the explicitly covariant Light-Front Dynamics

V. A. Karmanov\textsuperscript{a}, J. Carbonell\textsuperscript{b} and M. Mangin-Brinet\textsuperscript{c}\textsuperscript{*}

\textsuperscript{a}Lebedev Physical Institute, Leninsky Pr. 53, 119991 Moscow, Russia
\textsuperscript{b}Institut des Sciences Nucléaires, 53, avenue des Martyrs, 38 026 Grenoble, France

1. CONSTRUCTING THE ANGULAR MOMENTUM

We solve the bound state problem for two relativistic fermions in the ladder approximation. The problem is studied in the framework of the explicitly covariant light-front dynamics [1,2]. In this approach, the state vector is defined on an hyperplane given by the invariant equation $\omega \cdot x = 0$ with $\omega^2 = 0$. The standard light-front, reviewed in [3], is recovered for $\omega = (1, 0, 0, -1)$. Because of explicit covariance, the wave functions for a given angular momentum are decomposed in a few covariant spin structures multiplied by scalar spin components.

For two scalar particles with zero and non-zero angular momentum, this problem was solved in [4,5]. The light-front deuteron wave function was calculated, in a perturbative way, in [6]. Now we develop a general method, based on explicitly covariant light-front dynamics, to solve the two-fermion bound state problem. We calculate the spin components non-perturbatively for one-boson exchange kernels: scalar, pseudoscalar, vector and pseudovector exchanges [4,7,8].

The interaction kernel $V$ is a scalar, i.e., in c.m.-system, it depends on the scalar products of all available vectors, including the vector $\vec{n} = \vec{\omega}/|\vec{\omega}|$, determining the light-front orientation, that is: $V = V(\vec{k}, \vec{k}', \vec{s}_1, \vec{s}_2, \vec{n}, M^2)$. Here $\vec{s}_1, \vec{s}_2$ are fermion spin operators. Thus, for the two-fermion system, the angular momentum operator commuting with the kernel reads:

$$\vec{J} = -i[\vec{k} \times \partial/\partial \vec{k}] - i[\vec{n} \times \partial/\partial \vec{n}] + \vec{s}_1 + \vec{s}_2. \quad (1)$$

Besides, the three operators $J^2, J_z$ and $A^2 = (\vec{n} \cdot \vec{J})^2$ commute not only with the kernel, but also with each other. Therefore, one should construct the eigenstates of $J^2, A^2$ and $J_z$:

$$J^2 \psi = J(J + 1) \psi \quad (2)$$

$$A^2 \psi = a^2 \psi \quad (3)$$

and solve the bound state problem for given $(J, a)$. The equations for the functions with different $(J, a)$ are not coupled with each other. Then for given $J \geq 1$, one should construct from the states with different $a$, a superposition satisfying the angular condition.

1.1. The state with $J = 0^+$

The basis for the state with $J = 0^+$ contains two spin structures. The counting rule is based on the dimension of the spin matrix forming the two-fermion wave function: $N = (2s_1 + 1)(2s_2 + 1)/2 = 2 \times 2/2 = 2$. The division by 2 takes into account the parity conservation. The vector $\vec{n}$ enters in the wave function on the equal ground with the momenta of particles. In the reference system where $\vec{k}_1 + \vec{k}_2 = 0$, the wave function of two fermions in the state $J = 0^+$ is represented as:

$$\Phi_{\sigma_2 \sigma_1} = \sqrt{m_\sigma_2 w_{\sigma_2}^\dagger \sigma_1 \psi(\vec{k}, \vec{n}) w_{\sigma_1}^\dagger},$$

$$\psi(\vec{k}, \vec{n}) = \frac{1}{\sqrt{2}} \left( f_1 + \frac{i\vec{n} \times \vec{k}}{\sin \theta} f_2 \right), \quad (4)$$
where \( \vec{k} = \vec{k}_1 = -\vec{k}_2, \vec{k} = \vec{k}/k \). The functions \( f_{1,2} \) depend on \( k \) and \( \theta \), where \( k \cdot \hat{n} = \cos \theta \). It is convenient to represent this wave function in the four-dimensional form, in order to use in calculations the trace techniques of the Dirac matrices. Namely,

\[
\phi_{\sigma_2, \sigma_1}(k_1, k_2, p, \omega \tau) = \sqrt{m} \vec{u}_{\sigma_2}(k_2) \phi U_{\sigma} \vec{u}_{\sigma_1}(k_1) \tag{5}
\]

with \( \phi = f_1 S_1 + f_2 S_2 \) and

\[
S_1 = \frac{1}{2\sqrt{2\varepsilon_k}} \gamma_5, \\
S_2 = \frac{\varepsilon_k}{2\sqrt{2mk} \sin \theta} \left( \frac{2m \omega - m^2}{\varepsilon_k^2} \right) \gamma_5, \tag{6}
\]

where \( \omega = \omega_\nu \gamma^\nu, \ U_\nu = \gamma^\nu \gamma^0, \ \varepsilon_k = \sqrt{k^2 + m^2} \). When \( k_1 + k_2 = 0 \), eqs. (5), (6) turn into (4).

The basis (6) is orthonormalized.

1.2. The states with \( J = 1^+ \)

The wave function of the \( J = 1^+ \) state (deuteron, for example) is determined by six spin structures. They are split in two families with two spin structures for \( J = 1^+, a = 0 \) and with four spin structures for \( J = 1^+, a = 1 \).

One can easily check that the function \( \tilde{\psi}^0(\vec{k}, \vec{n}) \) satisfying the equation (3) with \( a = 0 \) is parallel to \( \vec{n} \), i.e., it satisfies the condition \( \tilde{\psi}^0 = \vec{n} \tilde{\psi}^0 \).

It has the following decomposition:

\[
\tilde{\psi}^0(\vec{k}, \vec{n}) = \sqrt{\frac{3}{2}} \left\{ g^{(0)}_1 \bar{\sigma} \hat{k} \\
+ g^{(0)}_2 \bar{\sigma} \left( \hat{k} \cos \theta - \vec{n} \right) / \sin \theta \right\} \vec{n} \tag{7}
\]

The function \( \tilde{\psi}^1(\vec{k}, \vec{n}) \) corresponding to \( a = 1 \) is orthogonal to \( \vec{n} \), i.e., it satisfies the condition \( \tilde{\psi}^1 \parallel \vec{n} \) \( \tilde{\psi}^1(\vec{k}, \vec{n}) \). It is convenient to introduce the vectors orthogonal to \( \vec{n} \):

\[
\hat{k}_\perp = \frac{\vec{k} - \cos \theta \hat{n}}{\sin \theta}, \quad \bar{\sigma}_\perp = \bar{\sigma} - (\vec{n} \cdot \bar{\sigma})\hat{n}.
\]

Then the function \( \tilde{\psi}^1 \) obtains the following form:

\[
\tilde{\psi}^1(\vec{k}, \vec{n}) = g^{(1)}_1 \sqrt{\frac{3}{2}} \bar{\sigma}_\perp \\
+ g^{(1)}_2 \sqrt{\frac{3}{2}} \left( 2\hat{k}_\perp(\hat{k}_\perp \cdot \bar{\sigma}_\perp) - \bar{\sigma}_\perp \right) \\
+ g^{(1)}_3 \sqrt{\frac{3}{2}} \hat{k}_\perp(\bar{\sigma} \cdot \hat{n}) + g^{(1)}_4 \sqrt{\frac{3}{2}} [\hat{k} \times \vec{n}] 
\tag{8}
\]

Both wave functions (7) and (8) can be also represented in the four-dimensional form [7,8], similarly to (5). The functions \( f_{1,2} \) for \( J = 0 \) and \( g^{(0)}_{1,2} \) for \( J = 1^+, a = 0 \) states are determined by a system of two equations. The functions \( g^{(1)}_{1-4} \) for \( J = 1^+, a = 1 \) state are determined by a system of four equations. The equations are given in [4,7–9].

2. SATISFYING THE ANGULAR CONDITION

The angular momentum operator in light-front dynamics is a dynamical one. Its equivalence to the kinematical operator (in particular, to the operator (1) for two fermions) is ensured by the so-called angular condition (see [2]). This condition cannot be satisfied for the state with fixed \( a \neq 0 \), its solution is a superposition of the states with different \( a \). Therefore, the wave function for \( J = 1^+ \), satisfying the angular condition, is represented as:

\[
\tilde{\psi}(\vec{k}, \vec{n}) = c_0 \tilde{\psi}^0(\vec{k}, \vec{n}) + c_1 \tilde{\psi}^1(\vec{k}, \vec{n}) \tag{9}
\]

Since we don’t solve the full field theory, but restrict ourselves by a particular (OBE) interaction, the angular condition can be also imposed in an approximate form. One can require that the wave function \( \tilde{\psi}(\vec{k}, \vec{n}) \) at \( \vec{k} = 0 \) does not depend on \( \vec{n} \). The coefficients \( c_0, c_1 \) in (9) are unambiguously determined by this requirement.

The masses \( M_0 \) for \( a = 0, 1 \) are split, though they should be degenerate in the exact solution. In standard LFD, this split – the mass dependence on the angular momentum projection – is the violation of the rotational invariance. Solution (9) corresponds to average mass \( M^2 = c^2_0 M_0^2 + c^2_1 M_1^2 \). For two scalar particles with massless exchange, this formula gives results close to the Bethe-Salpeter solution and restores with high accuracy the Coulombian degeneration [5].

One can show that at \( k = 0 \), the components \( g^{(0)}_{1,2} \) of the wave function \( \tilde{\psi}^0 \), eq. (7), depend on \( \theta \) as:

\[
g^{(0)}_1(k = 0, \theta) = b_0 \cos \theta, \quad g^{(0)}_2(k = 0, \theta) = -b_0 \sin \theta,
\]

where \( b_0 \) is a constant. In the wave function \( \tilde{\psi}^1 \), eq. (8), the component
$g_1^{(1)}(k = 0, \theta) = b_1$ does not depend on $\theta$, whereas $g_2^{(1)}(k = 0, \theta) = 0$. The coefficients in the solution (9) are expressed through $b_0, b_1$:

$$c_0 = \frac{b_1}{\sqrt{2b_0^2 + b_1^2}}, \quad c_1 = \frac{\sqrt{2b_0}}{\sqrt{2b_0^2 + b_1^2}}$$

(10)

The wave function (9) is now can be rewritten as:

$$\psi(\tilde{k}, \tilde{n}) = f_1 \frac{1}{\sqrt{2}} \phi + f_2 \frac{1}{2} (3\tilde{k} \cdot \phi - \phi)$$

$$+ f_3 \frac{1}{2} (3\tilde{n} \cdot \phi - \phi)$$

$$+ f_4 \frac{1}{2} (3\tilde{k} \cdot \tilde{n} \cdot \phi - 2(\tilde{k} \cdot \tilde{n}) \phi)$$

$$+ f_5 \frac{1}{2} \sqrt{\frac{3}{2}} \tilde{k} \times \tilde{n} + f_6 \frac{\sqrt{3}}{2} (\tilde{n} \cdot \phi - \tilde{k} \cdot \phi - \tilde{k} \cdot \phi).$$

The six components $f_i$ are expressed through $g_{1,2}^{(0)}$ and $g_1^{(1)}$.

3. NUMERICAL RESULTS

For illustration, we give here the results of numerical calculation with the kernel corresponding to the exchange of a scalar meson with mass $\mu$. The form factor in the fermion-meson vertex is not introduced, but the calculation is done on the finite interval $k \leq k_{\text{max}}$. Note that for $J = 0^+$ and for the coupling constant $\alpha = \frac{g^2}{(4\pi)} < \sqrt{\frac{\alpha_c}{3.72}}$ the solution is stable at $k_{\text{max}} \to \infty$ [9].

For $J = 1^+$, $m = 1$, $\mu = 0.25 \alpha = 1.18$, $B = 2m - M = 0.0501$, $k_{\text{max}} = 10$, the energy split between the states $a = 0$ and $a = 1$ is rather small: 2%, in comparison to 20% for two scalar particles [5]. For two fermions interacting by pseudoscalar exchange, the energy split is much more significant: for $\alpha = 60$, $B_{a=0} = 0.103$ and $B_{a=1} = 0.0494$, that is more than a 100% splitting in energy [8].

The components $f_i(k, \theta = 30^\circ)$ are shown in figure 1. One can see that for scalar exchange the dominating component is $f_1$ (turning into the S-wave in the nonrelativistic limit). For the coefficients (10) the following values were found: $c_0 = 0.5817, c_1 = 0.8134$, that is very close to $1/\sqrt{3} \approx 0.5773, \sqrt{2/3} \approx 0.8165$.

The details of approach and the numerical results for all exchanges (S, PS, V and PV) will be published in [8].

REFERENCES