Phase Space Geometry in Classical and Quantum Mechanics *

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Abstract

Phase space is the state space of classical mechanics, and this manifold is normally endowed only with a symplectic form. The geometry of quantum mechanics is necessarily more complicated. Arguments will be given to show that augmenting the symplectic manifold of classical phase space with a Riemannian metric is sufficient for describing quantum mechanics. In particular, using such spaces, a fully satisfactory geometric version of quantization will be developed and described.

1 Introduction

What is the difference between classical mechanics and quantum mechanics? From a certain perspective, it is surely true that the differences appear to be vast. One theory is deterministic, the other is stochastic. One theory involves point particles and their trajectories, the other involves wave functions generally spread out over space. One theory involves commuting algebraic expressions, the other involves generally noncommuting algebraic

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expressions corresponding, in each case, to observables. These differences are indeed vast and well known, and of course they are all true. In brief, if one wants to find big differences, then it is not too difficult to do so. However, let us take a different perspective.

Instead of focusing on vast distinctions, let us see how close we can bring the formulations of classical and quantum mechanics to each other. If we can bring them close to each other, then we may be able to shed comparative light on one side from the vantage point of the other side, a procedure which may be rather useful in gaining further understanding of both theories.

Prior to the discovery of quantum mechanics, there was a wealth of significant developments in the arena of classical mechanics. The range of applicability of classical mechanics is enormous, and, for many sorts of problems, a classical description is frequently all that is needed. With the advent of quantum mechanics, a wider family of problems may be successfully addressed. Sometimes one reads that it is necessary to let $\hbar$, Planck’s constant/2π, go to zero, i.e., $\hbar \rightarrow 0$, if one wishes to describe classical mechanics. Our viewpoint is quite the opposite in that $\hbar$ is, after all, not zero in the real world, and it should be accepted for the nonzero value that it has, $\hbar \simeq 10^{-27}$ erg-sec. Consequently, it should not be a case of classical or quantum descriptions, but, instead, classical and quantum descriptions. In other words, classical and quantum formulations should coexist.

The first part of the present article is devoted to a picture of classical and quantum physics fully consistent with the view that both formulations coexist. The second part of this article discusses the meaning of the variables used in phase space path integrals of different types, and shows how quantum mechanics impacts on the meaning of phase space variables that are used in classical mechanics. In the third part of this article, we show how the addition of a metric to the phase space of classical mechanics can be considered the key concept in defining the basic ingredients in quantum mechanics, and in particular the impact that the addition of the metric to the classical phase space has on the phase space of quantum mechanics.

Throughout this article we will rely on the concept of coherent states and their important application in building a bridge between the classical and the quantum world views. It is perhaps appropriate, therefore, that at this moment we present a mini review of the definition and some properties of coherent states before we put them to use in what follows.
1.1 Coherent states: Definitions and properties

Quantum mechanics deals with vectors, let us call them $|\psi\rangle$ following the notation of Dirac, in a Hilbert space $\mathcal{H}$, $|\psi\rangle \in \mathcal{H}$, with an inner product denoted by $\langle \phi | \psi \rangle$ taken to be linear in the right hand vector and antilinear in the left hand vector. Linear operators act on vectors and yield new vectors, and in the quantum mechanics of a single canonical system, two basic operators are $P$ and $Q$, an irreducible pair, which satisfy the Heisenberg commutation relation, $[P, Q] = -i \hbar \mathbb{1}$, where $\mathbb{1}$ denotes the unit operator. These operators are to be taken as self adjoint, which means that they can be used to generate unitary groups of transformations acting in the Hilbert space. In particular, we introduce the family of unitary operators given by

$$U[p, q] \equiv e^{-iqP/\hbar} e^{ipQ/\hbar},$$

(1)
defined for all $(p, q) \in \mathbb{R}^2$, and which, we state, have the following multiplication rule

$$U[p, q] U[p', q'] = e^{i(p'q' - pq')/2\hbar} U[p + p', q + q'] .$$

(2)

Next, we choose a distinguished vector $|\eta\rangle$, called the fiducial vector, which is normalized such that $\| |\eta\rangle \| \equiv \sqrt{\langle \eta | \eta \rangle} = 1$, and consider the set of vectors each of which is defined as

$$|p, q\rangle \equiv U[p, q] |\eta\rangle$$

(3)
for all $(p, q) \in \mathbb{R}^2$. As defined, every vector in this set has a unit norm, $\| |p, q\rangle \| = 1$. The set of states defined in this manner constitutes a set of coherent states; see, e.g., [1].

Any set of coherent states as defined above satisfies two fundamentally important properties. The first property is that the coherent state vectors are continuously parameterized. This means that if the parameters $(p', q') \rightarrow (p, q)$ in the sense of convergence in $\mathbb{R}^2$, i.e., $|p' - p| + |q' - q| \rightarrow 0$, then it follows that the vectors $|p', q'\rangle \rightarrow |p, q\rangle$ in the strong sense, that is $\| |p', q'\rangle - |p, q\rangle \| \rightarrow 0$. The second property is that the vectors not only span the full Hilbert space, but that they admit a resolution of unity as an integral over one dimensional projection operators given by

$$\mathbb{1} = \int |p, q\rangle \langle p, q| dp\,dq/(2\pi\hbar) .$$

(4)
Here the integral runs over all points \((p, q) \in \mathbb{R}^2\), and the integral converges in the weak sense (as well as in the strong sense).

As minimum requirements in the choice of \(|\eta\rangle\) it is generally convenient to require that

\[
\langle \eta | P | \eta \rangle = 0 , \quad \langle \eta | Q | \eta \rangle = 0 , \quad (5)
\]

which has the virtue that

\[
\langle p, q | P | p, q \rangle = p , \quad \langle p, q | Q | p, q \rangle = q , \quad (6)
\]

leading to the physical interpretation of the variables \(p\) and \(q\) as *expectation values* in the coherent states. It is furthermore common to choose \(|\eta\rangle = |0\rangle\), the ground state of an harmonic oscillator, typically satisfying the relation \((Q + iP)|0\rangle = 0\).

## 2 Equations of Motion, and Boundary Data

Classical mechanics is described by time dependent phase space paths, \(q(t)\) and \(p(t)\), where \(q\) denotes position and \(p\) denotes momentum. The equations that govern the time dependence may be determined as the extremal equations—also known as the Euler-Lagrange equations—that arise from stationary variation of the action functional

\[
I = \int [p\dot{q} - H(p, q)] \, dt \quad (7)
\]

holding \(q\) fixed at both the initial and final times, say, \(t = 0\) and \(t = T\), respectively. In this expression, \(H(p, q)\) denotes the all-important Hamiltonian function, which, for convenience, we have assumed to be time independent. Variation of the action leads to

\[
\delta I = \int [(\dot{q} - \partial H/\partial p)\delta p - (\dot{p} + \partial H/\partial q)\delta q] \, dt \quad (8)
\]

since the surface term, \(p\delta q\), vanishes. Setting \(\delta I = 0\) for general variations \(\delta p(t)\) and \(\delta q(t)\), we arrive at Hamilton’s equations of motion

\[
\dot{q} = \frac{\partial H}{\partial p} , \quad \dot{p} = -\frac{\partial H}{\partial q} . \quad (9)
\]
How these equations are derived and whether or not they have solutions are two fundamentally different issues. In other words, holding \( q \) fixed at two different times, as was used in this derivation, does not imply there is necessarily one and only one solution to these equations. Sometimes there may be a unique solution, but there also may be no solution, or even many solutions. As examples, consider \( \dot{q} = p \) and \( \dot{p} = -q \), \( q(0) = 0 \) and \( q(2\pi) = 1 \) (no solution) or \( q(2\pi) = 0 \) (many solutions). In higher dimensions, there may be no solution for a path that needs to pass through a wall, etc.

Accepting the difference between the derivation and the solution of the equations of motion allows us to consider more general situations. Suppose the action functional had the form

\[
I' = \int \left[ \frac{1}{2} (p\dot{q} - q\dot{p}) - H(p,q) \right] dt
\]

which differs by a total derivative from the previous expression, specifically

\[
I' = I - \frac{1}{2} \int [d(pq)/dt] \, dt.
\]

We now propose to derive the same basic equations of motion from \( I' \) that we obtained from \( I \). To do so requires that we hold both \( q \) and \( p \) fixed at the beginning and end of the time interval in order that, after integration by parts, we again find that

\[
\delta I' = \int [(\ddot{q} - \partial H/\partial p)\delta p - (\dot{p} + \partial H/\partial q)\delta q] \, dt.
\]

Insisting that \( \delta I' = 0 \) leads again to the equations of motion

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.
\]

These equations are the same as before, but now we are asked to seek solutions based on the boundary conditions \( p(0), q(0) = p', q' \) and \( p(T), q(T) = p'', q'' \). Clearly, in the general case the proposed solution is over specified and does not exist. However, in the present case—and unlike the previous case—we know how to limit the specified data so as to ensure a unique solution. One rule to obtain a unique solution would be to choose only the initial data \( p(0), q(0) \), while another choice would be to choose only the final data \( p(T), q(T) \).
3 On the Meaning of the Variables \( p \) and \( q \)

In the phase space path integral formulation of quantum mechanics, Feynman [2] states that we may compute the propagator according to the formula

\[
\langle q'' | e^{-iHT/h} | q' \rangle \equiv M \int e^{(i/h) \int [p\dot{q} - H(p,q)] dt} \mathcal{D}p \mathcal{D}q.
\]  

(14)

Here one is instructed to integrate over all functions \( p(t) \) and \( q(t) \), \( 0 < t < T \), subject only to the conditions that \( q(T) = q'' \) and \( q(0) = q' \). For present purposes it is not too important just how the integral on the right side is to be defined; rather we are more interested in the simpler question of what is the meaning of \( q \). The meaning of the variable \( q \) can be determined from the eigenvalue equation \( Q|q''\rangle = q''|q''\rangle \), which asserts that \( q \) has the meaning of the sharp value of position associated with an eigenvalue. We can determine the meaning of the variable \( p \) by using a related phase space path integral given by

\[
\langle p'' | e^{-iHT/h} | p' \rangle \equiv M \int e^{(i/h) \int [-\dot{q}p - H(p,q)] dt} \mathcal{D}p \mathcal{D}q.
\]  

(15)

In this case the integrals run over all functions \( p(t) \) and \( q(t) \), \( 0 < t < T \), subject only to the conditions that \( p(T) = p'' \) and \( p(0) = p' \). Consequently, the meaning of \( p \) is that of a sharp eigenvalue associated with the equation \( P|p''\rangle = p''|p''\rangle \).

It may seem reasonable that both \( p \) and \( q \) have the meaning of sharp values until one realizes that this situation refers to quantum mechanics and not classical mechanics since \( \hbar \neq 0 \). In fact, this interpretation asserts that we can specify both \( p \) and \( q \) simultaneously for all intermediate \( t \) values, while the uncertainty relation asserts that this is impossible for any \( t \) value.

It is interesting to observe that there is another interpretation of such phase space path integrals [3, 1], namely, that

\[
\langle p'', q'' | e^{-iHT/h} | p', q' \rangle \equiv M \int e^{(i/h) \int [p\dot{q} - H(p,q)] dt} \mathcal{D}p \mathcal{D}q.
\]  

(16)

Here, unlike the previous cases, one integrates over all \( p(t) \) and all \( q(t) \), \( 0 < t < T \), subject to the conditions \( p(T), q(T) = p'', q'' \) and \( p(0), q(0) = p', q' \). The initial and final eigenstates in this case are not sharp eigenstates but
rather are the coherent states $|p,q\rangle$ discussed above for a general $|\eta\rangle$ which satisfies (5). For coherent states, the meaning of $p$ and $q$ is not that of sharp eigenvalues but rather is that of mean values, as already noted previously, namely,

$$
\langle p, q|P|p, q\rangle = p, \quad \langle p, q|Q|p, q\rangle = q .
\tag{17}
$$

As mean values, it is perfectly acceptable to specify values of $p(t)$ and $q(t)$ simultaneously for all $t$, $0 < t < T$, and thus there is absolutely no contradiction with the uncertainty relation.

One should wonder how it is that the same formal path integral has two different evaluations; cf. Eqs. (14) and (16). The key word in the previous sentence is “formal”, which implies that the so-called path integrals are in fact undefined as they stand and these formal expressions need to be defined. That the results given in (14) and (16) are different, means that the individual formal expressions have received their definition by different rules. These matters are well spelled out elsewhere [4] and need not be repeated here.

The conclusion of the discussion in this section is that the only reasonable interpretation of the variables $p$ and $q$ is as mean values rather than truly sharp values since we live in a world where $\hbar \neq 0$. This interpretation has important implications and is not changed by any attempt to define (14) or (15) by lattice limits as are customarily used to give some level of proper definition to such expressions [5].

4 Shadow Metric

Classical mechanics takes place on a phase space manifold which is equipped with a symplectic form $\omega$ leading to a symplectic manifold. The symplectic form takes as its argument two vectors and returns a number. For example,

$$
\omega(dp, dq) = \omega_{ab} dp^a dq^b ,
\tag{18}
$$

where $\omega_{ba} = -\omega_{ab}$ and $\det[\omega_{ab}] \neq 0$. A symplectic manifold has what is called a symplectic geometry induced by the symplectic form. Such a geometry is rather loose, like that of a rubber sheet which may be stretched by different amounts in different directions and still retain its “geometry”. This kind of geometry is quite distinct from the more familiar Riemannian geometry determined by a Riemannian metric, and it is basic that a symplectic manifold
appropriate for classical mechanics should not be assumed to be a metric space endowed with a Riemannian metric.

Our point of view is that one nevertheless needs a metric in order to give physics to the usual mathematical expressions that appear in classical mechanics [6]. For example, one can make canonical coordinate transformations such that the mathematical expression for the Hamiltonian assumes the simple form $p$ for essentially any system. How is one to read out of the universal expression $H = p$ that the given expression actually refers to, say, an oscillator, or perhaps an anharmonic oscillator, etc? Clearly, one needs more information to make that choice correctly. And make no mistake, one definitely needs to make that choice because in quantum mechanics one solves for the spectrum of an oscillator or an anharmonic oscillator, etc., and no ambiguity in that situation can be admitted. To quantize a given classical theory one must not only know its mathematical formulation in terms of the phase space variables, but, at the same time, one must know to what physical system it refers. Unfortunately, this kind of information is simply not part of a traditional classical theory.

We must then seriously consider augmenting traditional classical mechanics with another structure the purpose of which is to keep track of the physics of the various mathematical expressions that enter into the theory. The structure we propose is that of a Riemannian metric, generally having the form

$$d\sigma^2 = A(p,q)dp^2 + 2B(p,q)dpdq + C(p,q)dq^2,$$

with $A > 0$, $C > 0$, and $AC > B^2$. For the sake of illustration, let us choose the simplest example given by

$$d\sigma^2 = dp^2 + dq^2,$$

which by its very form describes a two dimensional flat space expressed in Cartesian coordinates. This metric does not enter into the analysis of classical mechanics; specifically, it does not enter at all into the classical equations of motion. Its sole purpose is “to stand on the sidelines” and to determine the physics of any given expression. How do we intend to do this? Let us adopt the expression $\frac{1}{2}(p^2 + q^2) + \lambda q^4$ to correspond to the physics of an anharmonic oscillator provided that in the same coordinates the metric reads $d\sigma^2 = dp^2 + dq^2$. As the coordinates change, the expression for the Hamiltonian of this system changes form, but so does the expression for
the metric! To understand the physical meaning of the new expression for
the Hamiltonian, one need only find the transformation that restores the
metric to Cartesian form and—presto!—the Hamiltonian in these coordinates
assumes its usual physical form. In short, the metric encodes the physical
meaning of the mathematical expression that represents the Hamiltonian. For
the classical theory, this metric is separate from the theory itself, and for
that reason it has been called a “shadow metric”, one that is not part of
the necessary mathematical formalism. However, as we have just argued, it
is needed so as to complete the whole theory by ensuring that the physical
meaning of the given expressions are not lost. In point of fact—and perhaps
without even being aware of it—one intuitively introduces a “shadow metric”,
or its equivalent, so one can maintain a proper physical interpretation of the
expressions that appear; otherwise the mathematics would be disconnected
from the physics, a situation that would be untenable.

5 Continuous Time Regularization

As remarked earlier, path integrals in general, and phase space path integrals
in particular, are formal expressions that need some form of regularization in
order for them to become defined. The most common regularization is a so-
called lattice regularization which replaces the action integral by a Riemann
sum and integrates over the path values at the discrete times steps that are
thereby introduced. The result of interest arises in the limit that the lattice
spacing goes to zero, a limit which is taken as the final step in the calculation.
Of course, there are many different possible choices that one can make for the
discrete form of the classical action all of which yield the usual classical action
for continuous and differentiable paths in the limit that the lattice spacing
vanishes. However, and this is the important point, there is no guarantee
that the limit of the integral exists as the lattice spacing vanishes, and even
when it exists, there is no guarantee that the result is physically acceptable
for any of the various different choices of lattice action. This is an issue of
great importance that needs to be analyzed whenever one is in any doubt
about such problems.

On the other hand, a lattice form of regularization is not the only choice
that can be used, and in this section we wish to discuss a form of regulariza-
tion that maintains the classical action as an integral over time, and which is
called a continuous time regularization to distinguish it from the lattice form previously discussed. The key idea here is the introduction of an additional factor that serves as a regularization device. This regularization factor—and its removal—takes the form 

$$\lim_{\nu \to \infty} M \int e^{i / \hbar} \int [p \dot{q} - H(p,q)] dt \ e^{-(1/2\nu)} \int [\dot{p}^2 + \dot{q}^2] dt \ Dp \ Dq.$$  

(21)

Observe what has been done: We have inserted a real damping factor in the integrand that formally tends to unity as \( \nu \to \infty \). But, that limit is reserved until the integral over paths has actually been performed. It is important to observe that the regularized expression (21) can actually be given an unambiguous mathematical version as

$$\lim_{\nu \to \infty} 2\pi e^{\nu T/2\hbar} \int e^{i / \hbar} \int [p dq - H(p,q) dt] \ d\mu^\nu_W (p,q),$$

(22)

where \( \mu^\nu_W \) denotes a Wiener measure on a flat two-dimensional phase space expressed in Cartesian coordinates and for which the parameter \( \nu \) denotes the diffusion constant. Note well that this expression has no formal prefactor, and with \( \int pdq \) interpreted as a (Stratonovich) stochastic integral [8], (22) defines, for each \( \nu < \infty \), a completely unambiguous path integral over continuous phase space trajectories. Moreover, it can be shown [7] for a wide class of Hamiltonians that the limit exists and that the limit actually provides a solution to Schrödinger’s equation.

6 Existence of a Flat Space Metric in Any Canonical Quantum Theory

There are several standard rules of quantization—such as those of Heisenberg and Schrödinger—that do not explicitly use a metric on a flat phase space in their construction. On the other hand, the fact that such rules do not use such a flat space metric does not mean that there is no such metric. In fact, perhaps surprisingly, there is always such a metric implicitly present, as we now proceed to demonstrate. Everyone would agree that any canonical quantization procedure leads to vectors in a Hilbert space and to canonical self adjoint operators \( Q \) and \( P \) that satisfy the Heisenberg commutation
relation, \([Q, P] = i\hbar \mathbb{I}\). With the aid of these basic elements, we can always construct vectors of the form
\[
|p, q; \psi\rangle \equiv e^{-ipP/\hbar} e^{ipQ/\hbar} |\psi\rangle,
\]
for a general vector \(|\psi\rangle\), in whatever representation is involved. In terms of \(d|p, q; \psi\rangle \equiv |p + dp, q + dq; \psi\rangle - |p, q; \psi\rangle\), we next form the expression
\[
\|hd|p, q; \psi\rangle\|^2 - |\langle p, q; \psi|hd|p, q; \psi\rangle|^2,
\]
which is evidently quadratic in the differentials \(dp\) and \(dq\). Granting minimal domain requirements, it readily follows that (24) becomes
\[
\langle (\Delta Q)^2 \rangle dp^2 + \langle (\Delta P \Delta Q + \Delta Q \Delta P) \rangle dpdq + \langle (\Delta P)^2 \rangle dq^2,
\]
where in this expression, we have used the notation \(\langle (\cdot) \rangle \equiv \langle p, q; \psi|\langle \cdot\rangle|p, q; \psi\rangle\), and \(\Delta Q \equiv Q - \langle Q\rangle\), etc. It is clear that the given expression generates a metric on phase space, and since the metric coefficients \(\langle (\Delta Q)^2\rangle\), etc., are constants, then the metric characterizes a flat two dimensional phase space. This feature is embedded into any version of canonical quantization one can imagine. A flat space metric may not be explicitly used in arriving at your favorite quantization, but it is nevertheless present in the multitude of consequences that follow from the quantization itself.

Moreover, for many states of physical interest, e.g., a typical harmonic oscillator ground state, it follows that the coefficients appearing in (25) are all proportional to \(\hbar\). Thus, as seems entirely natural, the phase space metric induced by any quantization procedure is fully of a quantum character itself.

### 7 Metrical Quantization

It is clear that one may sometimes choose a different set of axioms to derive a common body of knowledge, and that dictum certainly applies to the process of quantization, as exemplified by the approaches of Heisenberg, Schrödinger, and Feynman. We have observed that traditional canonical quantization procedures do not make use of a flat phase space metric, but that such a metric nevertheless arises in a natural way within any quantization scheme. Suppose we took that metric, normally a secondary feature, and promoted it to primary status, namely postulating its existence as one of our axioms of...
quantization, indeed, as our primary axiom of quantization. The existence of a flat phase space metric and its elevation to prominence has another advantage for, as we have observed above, a metric we have called the “shadow metric” is an all but essential ingredient in order to keep track of the physics of the mathematical expressions under consideration.

Thus as our first axiom in the program of metrical quantization for canonical systems, we postulate the existence of a flat space metric $d\sigma(p,q)^2$ on our classical phase space which for the sake of the present discussion we take in the form

$$d\sigma(p,q)^2 = \hbar(dp^2 + dq^2), \quad (26)$$

a relation that evidently characterizes a flat two dimensional phase space expressed in Cartesian coordinates. Observe that this is a quantum object since it is proportional to $\hbar$. In the formal classical limit in which $\hbar \to 0$, it follows that the metric vanishes; this property ensures that the metric is strictly quantum in character and not part of the usual classical theory as conventionally interpreted. On the other hand, since $\hbar > 0$ in the real world, we can still use (26) as our shadow metric to give physics to our expressions.

Quantum mechanics needs to know what system is under discussion, it needs to know so that it can determine the energy spectrum, for example, of one specific system and not that of any other system. Thus it seems absolutely natural that we must combine the metric directly with the quantization formalism so that in fact the quantization will “know” what system is under consideration.

In an earlier section we discussed a continuous time regularization as a mathematical device to provide a regularization to an otherwise ill defined formal phase space path integral. Now, as we look at that expression once again, we see that the regularization itself involved the very same flat phase space metric we have in mind. That is, the given regularization accomplished the further goal of adding to the formalism an appropriate metric as the “keeper of the physics”. In short, not only does the continuous time regularization of the phase space path integral presented in (21) or (22) solve the issue of giving a proper mathematical meaning to the path integral, it simultaneously solves the problem of maintaining a proper physical interpretation throughout the quantization procedure.

With this discussion as background, we are in a position to formalize the two axioms of Metrical Quantization [5]. First we adopt the phase space
metric (26). The second, and final step, is to say how we use this metric and that is to postulate that the propagator is defined by

\[ \langle p'', q'' | e^{-iHT/\hbar} | p', q' \rangle \equiv \lim_{\nu \to \infty} 2\pi e^{\nu T/2\hbar} \int e^{(i/\hbar) \int [p dq - H(p, q) dt]} d\mu_W(p, q), \]  

(27)

where the Brownian motion paths are carried by the metric (26) in the sense of (21). This is all that is needed since, according to the Gel’fand-Naimark-Segal reconstruction theorem [9], everything quantum in character about (27) is a consequence of the functional form of the right hand side. In particular, and as the notation suggests, it follows that the right hand side generates the propagator in a canonical coherent state representation based on self adjoint Heisenberg operators \( Q \) and \( P \) that satisfy \( [Q, P] = i\hbar \). Moreover, in the present case it follows that the fiducial vector necessarily satisfies the relation \((Q + iP)|0\rangle = 0\). Additionally, it also follows that the Hamiltonian operator \( \mathcal{H} \) is related to the classical Hamiltonian \( H(p, q) \) by antinormal ordering; that is, the very act of introducing the regularization and making the formal phase space path integral well defined has removed all ambiguity including the usual factor ordering ambiguity. In short, the very form of the regularization has already implicitly chosen an ordering prescription for the quantization!

8 Covariance under Canonical Coordinate Transformations

The fact that (27) [or more loosely, (21)] is well defined as an integral means that we are free to change the variables of integration as usual. Let us discuss some of the changes we might make and see how they effect the form of the integrand.

From a classical point of view, recall that a classical canonical transformation associates the old and new canonical coordinates via a one form such as

\[ \eta d\eta = pdq + dF(\eta, q), \]  

(28)

where \( \eta, \eta \) denote the new canonical coordinates while \( p, q \) denote the old canonical coordinates. Furthermore, the Hamiltonian transforms as a scalar
in classical mechanics, which means that
\[ H(p, q) \equiv H(p(p, q), q(p, q)) = H(p, q). \] (29)

In classical mechanics, such expressions make sense because one is dealing with smooth functions of time that are both continuous and have continuous derivatives.

However, in the well defined path integral of (27) we are dealing with Brownian motion paths \( p(t), q(t) \) which are continuous paths but which are nowhere differentiable! This fact means that expressions like \( \int p dq \) cannot be defined as ordinary integrals, but rather they should be interpreted as stochastic integrals [8]. For the case at hand, we may adopt a Riemann sum prescription based on the mid-point rule of definition, namely
\[ \int p dq \equiv \lim_{\nu \to \infty} \frac{1}{2} \sum (p_{l+1} + p_l)(q_{l+1} - q_l), \] (30)
in the limit that the lattice spacing goes to zero. This is the so-called Stratonovich prescription, which has the virtue that it satisfies the rules of the ordinary calculus [8] despite the fact that the paths \( p(t) \) and \( q(t) \) are nowhere differentiable!¹

This correspondence with the ordinary rules of calculus means that in the continuous time regularized path integral the classical action will transform just as it does in the classical theory. The Wiener measure will also undergo a coordinate transformation, but it will still describe Brownian motion on a flat two dimensional phase space, although in general it will now do so in curvilinear coordinates rather than the initial Cartesian coordinates. Therefore under a canonical coordinate transformation (27) becomes [7, 10]
\[ \langle \bar{p}''', \bar{q}''' | e^{-i \mathcal{H} T / h} | \bar{p}', \bar{q}' \rangle = \lim_{\nu \to \infty} \mathcal{M} \int e^{(i/\hbar) \int [\vec{p}'' \delta \vec{q}'' - \vec{q}'' \delta \vec{p}'' + dG(p, q) - H(p, q)] dt} d\mu_W(p, q). \] (31)

In this expression, the function \( G(p, q) \) [which arises from \( F(q, q) \)] appears as a total derivative and so amounts, in effect, to the addition of a phase factor to each of the coherent states.

¹The commonly used alternative rule, which is the so-called Itô prescription, and given by \( \lim \sum p_l (q_{l+1} - q_l) \), has other virtues but it generally does not satisfy the ordinary rules of calculus. Due to our initial choice of coordinates, we are free to choose either the Stratonovich or the Itô rule with which to make our coordinate transformations since both rules lead to the same result in Cartesian coordinates for the integral in question.
Here, at last, we can see the virtue of the present formulation very clearly. Although the coordinate form of the Hamiltonian may well have changed, and thus the physical meaning of the Hamiltonian cannot be read directly from its functional form in the non-Cartesian coordinates, the phase space path integral nevertheless still refers to the original physical system and what ensures this is the fact that the metric on flat space needed to carry the Brownian motion paths acts as a shadow metric maintaining control over the proper physical interpretation. Moreover, we can also see that the quantum Hamiltonian $\mathcal{H}$ has in no way changed so that indeed the original physical system is still under discussion. Observe also that the result is still expressed in terms of the original coherent states—apart from a trivial phase factor—the only difference being the way in which they are parametrized. In particular,

$$|\bar{p},\bar{q}\rangle \equiv e^{-iG(p,q)/\hbar} e^{-iq\bar{p}\mathcal{P}/\hbar} e^{ip\bar{q}\mathcal{Q}/\hbar} |0\rangle . \quad (32)$$

These states still enjoy a resolution of unity, but which is now expressed in the form

$$\mathbb{1} = \int |\bar{p},\bar{q}\rangle \langle \bar{p},\bar{q}| d\bar{p}d\bar{q} / (2\pi\hbar). \quad (33)$$

In summary, and in a manner of speaking, our main point is that quantum phase space may be said to arise from classical phase space simply by the introduction of a metric. Thus, viewed in the right way, perhaps classical and quantum mechanics are closer to each other than is commonly believed!

**References**


