It is shown that a nonequilibrium environment can be instrumental in suppressing decoherence in quantum registers. The effect is found in the framework of exact coherent product solutions for model registers decoherring in a bath of degenerate harmonic modes, through couplings linear in bath coordinates. Such solutions represent a natural nonequilibrium extension of the standard solution for a decoupled initial register state and a thermal environment. Under appropriate conditions, the corresponding evolutions can propagate the reduced register distribution in an unperturbed manner, despite strong, persistent bath entanglement. As a byproduct, we also suggest a straightforward refinement of bang-bang decoherence control.

I. INTRODUCTION

At a very fundamental level the control of decoherence in a quantum register amounts to the engineering of its entanglement with the surrounding environment. When the register Hilbert space supports decoherence-free subspaces [DFS] [1], the problem is known to have at least the passive solution of constraining all quantum computation processes within a selected DFS [2]. In this situation, any entanglement with the environment is theoretically avoided. Alternatively, the register elements can be actively manipulated to the effect of maintaining an 'error-free' reduced register state. This philosophy underlies the design of quantum error correction codes [3] and dynamical decoupling techniques [4,5]. In the latter case, entanglement with the environment may not necessarily cancel at the end of the correction cycle [6], but contributes in such a way as to leave the reduced register state unchanged. In other words, decoherence control aims to maintain, or recover periodically, a given reduced register state in the presence of time-dependent entanglement with environmental modes.

For a typical model register of N two-level elements, immersed in a quantized environment and described by

$$H = \sum_{n=0}^{N-1} \varepsilon \sigma_z^{(n)} + \sum_q \hbar \omega_q b_q^\dagger b_q + \sum_{n=0}^{N-1} \sum_q \sigma_z^{(n)} \left( \chi_{qn} b_q^\dagger + \chi_{qn}^* b_q \right),$$

(1)

the passive or active character of the control strategy seems to be conditioned by the number of DFSs accessed by the encoding method. Indeed, since the associated Hilbert space is always decomposable into a direct sum of elementary DFSs [at the very least, direct products of the 1-dimensional eigenspaces for each $\sigma_z^{(n)}$], the corresponding reduced register state is necessarily distributed over a [finite] number of disjoint DFSs. If the encoding state is confined to a single DFS, decoherence is passively suppressed, but if the encoding involves multiple DFSs, active control seems mandatory.

The main intention of this paper is to point out that decoherence-free states, and therefore passive control, may be possible as well for distributions over multiple DFS and under entanglement with the environment. For this purpose we exploit an extended class of exact, closed-form density matrix solutions for systems with decoherence-free subspaces under interactions linear in environmental degrees of freedom, which has been retrieved while pursuing a seemingly unrelated problem in ref. [7]. Such solutions involve a nonequilibrium environment in a statistical superposition of Gaussian states and provide a natural extension of the standard solution currently employed in discussions of the decoherence process [8,9], which is based on an environment in thermal equilibrium and an initially uncorrelated register state. Here we introduce these solutions in the specific context of quantum registers, and extend the analysis of the decoherence process to the corresponding nonequilibrium regime of the environmental reservoir. Aside from establishing the existence of decoherence-free propagation under nonvanishing entanglement with the surroundings, it is also shown that such solutions prove instrumental in exploring dynamical suppression of decoherence, e.g., in the

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optimization of bang-bang control [4]. In particular, it turns out that in the high precision domain the bang-bang
operating frequency can be reduced by a factor of $2 \div 4$ or more.

II. EXACT NONEQUILIBRIUM EXTENSION OF THE STANDARD SOLUTION

To avoid irrelevant specifics, we begin by referring to a generalized version of the model Hamiltonian (1), where the
register Hamiltonian $H_R$ and the couplings $w_{R,q}$ to the bath modes are not necessarily specified, i.e.,

$$H = H_R + \sum_{q} \hbar \omega_q b_q^\dagger b_q + \sum_{q} (w_{R,q} b_q^\dagger + w_{R,q} b_q),$$

but are such that the register dynamics supports multiple decoherence-free subspaces. That is, the kernels of all
commutators $[H_R, w_{R,q}]$, $[H_R, w_{R,q}^\dagger]$ and $[w_{R,q}, w_{R,q}^\dagger]$ have a nontrivial intersection [e.g., $w_{R,q} = w_{R,q}^\dagger$ and $[H_R, w_{R,q}] = 0$, as in model (1)]. Similarly to the corresponding case for the spin-boson Hamiltonian (1) [10], the von Neumann
problem for Hamiltonian (2) is exactly solvable for an initially uncorrelated state $\rho(0) = \rho_R(0) \otimes \rho_{e,T}$, where the
register state $\rho_R(0)$ represents a statistical distribution over [several] distinct, orthogonal DFS, and the environment
is in thermal equilibrium at temperature $T$, with a density matrix

$$\hat{\rho}_{e,T} = \frac{1}{Z} \exp \left[ -\frac{1}{k_B T} \sum_{q} \hbar \omega_q b_q^\dagger b_q \right].$$

Although the standard solution makes use of the interaction picture, here we retain the Schroedinger picture, and
write the exact evolved state $\rho(t)$ in the closed operatorial form [7]

$$\hat{\rho}(t) = \left[ \sum_{\alpha} \hat{\alpha}_R(t) \otimes \hat{\gamma}_{e,\alpha}(t) \right] \cdot \left[ \sum_{\alpha} \hat{\alpha}_R^\dagger(t) \otimes \hat{\gamma}_{e,\alpha}^\dagger(t) \right],$$

where the environment components read

$$\hat{\gamma}_{e,\alpha}(t) = \exp \left[ \sum_{q} \left( \beta_{q\alpha}(t) b_q^\dagger - \beta_{q\alpha}^*(t) b_q \right) \right] \cdot [\hat{\rho}_{e,T}]^{1/2},$$

and the nonhermitian operators

$$\hat{\alpha}_R(t) = \exp \left[ \frac{i}{\hbar} \int_{0}^{t} d\tau \Omega_{\alpha}(\tau) \right] e^{-\gamma^R \hat{\alpha}_R(0)}$$

are so-called unnormalized state operators [7] on the register DFSs, such that

$$\rho_R(0) = \left[ \sum_{\alpha} \hat{\alpha}_R(0) \right] \cdot \left[ \sum_{\alpha} \hat{\alpha}_R(0) \right]^\dagger.$$

That is, the operator set $\{\hat{\alpha}_R\}$ provides an orthogonal, but not orthonormal, basis in the space of linear operators on
the direct sum of register DFSs, $Tr \left[ \hat{\alpha}_{R'}^\dagger \cdot \hat{\alpha}_R \right] \sim \delta_{R'R}$, and the sum $\sum_{\alpha} \hat{\alpha}_R(0)$ realizes an expansion in this basis of a
generalized, nonhermitian square-root $\hat{\gamma}_R(0)$ of $\hat{\rho}_R(0)$ [$\hat{\rho}_R(0) = \hat{\gamma}_R(0) \cdot \hat{\gamma}_R^R(0)$]. Each $\hat{\alpha}_R$ carries into $\hat{\rho}_R$ contributions from a single DFS, such that at all times it satisfies

$$w_q \cdot \hat{\alpha}_R(t) = \mu_{q,\alpha} \hat{\alpha}_R(t),$$

and evolves according to [see Eq. (5) above]
with $\Omega_\alpha = -\frac{1}{2} \sum_q \left( \mu_{qa} \beta_{qa}(t) + \mu^*_{qa} \beta_{qa}(t) \right)$ a scalar energy shift and the eigenvalues $\mu_{q,\alpha}$ defined through Eqs. (7).

Since any square root $\hat{\gamma}_R(0)$ is determined up to a gauge unitary factor [i.e., a transformation $\hat{\gamma}_R(0) \rightarrow \hat{\gamma}_R(0) \cdot U$, with $UU^\dagger = U^\dagger U = I$, leaves $\hat{\rho}_R(0)$ unaffected], the basis operators $\hat{\alpha}_R$ display the same gauge dependence. Their explicit structure becomes transparent in terms of a [orthonormal] wave function basis $\{ |\chi_\alpha\rangle \}$ on the corresponding DFS. By definition, the states $|\chi_\alpha\rangle$ are simultaneous eigenfunctions of all $w_q$ and $w^*_q$, i.e., $w_q |\chi_\alpha\rangle = \mu_{q,\alpha} |\chi_\alpha\rangle$ and $w^*_q |\chi_\alpha\rangle = \mu^*_{q,\alpha} |\chi_\alpha\rangle$. Then a state operator $\hat{\alpha}_R$ satisfying Eqs. (7) has the general form $\hat{\alpha}_R = \sum_{\alpha} |\chi_\alpha\rangle \langle \varphi_{X,\alpha}|$, where the vectors $|\varphi_{X,\alpha}\rangle$ are arbitrary register states, not confined to the DFS spanned by the basis $\{ |\chi_\alpha\rangle \}$. Note that each DFS can contribute multiple orthogonal $\hat{\alpha}_R$ terms to $\hat{\gamma}_R(0)$, although in the present context this formal distinction bears no consequence.

The time-dependent bath displacements $\beta_{q,\alpha}(t)$ perform harmonic oscillations according to

$$i\hbar \dot{\beta}_{q,\alpha} - \hbar \omega_q \beta_{q,\alpha} = \mu_{q,\alpha}, \quad (9)$$

with vanishing initial conditions and read, as usual, $\beta_{q,\alpha}(t) = \frac{\mu_{q,\alpha}}{\hbar \omega_q} \left( e^{-i \omega_q t} - 1 \right)$. Form (3) for the exact solution makes it easy to recognize that the initially equilibrated environment evolves at later times into a statistical superposition of nonstationary Gaussian states, with a reduced density matrix

$$\hat{\rho}_c(t) = \text{Tr}_R (\hat{\rho}) = \sum_{\alpha} \frac{\langle \hat{\alpha}_R | \hat{\alpha}_R \rangle}{Z} \exp \left[ -\frac{1}{k_B T} \sum_q \hbar \omega_q \left( b_q^\dagger - \beta_{q,\alpha}(t) \right) \left( b_q - \beta_{q,\alpha}(t) \right) \right], \quad (10)$$

entangled with a register in a reduced state

$$\hat{\rho}_R(t) = \text{Tr}_c (\hat{\rho}(t)) = e^{-\frac{i}{\hbar} H_R t} \cdot \left[ \sum_{\alpha',\alpha} \eta_{\alpha',\alpha}(t) \hat{\alpha}_{\alpha'}^\dagger(0) \cdot \hat{\alpha}_{\alpha}(0) \right] \cdot e^{\frac{i}{\hbar} H_R t}, \quad (11)$$

where the bath-mediated correlation $\eta_{\alpha',\alpha}(t)$ reads

$$\eta_{\alpha',\alpha}(t) = \exp \left[ \frac{i}{\hbar} \int_0^t \text{d}\tau (\Omega_{\alpha'}(\tau) - \Omega_{\alpha}(\tau)) \right] \text{Tr}_{\alpha'} \left[ \hat{\gamma}_{\alpha',\alpha}(t) \cdot \hat{\gamma}_{\alpha,\alpha}^\dagger(t) \right]. \quad (12)$$

The extension of solution (3) can be achieved now simply by acknowledging that its validity is not limited to null initial displacements for the environmental modes [7]. In particular, any entangled state evolved from a standard uncorrelated state can serve as a valid initial condition. Allowing the bath displacements to sample arbitrary, nonvanishing initial displacements, so that

$$\beta_{q,\alpha}(t) = \left[ \beta_{q,\alpha}(0) + \frac{\mu_{q,\alpha}}{\hbar \omega_q} \right] e^{-i \omega_q t} - \frac{\mu_{q,\alpha}}{\hbar \omega_q}, \quad (13)$$

identifies the complete range of initial conditions compatible with the general form (3) as

$$\hat{\rho}(0) = \left[ \sum_{\alpha} \hat{\alpha}_R(0) \otimes \hat{\gamma}_{c,\alpha}(0) \right] \cdot \left[ \sum_{\alpha} \hat{\alpha}_{R}^\dagger(0) \otimes \hat{\gamma}_{c,\alpha}^\dagger(0) \right], \quad (14)$$

i.e., as a particular set of register states nontrivially entangled with an environment in a nonequilibrium superposition of Gaussian states. From a formal point of view, expression (14) requires that the square-root $\hat{\gamma}(0)$ of the initial density matrix $\hat{\rho}(0)$ be a superposition of separable and orthogonal state operators $[\text{Tr} \left[ (\hat{\alpha}_R^\dagger \hat{\gamma}_{c,\alpha}) \cdot (\hat{\alpha}_R \hat{\gamma}_{c,\alpha}^\dagger) \right] \sim \text{Tr} \left( \hat{\alpha}_R^\dagger \cdot \hat{\alpha}_R \right) \sim \delta_{\alpha',\alpha}]$, with environmental factors given by state operators $\hat{\gamma}_{c,\alpha}(0)$ for Gaussian distributions. Such a superposition is called in the following a coherent product superposition. Solution (3) shows that the evolution driven by a Hamiltonian of type (2) conserves such a decomposition at all times, i.e., the square root $\hat{\gamma}(t)$ of the evolved density matrix $\hat{\rho}(t)$ remains a coherent product superposition. In this general case distinct $\hat{\alpha}_R$ terms defined on the
corresponding to the time-dependent environment state with where the correlation amplitudes of overlap amplitudes of the coherent environment states entangled with \( \hat{\rho}_q(t) \) correspond to high energy. In other words, the bath factors represent, up to a phase factor, the thermally weighted sum

\[
\sum_n \left| \{ | \chi_n(t) \rangle \} \right|^2 \coth \left( \frac{\hbar \omega_n}{2 k_B T} \right) .
\]

The above Eqs.(15) are identical to the 'equilibrium' expressions, up to the modified time-dependence of the displacement parameters under nonthermal initial conditions. For the spin-boson model (1) with an uncorrelated, thermal initial condition, one can recover straightforwardly the expressions recently derived in ref. [9]. It is also useful, for later reference, to write the bath factors in the more transparent form

\[
\eta_{\alpha \alpha} (t) = \exp \left[ \frac{i}{\hbar} \int_0^t d\tau \left( \Omega_{\alpha \alpha} (\tau) - \Omega_{\alpha} (\tau) \right) \right] \sum_{n_q} \exp \left( -E_{n_q} / k_B T \right) \langle n_q, \alpha \rangle (t) \langle n_q, \alpha \rangle (t) ,
\]

where \( | \{ n_q, \alpha \rangle \rangle = \exp \left[ \sum_q \left( \beta_{q\alpha} (t) b_q^\dagger - \beta_{q\alpha}^* (t) b_q \right) \right] \langle n_q \rangle = \prod_q \sqrt{n_q!} \left( b_q^\dagger (t) \right)^n_q \left| \{ \beta_{q\alpha} (t) \rangle \right) , \) with \( b_q^\dagger (t) = b_q^\dagger - \beta_{q\alpha}^* (t) \), is the time-dependent environment state with \( n_q \) displaced quanta of mode \( q \) excited over the displaced vacuum \( | \{ \beta_{q\alpha} (t) \rangle \rangle \) corresponding to \( \hat{\alpha}_R \) \( | \{ n_q \rangle \rangle \) is the standard excited state with \( n_q \) quanta in mode \( q \) \( \rangle \) , and \( E_{n_q} = \sum_q \hbar \omega_q n_q \) is the corresponding energy. In other words, the bath factors represent, up to a phase factor, the thermally weighted sum of overlap amplitudes of the coherent environment states entangled with \( \hat{\alpha}_R \) and \( \hat{\alpha}'_R \).

III. DECOHERENCE FREE STATES ENTANGLED WITH A NONEQUILIBRIUM ENVIRONMENT

Let the basis \( | \{ \chi \rangle \rangle \) provide an irreducible representation of the direct sum of register DFS. If every state operator \( \hat{\alpha}_R (0) \) is expressed in terms of the states \( | \chi \rangle \) as \( \hat{\alpha}_R (0) = \sum_{\chi} \langle \chi \rangle (1) | \phi_{\alpha, \chi} \rangle \langle \phi_{\alpha, \chi} | \), the reduced register state (11) assumes the form

\[
\hat{\rho}_R (t) = e^{-\hat{\pi} H_R t} \left( \sum_{\chi'} \kappa_{\chi' \chi} (t) | \chi' \rangle \langle \chi' | \right) e^{\hat{\pi} H_R t} ,
\]

where the correlation amplitudes \( \kappa_{\chi' \chi} (t) \) read

\[
\kappa_{\chi' \chi} (t) = \sum_{\alpha' \alpha} \eta_{\alpha' \alpha} (t) \langle \phi_{\alpha', \chi} | \phi_{\alpha, \chi} \rangle ,
\]

and the orthogonality of the \( \hat{\alpha}_R \)-s requires that \( \sum_{\chi} \langle \phi_{\alpha', \chi} | \phi_{\alpha, \chi} \rangle \sim \delta_{\alpha' \alpha} \). According to expressions (17) and (18), the evolution of the reduced state \( \rho_R \) deviates from the unperturbed, unitary dynamics due to the modulation of the intrinsic correlations represented by the factors \( \langle \phi_{\alpha', \chi} | \phi_{\alpha, \chi} \rangle \) by environment-mediated processes, which contribute the bath factors \( \eta_{\alpha' \alpha} \). The occurrence of multiple terms in the amplitudes \( \kappa_{\chi' \chi} \) is related to the environment-entangled nature of the initial register state. When the initial state is not entangled, all wave functions from a given DFS develop entanglement with the same Gaussian distribution of the environment [formally \( \alpha \) labels a DFS] and the sum in expression (18) reduces to a single term. Moreover, the correlation amplitudes for states pertaining to the same DFS do not experience environmental decoherence, since the corresponding bath factor reduces to unity. For this reason a nonentangled register state distributed on a single DFS evolves in an unperturbed, unitary manner. Surprisingly enough, it turns out that time-independent correlation amplitudes \( \kappa \) leading to unitary, unperturbed propagation, can occur as well for entangled states, eventually distributed over multiple DFS.
A first hint in this direction is seen by restricting the register state operators \( \hat{\alpha}_R \) to the same degenerate DFS, such that \( \mu_{q,\alpha} = \mu_q \) for all \( \alpha \), but dephasing the entangled bath components through different initial displacements, \( \beta_{q\alpha} (0) \neq \beta_{q\alpha'} (0) \). The corresponding register state is so distributed over a single DFS, but is also entangled with the environmental modes. In this case the time-dependence of the dissipative rates \( \Gamma_{\alpha'\alpha} \) is unconditionally suppressed, while the phase factors \( \Phi_{\alpha'\alpha} \) and the relative energy shifts (\( \Omega_{\alpha'} - \Omega_\alpha \)) show a time-dependence governed by the form \( \sum_q \frac{\mu_q}{\hbar \omega_q} [\beta_{q\alpha} (0) - \beta_{q\alpha'} (0)] e^{-i\omega_q t} \) and the complex conjugated. If one recalls that the environment is ordinarily symmetric under space inversions, such that \( \omega_q = -\omega_q \), it is easily noted that the latter terms may disappear provided \( \mu_q^* [\beta_{q\alpha} (0) - \beta_{q\alpha'} (0)] + \mu_q^* [\beta_{q\alpha'} (0) - \beta_{q\alpha} (0)] = 0 \) or, equivalently, \( \mu_q^* \beta_{q\alpha} (0) + \mu_q^* \beta_{q\alpha'} (0) = \lambda_q \) for all \( \alpha \) and some arbitrary \( \lambda_q \). In this case the bath correlation factors \( \eta_{\alpha'\alpha} \) become stationary and the register state propagates unitarily, in an unperturbed fashion.

A more powerful indication comes from the straightforward observation that if the environmental modes are displaced over the equilibrium positions \( \frac{\mu_q}{\hbar \omega_q} \), their state becomes stationary and the bath factors \( \eta_{\alpha'\alpha} \) vary only through time-dependent phase factors, as \( \exp \left[ \frac{i}{\hbar} (\Omega_{\alpha'} - \Omega_\alpha) t \right] \eta_{\alpha'\alpha} (0) \). If all state operators \( \hat{\alpha}_R \) are defined on nonoverlapping, orthogonal subspaces, each included in, but not necessarily identical to a DFS, then each phase factor \( \exp \left[ -\frac{i}{\hbar} \Omega_{\alpha} t \right] \) becomes characteristic of the associated subspace. Accordingly, the oscillatory part of \( \eta_{\alpha'\alpha} \) can be absorbed into a renormalized register Hamiltonian, via energy shifts specific to each subspace, and the propagation of the reduced register state is seen to be manifestly unitary, although the register remains entangled with the environment. In the renormalized register Hamiltonian, via energy shifts specific to each subspace, each included in, but not necessarily identical to a DFS, then each phase factor \( \exp \left[ -\frac{i}{\hbar} \Omega_{\alpha} t \right] \) becomes characteristic of the associated subspace. Accordingly, the oscillatory part of \( \eta_{\alpha'\alpha} \) can be absorbed into a renormalized register Hamiltonian, via energy shifts specific to each subspace, and the propagation of the reduced register state is seen to be manifestly unitary, although the register remains entangled with the environment. In the situation when \( \eta_{\alpha'\alpha} (0) \to 0 \) for all \( \alpha' \neq \alpha \), as happens for model (1) in an environment with a one-dimensional spectral density, the presence of the bath is effectively erased in \( \rho_R (t) \). The latter evolves unperturbed, as a block diagonal distribution on disjoint DFS, and individual decoherence-free states may become eventually pointer states [11]. But unless the register state is confined within a single DFS, entanglement with the environment never vanishes, and the overall state does not reduce to an uncorrelated, factored form.

Let us seek the general conditions for the unitary and unperturbed propagation of the reduced register state, starting from the requirement of time-independent bath factors, that is, from

\[
\frac{d}{dt} \ln \eta_{\alpha'\alpha} = \frac{i}{\hbar} (\Omega_{\alpha'} - \Omega_\alpha) t \eta_{\alpha'\alpha} (0) - \frac{i}{\hbar} \frac{d \Phi_{\alpha'\alpha}}{dt} - \frac{d \Gamma_{\alpha'\alpha}}{dt} = 0 .
\]  

Substitution of the corresponding expressions, including the explicit time-dependence (13) for the bath displacements, and a little algebra yields

\[
\frac{i}{\hbar} \frac{d}{dt} \ln \eta_{\alpha'\alpha} = F_0 + \sum_q \left( \frac{|\mu_{q\alpha'}|^2}{\hbar \omega_q} - \frac{|\mu_{q\alpha}|^2}{\hbar \omega_q} \right) ,
\]

where

\[
F_q = \sum_q \left( \frac{|\mu_{q\alpha'}|^2}{\hbar \omega_q} - \frac{|\mu_{q\alpha}|^2}{\hbar \omega_q} \right) .
\]

\[
F_{q,+} = (\mu_{q\alpha'}^* \beta_{q\alpha'}^0 - \mu_{q\alpha}^* \beta_{q\alpha}^0) + 2 (\mu_{q\alpha'}^* \beta_{q\alpha}^0 - \mu_{q\alpha'}^* \beta_{q\alpha'}^0) - \coth \left( \frac{\hbar \omega_q}{2k_B T} \right) (\beta_{q\alpha'}^0 - \beta_{q\alpha}^0) (\mu_{q\alpha'}^* - \mu_{q\alpha}^*),
\]

\[
F_{q,-} = (\mu_{q\alpha'}^* \beta_{q\alpha'}^0 - \mu_{q\alpha}^* \beta_{q\alpha}^0) + 2 (\mu_{q\alpha'}^* \beta_{q\alpha}^0 - \mu_{q\alpha'}^* \beta_{q\alpha'}^0) + \coth \left( \frac{\hbar \omega_q}{2k_B T} \right) (\beta_{q\alpha'}^0 - \beta_{q\alpha}^0) (\mu_{q\alpha'}^* - \mu_{q\alpha}^*) ,
\]

with the labels \( \alpha \) and \( \alpha' \) dropped for simplicity, and we have denoted \( \beta_{q\alpha}^0 = \beta_{q\alpha} (0) + \frac{\mu_{q\alpha}}{\hbar \omega_q} \). Passing now to the continuum limit, let us separate the contributions from degenerate modes in expression (20) and rewrite condition (19) in the form

\[
F_0 + \frac{1}{2} \int_0^\infty d\omega \frac{d |q|}{d\omega} \left[ \left( \oint_{\mathbb{S}_\omega} dS_q F_{q,+} \right) e^{i\omega t} + \left( \oint_{\mathbb{S}_\omega} dS_q F_{q,-} \right) e^{-i\omega t} \right] = 0 ,
\]

where \( S_\omega \) denotes the surface \( \omega_q = \omega \) in reciprocal space. Since the cancellation of a constant term \( F_0 \neq 0 \) demands
and, in all likelihood, divergent displacements for the static ground mode $\omega = 0$, it is seen that an unperturbed evolution of the register requires $F_0 = 0$, i.e.,

$$
\int dq \frac{|\mu_{q\alpha '}|^2}{\hbar \omega_q} = \int dq \frac{|\mu_{q\alpha}|^2}{\hbar \omega_q},
$$

(24a)
as well as

$$
\oint_{S_\omega} dS_q F_{q,\pm} = 0
$$

(24b)
for all $\alpha$ and $\alpha'$. Conditions (24b) reduce eventually to the simpler linear system

$$
\oint_{S_\omega} dS_q \left( \beta_{q\alpha'}^0 - \beta_{q\alpha}^0 \right) \left( \mu_{q\alpha'}^* - \mu_{q\alpha}^* \right) = 0,
$$

(25a)
and

$$
\oint_{S_\omega} dS_q \left[ \left( \mu_{q\alpha'}^0 \beta_{q\alpha'}^0 - \mu_{q\alpha}^* \beta_{q\alpha}^0 \right) + 2 \left( \mu_{q\alpha'}^* \beta_{q\alpha}^0 - \mu_{q\alpha}^0 \beta_{q\alpha'}^0 \right) \right] = 0,
$$

(25b)
when noted that according to the explicit expressions (21) the terms in $\coth \left( \frac{\hbar \omega_q}{2k_B T} \right)$ contributed by the dissipative exponent $\Gamma_{\alpha'\alpha}$ cancel separately.

It is seen here that the unperturbed evolution of a register distribution requires both a specific phasing of the entangled environmental modes, through proper displacements $\beta_{q\alpha}$ [Eqs. (25)], and a particular type of coupling to the bath for all register states involved [Eq. (24a)]. For any register of type (1) such states always exist, because for any DFS characterized by $\mu_{q\alpha}$ there exists another DFS characterized by $(-\mu_{q\alpha})$ and related to the former by a reversal of all qubits along direction $z$ ($|1\rangle_n \rightarrow |1\rangle_n$, $|\rangle_n \rightarrow |\rangle_n$), in every state. The trivial solution to Eqs. (25), $\beta_{q\alpha}^0 = \beta_{q\alpha'}^0 = 0$ for all $q$, corresponds to the case of stationary displacements discussed in the beginning of this section, while nontrivial solutions are essentially conditioned by the degeneracy of the environmental modes. It is worth noticing that the number of distinct DFS contributing to a decoherence-free distribution of the type discussed here is theoretically arbitrary [if finite], unless the environment is 1-dimensional and displays only two-fold degenerate modes $|\omega_q = \omega_q\rangle$. For the latter case, the number of distinct DFS involved cannot exceed 2, since the number of unknown $\beta_{q\alpha}$s in system (25) must exceed the number of constraints.

In addition to the conditions above, in a nontrivial environment-entangled state [some of] the bath factors must be nonvanishing, meaning that the corresponding dissipative factors $\Gamma_{\alpha'\alpha}$ must be finite. Assuming a nontrivial solution to system (24) does exist, substitution of the corresponding bath displacements in expression (15b) yields straightforwardly

$$
\Gamma_{\alpha'\alpha} \propto \frac{1}{2} \int d\omega \frac{dq}{d\omega} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \oint_{S_\omega} dS_q \left[ \left| \beta_{q\alpha'}^0 - \beta_{q\alpha}^0 \right|^2 + \left| \frac{\mu_{q\alpha'}^0 - \mu_{q\alpha}^0}{\hbar \omega} \right|^2 \right],
$$

(26)
and shows that $\Gamma_{\alpha'\alpha}$ cannot be finite unless its value

$$
\Gamma^0_{\alpha'\alpha} \propto \frac{1}{2} \int d\omega \frac{dq}{d\omega} (\hbar \omega)^{-2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \oint_{S_\omega} dS_q \left| \mu_{q\alpha'} - \mu_{q\alpha} \right|^2
$$

(27)
for equilibrium initial displacements $[\beta_{q\alpha}^0 = \beta_{q\alpha'}^0 = 0]$ is also finite. This intrinsic equilibrium term $\Gamma^0$ sets an upper limit on the magnitude of the corresponding bath coefficient, and therefore on the amplitude of the correlation between the associated register states. It increases with the temperature $[\Gamma^0 \rightarrow \infty$ as $T \rightarrow \infty$] regardless of the exact density of states or of the form of the coupling constants $\mu_{q\alpha}$, and gradually shrinks the set of register distributions compatible with unperturbed propagation toward trivial states, block diagonal on DFS. Moreover, the presence of properly phased coherent oscillations of the bath modes $[|\beta_{q\alpha}^0 - \beta_{q\alpha'}^0| \geq 0]$ results invariably in decreased bath coefficients and
decreased correlation amplitudes. Hence the preservation of an unperturbed register evolution seems to involve a trade-off between the stabilizing action of a nonequilibrium environment and the magnitude of the conserved register correlations. On the other hand, the finite or infinite character of $\Gamma_{alpha}^0$ at finite temperatures does depend on both the density of degenerate bath modes, and the coupling constants for the input states. To find the origin of this effect let us set $\beta_{\alpha}^0 = - (q_{\alpha}/\hbar \omega_q)$ and $\beta_{\alpha}^\prime = -(q_{\alpha}^\prime/\hbar \omega_q)$, and write the overlaps $\langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle$ in the form

$$\langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle = \langle n_q \rangle \exp \left[ \sum_q \left( \frac{\mu_{q_{\alpha}^0}}{\hbar \omega_q} b_q^\dagger - \frac{\mu_{q_{\alpha}^0}}{\hbar \omega_q} b_q \right) \right] \exp \left[ - \sum_q \left( \frac{\mu_{q_{\alpha}^\prime}}{\hbar \omega_q} b_q^\dagger - \frac{\mu_{q_{\alpha}^\prime}}{\hbar \omega_q} b_q \right) \right] \langle n_q \rangle$$

$$\times \nu_{\alpha^0} \prod_q \langle 0_q \rangle \exp \left[ - \frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right] \exp \left[ \frac{\mu_{q_{\alpha}^\prime} - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right] b_q^\dagger \rangle 0_q \rangle,$$  \hspace{1cm} (28)

where $\nu_{\alpha^0} = \exp \left[ - \frac{1}{2} \sum_q \left( \frac{\mu_{q_{\alpha}^0}}{\hbar \omega_q} - \frac{\mu_{q_{\alpha}^0}^\prime}{\hbar \omega_q} \right)^2 \right] \exp \left[ - \sum_q \frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right] \exp \left[ - \frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right]$. The vacuum averages in the latter expression can be calculated as

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \frac{\partial^n}{\partial (\lambda^*)^m} \langle 0_q | \exp [-\lambda^* b_q] \exp [\lambda b_q^\dagger] | 0_q \rangle = P_n (\lambda, \lambda^*) \exp [-|\lambda|^2],$$

with $P_n$ a polynomial expression, to the result that

$$\langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle \propto \exp \left[ - \sum_q \left( \frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right)^2 \right] \exp \left[ - \sum_q \frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q} \right]$$

or, equivalently,

$$\exp \left[ - \Gamma_{alpha}^0 (T = 0) \right] \propto \prod_q \langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle.$$  \hspace{1cm} (29b)

Here $\langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle$ denotes the displaced vacuum corresponding to the equilibrium displacements $[-\mu_{q_{\alpha}^0}/(\hbar \omega_q)]$. Thus the magnitude of $\Gamma_{alpha}^0 (T = 0)$ is set by the product of the overlap amplitudes for the displaced equilibrium vacua associated with the entangled register states. Clearly, $\Gamma_{alpha}^0 (T = 0)$ vanishes when $\langle \{n_{Q_{\alpha}^0}\}|\{n_{Q_{\alpha}^0}\}\rangle \propto \exp[-(1/2)\frac{\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}}{\hbar \omega_q}] \rightarrow 0$ as $q \rightarrow 0$, i.e., when $|\mu_{q_{\alpha}^0} - \mu_{q_{\alpha}^0}^\prime|/(\hbar \omega_q)$ diverges as $q \rightarrow 0$, unless the density of modes compensates for the contributions from low-frequency modes. At finite temperatures the effect is further amplified by the thermal excitation of low-frequency states. On the other hand, the situation can improve considerably when $|\mu_{q_{\alpha}^0} - \mu_{q_{\alpha}^0}^\prime|/(\hbar \omega_q)$ remains finite as $q \rightarrow 0$.

For a more precise discussion it is convenient to refer now to model (1). Under a common functional prescription for the density of states, it turns out that a single qubit register $[N = 1, \chi_{qn} = \chi(\omega_q)]$ can display a finite $\Gamma^0$ only in a 3-dimensional environment. Not surprisingly, a similar behavior is also seen in multiquit registers with collective decoherence $[N > 1, \chi_{qn} = \chi(\omega_q)]$. However, linear registers with individual decoherence $[\chi_{qn} = \chi(\omega_q) \exp(i q \cdot r_n)]$ present states for which $\exp[-(1/2)\mu_{q_{\alpha}^0}^\prime - \mu_{q_{\alpha}^0}^0/(\hbar \omega_q)]$ remains finite as $q \rightarrow 0$, and which generate finite $\Gamma^0$-s even in a 1-dimensional environment.

Indeed, a single qubit register can also only have two one-dimensional DFS, hence the state operators $\hat{\alpha}_R$ read simply $\hat{\alpha}_1 = \langle \uparrow | \langle \phi_1 |$, and $\hat{\alpha}_1 = \langle \downarrow | \langle \phi_1 |$, and the associated $\mu_{q_{\alpha}^0}$-s become $\mu_{q_{\alpha}^0} = -\mu_{q_{\alpha}^0} = \chi(\omega_q)$. As a result, condition (24a) is satisfied by default, system (25) reduces to

$$\oint_{S_{\omega}} dS_{\omega} \beta_{q_{\alpha}^0}^0 = \oint_{S_{\omega}} dS_{\omega} \beta_{q_{\alpha}^0}^0 = 0,$$  \hspace{1cm} (30a)

and a finite $\Gamma_{11}$ requires

$$\int d\omega \frac{d |q| G(\omega)}{d\omega} \left| \frac{\chi(\omega)}{\hbar \omega} \right|^2 \coth \left( \frac{\hbar \omega}{2 k_B T} \right) < \infty,$$  \hspace{1cm} (30b)
where \( G(\omega) \) is the density of modes at frequency \( \omega \). In general, the density \( G(\omega) \) grows as \( \text{area}(S_2) \sim \omega^{d-1} \), where \( d \) is the dimension of the environment, and is characterized by a natural ultraviolet cut-off frequency \( \omega_c \), which sets the upper limit for the rate of dissipation processes in the environment. This necessary feature is usually accounted for by setting \( G(\omega) \propto \omega^{d-1} \exp[-\omega/\omega_c] \). It is also common to assume a quasi-linear dispersion \( d[q]/d\omega \sim \text{const} \) and a coupling \( \chi(\omega) \propto \sqrt{\omega} \), so that the final prescription reads \( (d[q]/d\omega)G(\omega)\chi(\omega)^2 = \lambda(\hbar^2/2)(\omega/\omega_c)^d \exp[-\omega/\omega_c] \), where the scaling factors are chosen such that \( \lambda \) is an adimensional constant. In this case, \( \chi(\omega)/\omega \propto (1/\sqrt{\omega}) \to \infty \) as \( q \to 0 \), and \( \Gamma^0_{\text{class}} \) cannot be finite unless this singularity is balanced by the density of states. This is also evident from the integrand in Eq. (30b), which behaves in the low-frequency limit as \( \omega^{d-3} \). In particular, a 1-dimensional density of states proves insufficient to counteract the low-frequency contribution even at zero temperature. A 2-dimensional density suffices at zero temperature, but fails at finite temperatures. Only in a 3-dimensional environment does \( \Gamma^0_{\text{class}} \) acquire finite values at both zero and finite temperatures. In this case, its exact expression can be given an analytical form in terms of the generalized Riemann zeta function \( \zeta(g,z) = (1/\Gamma(g)) \int_0^\infty d\xi \xi^{g-1} e^{-\xi}(1-e^{-\xi}) \) [here \( \Gamma(g) \) denotes the Gamma function] as

\[
\Gamma^0_{\text{class}} \left( \frac{k_BT}{\hbar\omega_c} \right) = \Gamma^0_{\text{class}}(0) \left[ 2 - \frac{(k_BT)^2}{\hbar\omega_c^2} \zeta \left( 2, \frac{k_BT}{\hbar\omega_c} \right) - 1 \right],
\]

where \( \Gamma^0_{\text{class}}(0) = \lambda \). Expression (31) shows that \( \Gamma^0_{\text{class}} \) increases monotonously with the temperature, behaving as \( \Gamma^0_{\text{class}} \approx \Gamma^0_{\text{class}}(0) \left[ 1 + (\pi^2/3)(k_BT/\hbar\omega_c)^2 \right] \) in the low temperature limit \( k_BT < < \hbar\omega_c \) and as \( \Gamma^0_{\text{class}} \approx 2\Gamma^0_{\text{class}}(0) (k_BT/\hbar\omega_c) \) at high temperatures \( k_BT >> \hbar\omega_c \). Therefore the maximum amplitude of register correlations that can be carried in an environment-entangled unperturbed propagation decreases fast to zero in a classical environment, with temperatures \( k_BT \geq \hbar\omega_c \). A nontrivial unperturbed propagation is seen to require a low-temperature, quantum environment.

The situation is very similar in the closely related case of a multi-qubit register \( [N > 1] \) with collective decoherence \( |\chi_{qn} = \chi(\omega_q)| \), when \( \mu_{xn} = \chi(\omega_q) \sum_{n=0}^{N-1} s_n \), where \( s_n \) is the eigenvalue of \( \sigma_z^{(n)} \) labeling the corresponding decoherence-free states. For brevity we have discarded labels for individual states in the latter sum, since all states in the same DFS are characterized by the same \( \sum_{n=0}^{N-1} s_n \). Due to condition (24a), the register states contributing to a given environment-entangled decoherence-free distribution can only belong to either of a pair of DFS characterized by coupling constants \( \mu_{\alpha}, \alpha \) and \( -\mu_{\alpha}, \alpha \). Thus system (25) reduces again to Eqs. (30a) and condition (30b) for a finite \( \Gamma^0 \) remains unchanged, although the exact value of \( \Gamma^0 \) changes by a factor of \( \left| \sum_{n=0}^{N-1} s_n \right|^2 \). This implies that the correlation amplitudes allowed in an unperturbed propagation decrease strongly for register states with spin projections \( \left| \sum_{n=0}^{N-1} s_n \right| > 1 \).

Consider now a N-qubit register \( [N > 1] \) in an individual decoherence regime, such that \( \mu_{xn} = \chi(\omega_q) \sum_{n=0}^{N-1} e^{iqn}\sigma_{n,a} \), where \( q_n \) denotes the position vector of the \( n \)-th qubit. Because the DFS of this register are trivially 1-dimensional and correspond to the eigenstates \( \{|s_{n,a}\} \) of the unperturbed \( H_R \), the state operators \( \hat{\sigma}_R \) can only be of the form \( \{|s_{n,a}\}\langle \phi_{\alpha}\rangle \). Thus the index \( \alpha \) labels here the unperturbed eigenstates. Let the register have a linear geometry, such that \( r_n = na \) and select a specific eigenstate \( \{|s_{n,a}\} \). The set of states compatible with condition (24a) for \( \{|s_{n,a}\} \) includes the \( N \) states \( \{|s_{n,a}\} \] related to \( \{|s_{n,a}\} \) by a cyclic permutation modulo \( N \), i.e. \( s_{n,a} = s_{n-1,a} \). Indeed, the corresponding coupling constants differ from \( \mu_{aq_n} \) only by a phase factor, respectively \( \mu_{aq_n} = e^{i\eta m} \mu_{aq_n} \). For any pair \( \alpha_m \) and \( \alpha_{m'} \) of such states, the quantity \( |\langle \mu_{aq_m} - \mu_{aq_{m'}} \rangle/\hbar\omega_q|^2 = 4(|\chi(\omega)/\hbar\omega|) \sin(|m-m'|/2) \sum_{n=0}^{N-1} e^{in\omega t} |s_{n,a}|^2 \) behaves as \( \omega \) when \( q \to 0 \), such that all overlap amplitudes in Eq. (29b) remain finite. Here we made use of the transit time \( t_s \) defined by \( qa = \omega t_s \). Accordingly, the integrand in the equilibrium dissipative factor

\[
\Gamma^0_{\text{class},m,a} \propto 2 \int_0^{\infty} d\omega \, \frac{d[q]/d\omega}{\hbar\omega} \chi(\omega) \coth \left( \frac{\hbar\omega}{2k_BT} \right) \int_{S_\omega} dS_q \sin^2 \left( \frac{m-m'}{2} \right) \sum_{n=0}^{N-1} e^{i\eta m} |s_{n,a}|^2 \]

behaves as \( \omega^{d-1} \) in the low-frequency limit and \( \Gamma^0_{\text{class},m,a} \) remains finite in any environment, at any finite temperature. One can also consider the mirror permutation states with \( s_{n,\overline{m}} = s_{N-n,a} = s_{N-n+m,a} \) and \( \mu_{q,\overline{a}} = \ldots \).
The equilibrium dissipative factor of any pair of such states reads \( \Gamma^0_{\alpha,\alpha} = \Gamma^0_{\alpha,\alpha} \), while the factor corresponding to a direct permutation state and a mirror permutation state is given by

\[
\Gamma^0_{\alpha,\pi_m} \propto 2 \int_0^{2\pi} d\omega' |\psi(\omega')| \chi(\omega')^* \chi(\omega')^2 \coth \left( \frac{\hbar\omega}{2kBT} \right) dS' \sum_{n=0}^{N-1} s_{n,\alpha_0} \sin \left[ q \left( n + m' + m - N \right) \right] a^2 ,
\]

and again proves to be finite in any environment. Consequently, any environment-entangled distributions which involve the 2N direct and mirror permutation counterparts of \( \{ s_{n,\alpha_0} \} \) and comply with conditions (25) generate nontrivial register mixtures propagating in an unperturbed manner. Note however that in a 1-dimensional environment such a mixture can only accommodate two distinct pure states [see Eqs. (25)]. It is also worth noting that in this case the precise magnitude of \( \Gamma^0 \) is strongly dependent on the input states.

In concluding this section let us stress that the environment-entangled unperturbed evolution discussed above should be understood as one particular case of environment-entangled unitary dynamics. For instance, conditions (24b) or system (25) suffice to guarantee a unitary, quasi-unperturbed propagation of \( \rho_R(t) \) provided the state operators \( \hat{\alpha}_R \) correspond to distributions on orthogonal subspaces, such that \( \hat{\alpha}_R^\dagger \hat{\alpha}_R = (\hat{\alpha}_R^\dagger)^2 \cdot \hat{\alpha}_R = 0 \) and \( [\hat{\alpha}_R^\dagger, \hat{\alpha}_R^\dagger(\hat{\alpha}_R^\dagger)] = 0 \). This is because the stationary phase term \( F_0 \) in Eq. (13) is separable into contributions from individual \( \hat{\alpha}_R \)-s, hence from orthogonal subspaces, and contributes merely stationary energy shifts to the unperturbed Hamiltonian. Furthermore, for \( \hat{\alpha}_R \)-s defined on orthogonal subspaces it is also possible to obtain sufficient conditions for the unitary propagation of \( \rho_R(t) \) by requiring only that the dissipative factors of all \( \eta_{\alpha'\alpha} \) be stationary \( [d\Gamma_{\alpha'\alpha}/dt = 0] \) and that all phase factors be separable, which amounts in fact to \( d\Phi_{\alpha'\alpha}/dt = 0 \). The unperturbed Hamiltonian is modified then by time-dependent energy shifts and the bath displacements must satisfy constraints very similar to system (25), reading

\[
\sum_{\omega_q = \omega \neq 0} q (\beta_{q\alpha'}^0 - \beta_{q\alpha}^0) (\mu_{q\alpha'}^* - \mu_{q\alpha}^*) = 0 , \tag{34a}
\]

\[
\sum_{\omega_q = \omega \neq 0} (\mu_{q\alpha'}^0/\beta_{q\alpha}^0 - \mu_{q\alpha}^0/\beta_{q\alpha'}^0) = 0 . \tag{34b}
\]

The requirement that \( \Gamma^0_{\alpha'\alpha} \) be finite remains, of course, unchanged.

**IV. ELEMENTARY ENTANGLEMENT REPHASING THROUGH QUBIT FLIPPING: IMPROVED DYNAMICAL CONTROL OF DECOHERENCE**

A practical approach to the proposed stabilization of register states through proper (re)phasing of the environment does not necessarily require direct manipulation of the environment. As a counterexample, let us point out that at least one method of decoherence control available in the literature involves an elementary rephasing of entanglement of the type discussed here. We refer specifically to quantum bang-bang control [4], inspired by the multi-pulse decoupling techniques of NMR, which counteracts decoherence through a train of identical spin-flip cycles. Each cycle generates a revival of coherence through a pair of coherent \( \pi \)-pulses that alternately flip the state of the register. When this process is examined in the context of the extended density matrix solution of Sec. II, the cause of this revival effect is distinctly exposed as an elementary adjustment of the entanglement with the environment. In other words, spin flipping provides a simple working procedure for manipulating entanglement. In a supplementary outcome, the exact density matrix solution is also applied to a straightforward refinement of the bang-bang technique, which lowers the working cycle frequency by a factor of at least 2.

For simplicity, consider only model (1) with a single qubit. A bang-bang procedure applies a succession of resonant radiofrequency pulses \([\pi\text{-pulses}]\) which evolve one eigenstate of the qubit into the other, i.e., \(|\uparrow\rangle \rightarrow |\downarrow\rangle \) and \(|\downarrow\rangle \rightarrow |\uparrow\rangle \), on a time scale \( \tau_p \) short compared to the typical decoherence time. As before, throughout the following we assume the Schroedinger picture. If the rf field is strong enough, the interaction of the qubit with the environment can be neglected during the pulse, so that the bath coordinates remain unaffected. Therefore an environment-entangled state, which reads
\[ \hat{\rho}(t_-) = |↓\rangle \langle \downarrow| \hat{\rho}_{e,\uparrow} |\uparrow\rangle + |\uparrow\rangle \hat{\rho}_{e,\downarrow} |\downarrow\rangle + |\downarrow\rangle \langle \downarrow| \hat{\rho}_{e,\uparrow} |\uparrow\rangle + |\uparrow\rangle \hat{\rho}_{e,\downarrow} |\downarrow\rangle \]

immediately before the pulse, transforms into

\[ \hat{\rho}(t_+) = |\downarrow\rangle \hat{\rho}_{e,\uparrow} |\uparrow\rangle + |\uparrow\rangle \hat{\rho}_{e,\downarrow} |\downarrow\rangle + |\downarrow\rangle \langle \downarrow| \hat{\rho}_{e,\uparrow} |\uparrow\rangle + |\uparrow\rangle \hat{\rho}_{e,\downarrow} |\downarrow\rangle \]

immediately after the pulse. An elementary spin-flip cycle consists of two such pulses applied at a time interval \( \Delta t \), the first of which reverses the qubit state, while the second restores the original configuration. Between any two pulses the qubit-environment system resumes the dynamics described by Hamiltonian (1). For the purpose of illustration, it is sufficient to examine the process in the limit of infinitely narrow pulses, \( \tau \rightarrow 0 \) [4], when each cycle can be approximated by the following piecewise evolution: i) propagation under Hamiltonian (1) for a duration \( \Delta t \), starting at time \( t_n \); ii) instantaneous \( \pi \)-pulse and interchange of qubit eigenstates at time \( (t_n + \Delta t) \); iii) evolution under Hamiltonian (1) from time \( (t_n + \Delta t) \) to time \( t_{n+1} = t_n + 2\Delta t \); iv) second \( \pi \)-pulse and interchange of qubit eigenstates at time \( t_{n+1} \).

The analysis of this process in the framework of solution (3) is quite straightforward provided we shift focus from the qubit eigenstates to the entangled environment distributions. Indeed, the transformation of a qubit-environment state upon application of a \( \pi \)-pulse can be interpreted also as a switch of the environment distributions associated with the qubit states, in the sense that \( \hat{\rho}_{e,\uparrow} \rightarrow \hat{\rho}_{e,\downarrow} \), \( \hat{\rho}_{e,\downarrow} \rightarrow \hat{\rho}_{e,\uparrow} \), etc. In terms of state operators, this amounts to the simpler interchange \( \hat{\gamma}_{e,\uparrow} \rightarrow \hat{\gamma}_{e,\downarrow} \) and \( \hat{\gamma}_{e,\downarrow} \rightarrow \hat{\gamma}_{e,\uparrow} \). The latter implies that the associated bath displacements before the rf pulse for \( \hat{\gamma}_{e,\uparrow} \) become initial displacements for \( \hat{\gamma}_{e,\downarrow} \) after the pulse and vice versa, i.e., \( \beta \equiv \beta_{\uparrow}(n\Delta t + 0) = \beta_{\downarrow}(n\Delta t - 0) \) and \( \beta_{\downarrow}(n\Delta t + 0) = \beta_{\uparrow}(n\Delta t - 0) \). It follows that the interchange of qubit eigenstates following a \( \pi \)-pulse is equivalent to a rephasing of the entangled environment distributions. The time-dependence of the bath displacements for the entangled \( \hat{\gamma}_e \)'s ensues now without difficulty. Starting at time \( t_n = 2n\Delta t \) \( [t_0 = 0] \) with displacements \( \beta_{\uparrow}(t_n) \), one has: i) for \( t_n < t < (t_n + \Delta t) = (2n + 1)\Delta t \): \( \beta_{\uparrow}(t) = \beta_{\uparrow}(t_n) \pm x_q/\hbar\omega_q [\exp(-i\omega_q(t - t_n))] + x_q/\hbar\omega_q \); ii) for \( (2n + 1)\Delta t < t < t_{n+1} = 2(n + 1)\Delta t \): \( \beta_{\uparrow}(t) = \beta_{\uparrow}(t_n + \Delta t) \pm x_q/\hbar\omega_q [\exp(-i\omega_q(t - t_n - \Delta t)) + x_q/\hbar\omega_q] \).

Further, let the initial qubit-environment state be unentangled, with the environment in thermal equilibrium, such that \( \beta_{\uparrow}(0) = \beta_{\downarrow}(0) = 0 \). Since the unperturbed evolution in the first half-cycle drives same mode displacements to values of opposite sign, such that \( \beta_{\downarrow}(t) = -\beta_{\uparrow}(t) = (x_q/\hbar\omega_q) [\exp(-i\omega_qt) - 1] \) for \( 0 < t < \Delta t \), the corresponding evolved displacements will have opposite signs at any later time, that is, \( \beta_{\uparrow}(t) = -\beta_{\downarrow}(t) \) for all \( t \). As a result, all phase factors in the bath factor \( \eta_{\uparrow\downarrow}(t) \) for the reduced qubit state vanish, and the only relevant quantity remains the dissipative factor \( \Gamma(t) = 2 \sum_q |\beta_{\uparrow}(t)|^2 \coth(\hbar\omega_q/2k_BT) \). At the same time, the effect of a \( \pi \)-pulse is seen to amount to a change of sign of the bath displacements in the entangled environment distributions, such that \( \beta_{\uparrow}(k\Delta t + 0) = -\beta_{\downarrow}(k\Delta t - 0) \). Accounting for this into the time-dependence of the displacements, yields immediately stroboscopic recurrence relations of the form

\[ \beta_{\uparrow}(k\Delta t + 0) = \beta_{\uparrow}(k - 2\Delta t + 0) e^{-2i\omega_q\Delta t} + x_q/\hbar\omega_q \left(e^{-i\omega_q\Delta t} - 1 \right)^2 . \]

Solving the recurrence for \( k = 2n \) with initial condition \( \beta_{\uparrow}(0) = 0 \) recovers the stroboscopic result for \( \Gamma(t) \) obtained in ref. [4], which we write in the form

\[ \Gamma_{st}(2n\Delta t) = 2 \sum_q \coth\left(\frac{\hbar\omega_q}{2k_BT}\right) \left|\beta_{\uparrow,0}(2n\Delta t)\right|^2 \tan\left(\frac{\omega_q\Delta t}{2}\right)^2 , \]

where \( \beta_{\uparrow,0}(t) = (x_q/\hbar\omega_q) [\exp(-i\omega_qt) - 1] \) denotes the evolved displacement in the absence of pulses. Under the usual prescription for the density of states [see previous section], the above expression becomes

\[ \Gamma_{st}(2n\Delta t) = \lambda\hbar^2\omega_c \int_0^\infty \omega \exp\left(-\frac{\omega}{\omega_c}\right) \coth\left(\frac{\hbar\omega}{2k_BT}\right) \left|\beta_{\uparrow,0}(2n\Delta t)\right|^2 \tan\left(\frac{\omega_q\Delta t}{2}\right)^2 . \]
\[ \frac{d\Gamma_{st}}{dt} = 2i \sum_q \omega_q \coth \left( \frac{\hbar \omega_q}{2k_B T} \right) \left[ \frac{\chi_q^2}{\hbar \omega_q} \beta_{q1}(t) - \frac{\chi_q}{\hbar \omega_q} \beta_{q1}(t) \right]. \] (36)

In contrast, the exact time reversal of \( \Gamma(t) \) at time \( t \) requires that the bath displacements change according to \( \chi_q^2 \beta_{q1(t)}(t+) = \chi_q \beta_{q1(t)}(t-)^* \). Remarkably, this differs from the flip-induced change by a mere phase factor [see also [4]]. Thus if the bath factor \( \eta = \exp(-\Gamma) \) decays during the last moments of the first half-cycle, a change of sign of the displacements after the mid-cycle flip suffices to induce a subsequent revival of coherence. Formal evidence for this phenomenon is provided by the expression of the time-derivative \( d\Gamma/dt \) at the moment immediately following the mid-cycle pulse, which can be shown straightforwardly to amount to

\[ \frac{d\Gamma_{st}}{dt}(2n-1)\Delta t + 0 = -\lambda \hbar^2 \omega_c \int_0^\infty d\omega \left( \frac{\omega}{\omega_c} \right)^{d-1} \exp \left( -\frac{\omega}{\omega_c} \right) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \cos^2 \left( (n - \frac{1}{2})\omega \Delta t \right) \tan \left( \frac{\omega \Delta t}{2} \right) \] (37)

for a standard density of states. It is not difficult to observe that for \( d \geq 3 \) this expression is always negative, since the dominant contribution comes from the range \( 0 \leq \omega \Delta t \leq \pi \), where the integrand is negative [note that for \( \omega \Delta t \geq \pi \) the periodicity factor \( \cos^2((n - \frac{1}{2})\omega \Delta t) \tan(\frac{\omega \Delta t}{2}) \) is multiplied by a monotonously decreasing function if \( d \leq 4 \)]. Hence the factor \( \eta \) always increases after the mid-cycle flip.

Surprisingly, the detailed solution reveals that the second flip also induces a revival. Indeed, the derivative of \( \Gamma \) at the moment immediately preceding the second pulse of a cycle reads

\[ \frac{d\Gamma_{st}}{dt}(2n\Delta t - 0) = \lambda \hbar^2 \omega_c \int_0^\infty d\omega \left( \frac{\omega}{\omega_c} \right)^{d-1} \exp \left( -\frac{\omega}{\omega_c} \right) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \sin^2(n\omega \Delta t) \tan \left( \frac{\omega \Delta t}{2} \right) \] (38)

and, by the same argument as for expression (37), is noted to be always positive for \( d \leq 3 \), since the integrant is positive in the dominant range \( 0 \leq \omega \Delta t \leq \pi \). Because \( \eta \) is necessarily decreasing at this moment, the second pulse can only produce a revival. It also becomes apparent that \( \eta \) displays [at least] a maximum between any two consecutive pulses. The numerical integration of expression (35) confirms this effect, as shown in Fig. 1 for representative model parameters, cycle periods and temperatures [for brevity we display only results for an ohmic environment, \( d = 1 \); the superohmic \( d = 3 \) case is qualitatively similar]. Should the time reversal of \( \eta(t) \) be exact, the maxima would occur precisely at times \( t_n = 2n \Delta t \), since the derivative of \( \Gamma \) is null in the initial state. Because the reversal is only partial, the maxima of \( \eta \) are seen to shift gradually, with each cycle, toward the midpoint of the interval between consecutive pulses. Moreover, as the temperature increases, the read-out values at \( t_n = 2n \Delta t \) tend to become comparable to the lowest values in a cycle. One may infer that shifting the read-out times by \( \sim +\Delta t/2 \), so as to take advantage of these maxima, may yield a slight improvement of the overall outcome.

It turns out, in fact, that a minor rearrangement of the protocol results in a more significant gain. Consider the following version of the idealized bang-bang cycle: i) propagation under Hamiltonian (1) for a duration \( \Delta t/2 \), starting at time \( t_n \); ii) first \( \pi \)-pulse and interchange of qubit eigenstates at time \( t_n + \Delta t/2 \); iii) evolution under Hamiltonian (1) from time \( t_n + \Delta t/2 \) to time \( t_n + 3\Delta t/2 \); iv) second \( \pi \)-pulse and interchange of qubit eigenstates at time \( t_n + 3\Delta t/2 \); v) readout at \( t_{n+1} = t_n + 2\Delta t \). As for the standard protocol, it can be verified that under this optimized sequence the qubit coherence also displays a maximum between any two consecutive rf-pulses. But since the read-out is scheduled now halfway between two pulses, one can expect that the corresponding output values fall close to the coherence maxima following the second pulse in each cycle. In addition, we can anticipate that the optimized protocol also benefits from the halved lag \( \lfloor \Delta t/2 \rfloor \) between the initial configuration to be preserved and the first applied rf-pulse, which induces the first revival. Similarly to an increase in the cycle frequency, this reduced lag leads to faster clipping of the decoherence periods within each cycle, hence again to higher coherence maxima and higher read-out values.

For a quantitative assessment of these effects, let us resort once more to solution (3). If the read-out times are \( t_n = 2n \Delta t \), the bath displacements between consecutive \( \pi \)-pulses read as follows: i) for \( (t_n - \Delta t/2 < t < (t_n + \Delta t/2) \):
\[ \beta_{q1(t)}(t) = \beta_{q1(t)}(t - \Delta t/2 - 0) \pm \chi_q / \hbar \omega_q \exp[-i \omega_q(t - t_n + \Delta t/2)] \mp \chi_q / \hbar \omega_q ; \]
ii) for \( (t_n + \Delta t/2 < t < (t_n + 3\Delta t/2) \):
\[ \beta_{q1(t)}(t) = \beta_{q1(t)}(t_n + \Delta t/2 - 0) \pm \chi_q / \hbar \omega_q \exp[-i \omega_q(t - t_n + \Delta t/2)] \mp \chi_q / \hbar \omega_q . \]
As before, we assume an unentangled environment in thermal equilibrium, so that \( \beta_{q1}(0) = \beta_{q1}(0) = 0 \) and \( \beta_{q1}(t) = -\beta_{q1}(t) \) for all \( t \). Taking this into account leads to a stroboscopic recurrence of the form

\[ \beta_{q1}(t_{n+1}) = \beta_{q1}(t_n) e^{-i \omega_q \Delta t} + \frac{\chi_q}{\hbar \omega_q} \left( e^{-i \omega_q \Delta t/2} - 1 \right)^2 \left( e^{-i \omega_q \Delta t} - 1 \right) , \]

which solved for \( \beta_{q1}(0) = 0 \) yields, under a standard density of states,
\[ \Gamma_{\text{opt}}(2n\Delta t) = \lambda h^2 \omega_c \int_0^\infty d\omega \left( \frac{\omega}{\omega_c} \right)^{d-1} \exp \left( -\frac{\omega}{\omega_c} \right) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \beta_{q,0}(2n\Delta t) \left( \frac{1 - \cos(\omega_q \Delta t/2)}{\cos(\omega_q \Delta t/2)} \right)^2. \]  

Here \( \beta_{q,0}(2n\Delta t) \) is again the unperturbed evolved displacement. Note that expression (39) differs from the standard expression (35) only through the substitution of the factor \(|\tan(\omega_q \Delta t/2)|^2\) by a factor of \([|1 - \cos(\omega_q \Delta t/2)|/ \cos(\omega_q \Delta t/2)]^2\). Remarkably, the two factors have a qualitatively similar functional dependence on \((\omega_q \Delta t)\), although their periodicity differs from \(2\pi\) to \(4\pi\), respectively. Quantitatively, on the dominant range \(0 \leq \omega_q \Delta t \leq \pi\), one has \(|\tan(\omega_q \Delta t/2)|^2 \geq \left|\frac{|1 - \cos(\omega_q \Delta t/2)|}{\cos(\omega_q \Delta t/2)}\right|^2\) and this proves sufficient to render \(\Gamma_{st}(2n\Delta t) \geq \Gamma_{\text{opt}}(2n\Delta t)\), hence \(\eta_{st}(2n\Delta t) \leq \eta_{\text{opt}}(2n\Delta t)\). The magnitude of this effect has been evaluated by numerical integration of expression (39) and the results are displayed in Figs. 1 and 2, alongside the corresponding standard output.

As expected, the qubit coherence is always better preserved under the optimized protocol. In addition, the optimized outcome is considerably more stable under temperature changes, with coherence variations \(\leq 10^{-4}\) compared to \(\sim 10^{-2}\) under the standard protocol. The absolute magnitude of the improvement generally does not exceed \(5\%\) [over time] at identical cycle periods and temperatures in an ohmic environment, but may raise above \(\sim 10\%\) in the superohmic case [not shown]. However, the true advantage of the optimized sequence can be better appreciated in the high precision regime. Indeed, as can be observed from Fig. 2, in order to maintain the errors in \(\eta\) under, e.g., \(\sim 0.1\%\) over an extended period of time [say, at least an order of magnitude longer than the typical decoherence time \(\sim \omega_c^{-1}\)], a standard sequence must be applied at a frequency substantially higher than \(\omega_c\). To give a semiquantitative reference, under the choice of model parameters employed in Fig. 2 the rf-frequency must be at least \(4\omega_c\) at \(T = 1K\) and at least \(6\omega_c\) at \(T = 100K\). The optimized protocol is seen to deliver a similar precision with a sizable reduction in frequency, in an essentially temperature-independent manner. E.g., under the same model parameters the necessary frequency amounts to only \(\sim 2\omega_c\) at both \(T = 1K\) and \(T = 100K\). This reduction in the operating frequency becomes increases as the precision bounds on \(\eta\) increase. As a result, an optimized protocol applied at \(T = 100K\) with a frequency of \(2.5\ \omega_c\) can perform better [errors \(\leq 10^{-6}\)] than a standard protocol operating at \(T = 1K\) with a frequency of \(10\omega_c\) [errors \(\leq 10^{-3}\)].

V. CONCLUSION

Quantum registers with DFS under interactions linear in the coordinates of a harmonic environment [Eq. (2)] support a large class of so-called coherent-product distributions [Eq. (3)], that is, distributions on direct sums of register DFS nontrivially entangled at all times with a statistical superposition of environmental Gaussian states. In particular, any register distribution on a direct sum of DFS develops into a coherent-product distribution when brought in contact with an initially uncorrelated thermal environment. The present study was prompted by the straightforward observation that solution (3) yields a unitary register evolution in the special case when the associated bath displacements coincide with the equilibrium displacements characteristic of each register DFS. Here we showed that solution (3) also covers a large set of evolutions, which propagate the reduced register state in an unperturbed, decoherence-free manner, although the overall register-environment state remains constantly entangled. These states require an appropriate density of degenerate environmental modes [expression (28) must be finite] with a specific phasing of their Gaussian displacements [Eqs. (26)], and a particular constraint, not very restrictive, on the coupling constants of the register DFSs to the environmental modes [Eq. (25a)]. Somewhat unexpected, an ohmic environment cannot support such states in single qubits or registers with collective decoherence, but is marginally effective for linear registers with individual decoherence. A superohmic environment, on the other hand, is considerably more efficient in sustaining entangled, decoherence-free distributions in a variety of registers, including registers with collective decoherence. However, since the amplitude of correlations between distinct DFS compatible with such a propagation decreases significantly with increasing temperature, nontrivial entangled decoherence-free states likely entail a low temperature environment.

The practical worth of these distributions depends to a large extent on one’s ability to generate and control a proper phasing of the environmental entanglement. It turns out that qubit flipping through radiofrequency \(\pi\)-pulses, as in bang-bang suppression of decoherence, provides an elementary procedure for such a manipulation, which amounts to a change of sign of the associated Gaussian displacements. As an immediate corollary, examination of the bang-bang protocol in terms of solution (3) suggests a straightforward streamlining modification. It is found that shifting the entire sequence of rf-pulses toward the initial moment by half the time between consecutive pulses, while the read-out times are left unchanged, allows high precision control with cycle frequencies reduced by a factor of about \(2/4\) and with a temperature sensitivity diminished by two orders of magnitude.


[6] Quantum error-correcting codes purge unwanted environment entanglement onto an ancillary subsystem and recover the original uncorrelated register state. Dynamical decoupling, in particular quantum bang-bang control, does not aim to eliminate environment entanglement, but to recover, after each cycle, a reduced register state identical to the original uncorrelated state. Sec. IV herein gives a detailed account of this feature.


Figure captions

FIG. 1. Time-dependence of qubit decoherence in an Ohmic environment under standard [Eq. (35), hollow symbols] and optimized [Eq. (39), solid symbols] sequences of rf-pulses, for two typical temperatures corresponding to $k_B T / \hbar \omega_c = 0.01$ [squares] and $k_B T / \hbar \omega_c = 1.0$ [circles]. For a cut-off frequency $\omega_c \approx 10^{13} s^{-1}$, the respective temperatures are $T = 1K$ and $T = 100K$. Time is given in units of the read-out [cycle] period $\tau_{cycle} = 2 \Delta t$, and the model scaling constant was set to $\lambda \hbar \omega_c = 0.25$.

FIG. 2. Errors in qubit decoherence as a function of cycle frequency under standard [hollow symbols] and optimized [solid symbols] sequences applied for a time $t$, at temperatures $T = 1K$ [squares] and $T = 100K$ [circles]. Other parameters as in Fig. 1. Each point represents the read-out after a number of $t / \tau_{cycle}$ cycles.