Disjoint System Decoupling and Environmental Influence on the Measurement Process in Stochastic Reduction Models

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ABSTRACT

We consider the energy-driven stochastic state vector reduction equation for the density matrix, which for pure state density matrices can be written in two equivalent forms. We use these forms to discuss the decoupling of the noise terms for independent subsystems, and to construct pure state “environmental” stochastic density matrices whose time-independent expectations are the usual quantum statistical distributions. We then consider a measurement apparatus weakly coupled to an external environment, and show that in mean field (Hartree) approximation the stochastic equation separates into independent equations for the apparatus and environment, with the Hamiltonian for the apparatus augmented by the environmental expectation of the interaction Hamiltonian. We use the Hartree approximated equation to study a simple accretion model for the interaction of the apparatus with its environment. Our analysis supports the suggestion that a measurement takes place when the different outcomes are characterized by sufficiently distinct environmental interactions for the reduction process to be rapidly driven to completion.
1. Introduction

Understanding the measurement process has been a persistent problem since the inception of quantum mechanics. In the orthodox Copenhagen interpretation, measurements are accounted for by invoking a layer of classical, non-quantum mechanical reality; attempts to extend quantum mechanics to include the measuring apparatus itself lead to quandaries, such as the famous Schrödinger cat paradox. One approach to this problem that has been much studied recently [1-8] postulates that the Schrödinger equation is only an approximate description of reality, and must be modified by small, nonlinear stochastic terms. These terms drive the state vector reduction process in measurements, and account for the non-observation of macroscopic quantum superpositions.

The proposal that a stochastic, nonlinear Schrödinger equation provides the phenomenology of quantum measurement passes a number of consistency tests. In its energy-driven form, it leads exactly to probabilities given by the Born rule [4,8,9], and for measurements on degenerate systems leads to the Lüders projection rule [9]. There are plausible arguments [7,8] that with a Planckian magnitude of the stochastic term, coherence is maintained for microscopic systems, while state vector reduction proceeds for superpositions of macroscopically distinguishable alternatives. However, the stochastic Schrödinger equation is inherently nonrelativistic [10,11], involving the same stochastic differential at all spatial locations. This raises the issue of whether it is consistent with clustering - put simply, does the reduction of the state vector in a localized measuring apparatus proceed independently of what goes on far away from the laboratory? An affirmative answer to this question was given [8] under the assumption that all of the universe is governed by the pure state stochastic reduction equation. Our aim in this paper is to extend this analysis in two directions, first
by showing that the standard statistical distributions describing an equilibrium “environment” can arise as the stochastic expectations of appropriate pure state density matrices, and second by examining in detail the stochastic evolution of a “measurement” system weakly coupled to its environment.

This paper is organized as follows. In Sec. 2 we give two equivalent forms of the Itô noise term in the stochastic evolution equation for a pure state density matrix, and use these to discuss clustering for disjoint subsystems. In Sec. 3 we use one of these forms to prove the existence of pure state density matrices whose stochastic expectation gives the standard quantum statistical distributions. In Sec. 4 we consider a “measurement” subsystem weakly coupled to an “environment” subsystem, obeying overall the density matrix stochastic evolution equation, and derive the corresponding single system Hartree or mean field stochastic equations for the measurement and environment subsystems, working to first order accuracy in the interaction Hamiltonian. The mean field approximation is applied in Sec. 5 to a simple accretion model for the influence of environmental couplings on the measurement process. In Sec. 6 we state our conclusions regarding the implications of this analysis for the measurement process in quantum mechanics.

2. Stochastic Density Matrix Equations and Clustering

We begin by recalling some formulas from the theory of stochastic Schrödinger equations [1-8]. Letting \(|\chi\rangle\) be a normalized state vector, the standard stochastic evolution (“quantum state diffusion”) equation for \(|\chi\rangle\) takes the form

\[
d|\chi\rangle = [\alpha dt + \beta dW_t]|\chi\rangle ,
\]

\[(1a)\]
with $dW_t$ a real Itô stochastic differential obeying
\begin{equation}
\frac{dW_t^2}{dt} = dt, \quad dW_t dt = 0, \quad (1b)
\end{equation}
and with
\begin{align}
\alpha &= -iH - \frac{1}{8}\sigma^2[A - \langle A \rangle]^2, \\
\beta &= \frac{1}{2}\sigma[A - \langle A \rangle], \quad (1c)
\end{align}
where $\sigma$ is a numerical parameter governing the strength of the stochastic and drift terms, and $A$ is a self-adjoint operator with expectation $\langle A \rangle$ in the state $|\chi\rangle$,
\begin{equation}
\langle A \rangle = \langle \chi | A | \chi \rangle. \quad (1d)
\end{equation}
The energy-driven case of the stochastic evolution is obtained by taking $A$ to be the Hamiltonian $H$, which we shall assume henceforth. It is straightforward to show that the evolution of Eqs.(1a)-(1d) preserves the normalization of the state vector $|\chi\rangle$.

Defining the pure state density matrix $\rho = |\chi\rangle\langle \chi |$, it is easy to show that the state vector evolution of Eqs. (1a)-(1d) implies that the density matrix evolution is given by
\begin{equation}
\frac{d\rho}{dt} = -i[H, \rho]dt - \frac{1}{8}\sigma^2[H, [H, \rho]]dt + \frac{1}{2}\sigma N(\rho, H)dW_t. \quad (2)
\end{equation}
Direct calculation from Eqs. (1a)-(1d) gives the coefficient $N(\rho, H)$ of the Itô noise term $dW_t$ in Eq. (2) in the form
\begin{equation}
N(\rho, H) = \{\rho, H\} - 2\rho \text{Tr}\rho H, \quad (3a)
\end{equation}
which by use of the pure state condition $\rho^2 = \rho$ can be written in the equivalent form
\begin{equation}
N(\rho, H) = [\rho, [\rho, H]] \quad. \quad (3b)
\end{equation}
Both of these forms have the property that $\rho^2 = \rho$ implies that $\{\rho, d\rho\} + (d\rho)^2 = d\rho$, which can be rewritten as $(\rho + d\rho)^2 = \rho + d\rho$, and so they preserve the pure state condition.
Let us now consider a system for which the Hamiltonian $H$ is the sum of two Hamiltonians $H_1, H_2$ which depend on disjoint sets of variables, and investigate the conditions under which Eqs. (2) and (3a), (3b) admit factorized solutions $\rho = \rho_1\rho_2$, with $\rho_{1,2}$ obeying equations of similar form driven by the respective Hamiltonians $H_{1,2}$. Substituting $H = H_1 + H_2$ and $\rho = \rho_1\rho_2$ into Eqs. (3a), (3b), and using the facts that all variables in set 1 commute with all variables in set 2, and that $\text{Tr} = \text{Tr}_1\text{Tr}_2$, we find respectively from Eqs. (3a) and (3b) that

\[
N(\rho_1\rho_2, H_1 + H_2) = \rho_2[\{\rho_1, H_1\} - 2\rho_1\text{Tr}_2\rho_2\text{Tr}_1\rho_1H_1] + \rho_1[\{\rho_2, H_2\} - 2\rho_2\text{Tr}_1\rho_1\text{Tr}_2\rho_2H_2] ,
\]  
(4a)

\[
N(\rho_1\rho_2, H_1 + H_2) = \rho_2^2[\rho_1, [\rho_1, H_1]] + \rho_1^2[\rho_2, [\rho_2, H_2]] .
\]  
(4b)

Clustering requires that

\[
N(\rho_1\rho_2, H_1 + H_2) = \rho_2N_1(\rho_1, H_1) + \rho_1N_2(\rho_2, H_2) ,
\]  
(5)

with $N_{1,2}$ the restrictions of $N$ to the 1,2 subspaces. We see that Eq. (4a) obeys the clustering property by virtue of the trace conditions $\text{Tr}_1\rho_1 = 1$, $\text{Tr}_2\rho_2 = 1$, while Eq. (4b) satisfies the clustering property by virtue of the pure state conditions $\rho_1^2 = \rho_1$, $\rho_2^2 = \rho_2$.

Let us now examine the clustering properties of the remaining terms in Eq. (2). For the left hand side, we find by use of the Itô extension of the chain rule that

\[
d(\rho_1\rho_2) = \rho_2d\rho_1 + \rho_1d\rho_2 + d\rho_1d\rho_2 .
\]  
(6a)

Thus, in order to have $d\rho_1$ and $d\rho_2$ obeying equations of the same form as $d\rho$ but restricted to the 1,2 subspaces, the left hand side should take the form, using Eqs. (1b) and (2),

\[
d(\rho_1\rho_2) = \rho_2d\rho_1 + \rho_1d\rho_2 + \frac{1}{4}\sigma^2N_1(\rho_1, H_1)N_2(\rho_2, H_2)dt .
\]  
(6b)
For the $dt$ terms on the right hand side of Eq. (2), we have

$$-i[H_1 + H_2, \rho_1 \rho_2]dt - \frac{1}{8}\sigma^2[H_1 + H_2, [H_1 + H_2, \rho_1 \rho_2]]dt$$

$$= \rho_2\{-i[H_1, \rho_1]dt - \frac{1}{8}\sigma^2[H_1, [H_1, \rho_1]]dt\}$$

$$+ \rho_1\{-i[H_2, \rho_2]dt - \frac{1}{8}\sigma^2[H_2, [H_2, \rho_2]]dt\}$$

$$- \frac{1}{4}\sigma^2[H_1, \rho_1][H_2, \rho_2]dt \quad \text{(6c)}$$

Assuming the conditions for the clustering property of Eq. (5) to hold for the Itô noise term, comparing Eqs. (6a)-(6c) we see that the complete density matrix evolution equation will cluster if and only if

$$N_1(\rho_1, H_1)N_2(\rho_2, H_2) = -[H_1, \rho_1][H_2, \rho_2] \quad \text{(7)}$$

This condition does not hold as in identity for either of the two possible forms for $N(\rho, H)$ given in Eqs. (3a), (3b), and so the $\sigma^2 dt$ or drift term in the stochastic evolution equation does couple disjoint systems. However, there is an important special case [8] in which disjoint systems decouple asymptotically. Taking $N(\rho, H)$ as in Eq. (3b), Eq. (7) becomes

$$[\rho_1, [\rho_1, H_1]][\rho_2, [\rho_2, H_2]] = -[H_1, \rho_1][H_2, \rho_2] \quad \text{(8)}$$

which is satisfied, by virtue of both the left and right hand sides vanishing, whenever either $[\rho_1, H_1] = 0$ or $[\rho_2, H_2] = 0$, conditions that are respectively obeyed when system 1 or system 2 is at the endpoint of the state vector reduction process. In particular, if system 1 represents a measurement process, and system 2 represents a pure state environment at the endpoint of its reduction process, then the stochastic dynamics of system 1 is completely independent of the dynamics of its environment.
3. Pure State Martingale Construction of the Standard Quantum Statistical Distributions

In order for the measurement system to decouple from its environment, we have seen that the environment must be described by a pure state density matrix. This raises the question of how such a description can be compatible with the usual description of equilibrium environments in terms of the standard quantum statistical distributions, which are mixed state density matrices $\rho$ obeying the trace condition $\text{Tr}\rho = 1$ but not the pure state condition $\rho^2 = \rho$. The answer is that in the theory of stochastic state vector reduction, the role of the usual mixed state density matrix is played \[7\] by the stochastic expectation $E[\rho]$. Thus an equilibrium environment is described by a pure state density matrix obeying $\rho^2 = \rho$, the stochastic expectation of which has the form $E[\rho] = f(H)$, with $f$ one of the standard quantum statistical distribution functions of the Hamiltonian. Since Eq. (2) implies that $E[\rho]$ obeys the time evolution equation

$$dE[\rho] = -i[H, E[\rho]]dt - \frac{1}{8}\sigma^2[H, [H, E[\rho]]]dt ,$$

any $E[\rho]$ of the form $f(H)$ is time independent, as expected of the quantum statistical distributions.

To show that there are pure state density matrices with the required expectation, we proceed constructively by use of the density matrix evolution equation in the form

$$d\rho = -i[H, \rho]dt - \frac{1}{8}\sigma^2[H, [H, \rho]]dt + \frac{1}{2}\sigma[\{\rho, H\} - 2\rho\text{Tr}\rho H]dW ,$$

Although we derived this equation in Sec. 2 for pure state density matrices, we shall now use it as a stochastic evolution equation for density matrices $\rho$ that do not obey the pure state
condition $\rho^2 = \rho$. Taking the initial $\rho$ at time $t = 0$ as $\rho_0 = f(H)$, we see from Eq. (9), which follows by taking the expectation of Eq. (10), that $E[\rho] = f(H)$ for all times. Also, since Eq. (10) only involves the Hamiltonian $H$, the stochastically evolved $\rho$ is still a function of $H$, and so commutes with $H$ at all times. Thus, for the choice of initial condition $\rho_0 = f(H)$, Eq. (10) simplifies to

$$d\rho = \frac{1}{2} \sigma[[\rho, H] - 2\rho \text{Tr} \rho H] dW_t. \quad (11)$$

Equation (11) defines $\rho$ to be a martingale, for which the expectation $E_s$ conditional on information available up to time $s$ obeys $E_s[\rho_t] = \rho_s$, $s \leq t$, which reduces to $E[\rho] \equiv E_0[\rho_t] = \rho_0 = f(H)$ when $s = 0$. [Note that if instead of Eq. (10) we had used the stochastic equation obtained from Eqs. (2) and (3b), the initial $\rho_0 = f(H)$ would not evolve in time at all, since Eq. (3b) vanishes identically when $\rho$ commutes with $H$. This underscores again the fact that Eqs. (3a) and (3b) are equivalent only for pure state density matrices, but define different stochastic evolutions for density matrices not obeying the pure state condition $\rho^2 = \rho$.]

Let us now show, making the assumption that the Hamiltonian $H$ is nondegenerate, that at late times $\rho$ evolves by Eq. (11) into a pure state. The proof of this parallels the proof [8,9] that Eqs. (1a)-(1d) lead to state vector reduction. We consider the variance $V$ of the Hamiltonian, defined by

$$V = \text{Tr} \rho H^2 - (\text{Tr} \rho H)^2, \quad (12a)$$

which by the Itô extension of the chain rule evolves in time as

$$dV = \text{Tr} d\rho H^2 - 2 \text{Tr} \rho H d\rho H - (\text{Tr} d\rho H)^2. \quad (12b)$$
Using Eq. (11) for $d\rho$ to evaluate $\text{Tr}d\rho H^n$, we find

$$\text{Tr}d\rho H^n = \sigma[\text{Tr}\rho H^{n+1} - \text{Tr}\rho H^n\text{Tr}\rho H]dW_t . \quad (12c)$$

Thus, substituting Eq. (12c) into Eq. (12b) and taking the expectation, we get

$$dE[V] = -\sigma^2 E[V^2]dt . \quad (13a)$$

From here on the argument is identical to that of Refs. [8,9], and leads to the conclusion that as $t \to \infty$ the variance $V$ approaches 0 almost certainly. When the energy spectrum is nondegenerate, this implies that at late times only one density matrix element $\rho_E$ is nonzero, and so the initial density matrix $\rho_0 = f(H)$ has evolved to a pure state density matrix. Again, by the arguments of [8,9], the probability of finding each final eigenvalue $E$ is just the initial density matrix element $f(E)$, and so at large times the expectation of the evolved density matrix is

$$E[\rho_\infty] = \sum_E f(E)|E\langle E| = f(H) , \quad (13b)$$

in agreement with our conclusion that $E[\rho] = f(H)$ for all times. Thus, evolution of the initial density matrix $\rho_0 = f(H)$ by Eq. (10) leads to a late time pure state density matrix that obeys $\rho^2 = \rho$ and $E[\rho] = f(H)$. We take such density matrices as our model for the environment, and by the arguments of Sec. 2, are assured that the evolution of measurement systems uncoupled by Hamiltonian interaction terms to this environment are independent of the environmental dynamics.
4. Mean Field Approximation for a System

Weakly Coupled to its Environment

Let us now consider two subsystems with disjoint variables that are weakly coupled through an interaction term $\Delta H$ in the Hamiltonian, so that the total Hamiltonian appearing in Eq. (10) is $H = H_1 + H_2 + \Delta H$. We shall take subsystem 1 to be a measuring apparatus (including the microscopic system being measured), whose reduction dynamics we wish to follow, while we take subsystem 2 to be the external environment with which this measuring apparatus interacts. We shall derive a mean field approximation to the dynamics, in which each subsystem obeys an independent system stochastic equation with a modified Hamiltonian, that reflects the mean interaction with the other subsystem. To this end, we substitute the independent subsystem Ansatz $\rho = \rho_1\rho_2$ into Eq. (10), and take the partial trace $\text{Tr}_2$ to average over the subsystem 2 dynamics, giving an effective equation for subsystem 1, and similarly, with the roles of 1 and 2 interchanged, to get an effective equation for subsystem 2.

We shall assume that in the limit of vanishing coupling $\Delta H$, the environment subsystem 2 is in one of the ensembles constructed in Sec. 3 that is a function solely of $H_2$, so that in the presence of $\Delta H$ we have $[\rho_2, H_2] = O(\Delta H)$. We do not make a corresponding assumption for subsystem 1, since we will be interested in the case in which this is initially in a generic pure state.

We proceed with this calculation term by term. From the left hand side of Eq. (10), substituting Eq. (6a) we get

$$\text{Tr}_2 d\rho = \text{Tr}_2 \rho_2 d\rho_1 + (\rho_1 + d\rho_1) \text{Tr}_2 d\rho_2 = d\rho_1,$$

where we have used the condition $\text{Tr}_2 \rho_2 = 1$ which implies that $\text{Tr}_2 d\rho_2 = 0$. From the first
term on the right hand side of Eq. (10) we get

\[ \text{Tr}_2(-i[H, \rho]dt = -i\text{Tr}_2[H_1, \rho_1] - i\text{Tr}_2[\Delta H, \rho_1\rho_2] - i\text{Tr}_2[H_2, \rho_2]\rho_1. \] (15a)

The first term on the right of Eq. (15a) gives simply

\[ -i[H_1, \rho_1]dt . \] (15b)

Since \( \text{Tr}_2\Delta H\rho_2 = \text{Tr}_2\rho_2\Delta H \), the second term on the right of Eq. (15a) becomes

\[ -i[\text{Tr}_2\rho_2\Delta H, \rho_1]dt , \] (15c)

and the third term on the right of Eq. (15a) vanishes. So in sum, the first term on the right hand side of Eq. (10) gives

\[ -i[H_1 + \text{Tr}_2\rho_2\Delta H, \rho_1]dt . \] (16)

We turn next to the second term on the right hand side of Eq. (10), which gives

\(-\frac{1}{8}\sigma^2 dt\) times the partial trace of the double commutator,

\[ \text{Tr}_2[H, [H, \rho]] = \text{Tr}_2[H_1 + H_2 + \Delta H, [H_1 + H_2 + \Delta H, \rho_1\rho_2]] \]

\[ = \text{Tr}_2[H_1 + \Delta H, [H_1 + H_2 + \Delta H, \rho_1\rho_2]] \]

\[ = \text{Tr}_2\{[H_1 + \Delta H, [H_1, \rho_1]\rho_2] + [H_1, [H_2, \rho_2]\rho_1] + [H_1, [\Delta H, \rho_1\rho_2]] \]

\[ + [\Delta H, [H_2, \rho_2]\rho_1] + [\Delta H, [\Delta H, \rho_1\rho_2]] \} \]

\[ = [H_1 + \text{Tr}_2\rho_2\Delta H, [H_1 + \text{Tr}_2\rho_2\Delta H, \rho_1]] + O((\Delta H)^2) \]

where we have used the facts that (i) \( \text{Tr}_2[H_2, g(1, 2)] = 0 \) for any function \( g \) of variables 1,2, and that (ii) by our equilibrium assumption for the environment, \( [H_2, \rho_2] \) is of order \( \Delta H \). [Step (ii) is the only one which does not go through in the corresponding effective equation calculation for the environment subsystem 2, leading to an additional term in its effective equation of motion given in Eq. (20a) below.]
Finally, we turn to the third term on the right hand side of Eq. (10), which gives 
\[ \frac{1}{2} \sigma dW_t \] times 
\[
\text{Tr}_2[(\rho, H) - 2 \rho \text{Tr} \rho H] = \text{Tr}_2[(\rho_1 \rho_2, H_1 + H_2 + \Delta H) - 2 \rho_1 \rho_2 \text{Tr}_1 \rho_1 \rho_2 (H_1 + H_2 + \Delta H)]
\]
\[
= \text{Tr}_2[(\rho_1 \rho_2, H_1 + \Delta H) + \rho_1 \{\rho_2, H_2\} - 2 \rho_1 \rho_2 \text{Tr}_1 \rho_1 (H_1 + \text{Tr}_2 \rho_2 \Delta H) - 2 \rho_1 \rho_2 \text{Tr}_2 \rho_2 H_2]
\]
\[
= \{\rho_1, H_1 + \text{Tr}_2 \rho_2 \Delta H\} - 2 \rho_1 \text{Tr}_1 \rho_1 (H_1 + \text{Tr}_2 \rho_2 \Delta H)
\] ,
(18)
where no approximations have been made.

Putting everything together, we see that the mean field approximation for the “measurement” subsystem 1 is

\[
d\rho_1 = -i[H_1, \rho_1] dt - \frac{1}{8} \sigma^2 [H_1, [H_1, \rho_1]] dt + \frac{1}{2} \sigma \{\rho_1, H_1\} - 2 \rho_1 \text{Tr}_1 \rho_1 H_1^\prime] dW_t + O(\sigma^2 (\Delta H)^2 dt^2)
\]
(19a)
with the effective Hamiltonian

\[
H_1^\prime = H_1 + \text{Tr}_2 \rho_2 \Delta H
\] .
(19b)

The corresponding equation for the “environment” subsystem 2 is obtained by interchanging the labels 1 and 2, and restoring the term dropped in step (ii) leading to Eq. (17), giving

\[
d\rho_2 = -i[H_2, \rho_2] dt - \frac{1}{8} \sigma^2 [H_2, [H_2, \rho_2]] dt + \frac{1}{2} \sigma \{\rho_2, H_2\} - 2 \rho_2 \text{Tr}_2 \rho_2 H_2^\prime] dW_t
- \frac{1}{8} \sigma^2 [\text{Tr}_1 (\Delta H[H_1^\prime, \rho_1]), \rho_2] dt + O(\sigma^2 (\Delta H)^2 dt)
\]
(20a)
with the effective Hamiltonian

\[
H_2^\prime = H_2 + \text{Tr}_1 \rho_1 \Delta H
\] .
(20b)

The added term on the second line of Eq. (20a) vanishes through order $(\Delta H)^2$ when the reduction process for subsystem 1 has concluded, since then the density matrix for subsystem
1 obeys $[H'_1, \rho_1] = 0$ up to error terms of order $(\Delta H)^2$. As a consistency check on the calculation, we see that the mean field evolution equations obey $\text{Tr}_1 d\rho_1 = \text{Tr}_2 d\rho_2 = 0$, and so preserve the trace conditions $\text{Tr}_1 \rho_1 = \text{Tr}_2 \rho_2 = 1$.

5. Influence of Environmental Couplings on the Measurement Process

The principal result of the preceding section is Eqs. (19a) and (19b), which give the mean field approximation to the evolution of a measurement system which starts evolving from a generic pure state, coupled to an environment which (neglecting $\Delta H$) is at the pure state endpoint of its reduction process. In this section we apply Eqs. (19a) and (19b) to discuss the influence of the environment on the measurement process. Although in some cases [8] thermal fluctuations can play a significant role, usually the most important environmental effect on the measurement process is accretion of molecules from the environment onto the surface of the measuring apparatus [7,8]. Hence in this section we analyze a simple model for such accretion processes.

Consider a measuring apparatus which has $N$ surface accretion sites for molecules of mass $m$. In Fock space representation, its Hamiltonian can be written as

$$H_1 = H_0 + \sum_{j=1}^{N} ma_j^\dagger a_j,$$

with $H_0$ the bulk Hamiltonian for the apparatus, and with $a_j^\dagger$ and $a_j$ respectively the creation and annihilation operators for the accreted molecules. We assume the environment to contain a large number $M$ of molecules that can be accreted onto the surface, with creation and annihilation operators $b_k^\dagger, b_k$, $k = 1, ..., M$, and with a coupling to the accretion sites given
\[ \Delta H = \sum_{j=1}^{N} \sum_{k=1}^{M} [A_{jk} a_j^\dagger b_k + \text{adjoint}] . \tag{21b} \]

This interaction Hamiltonian conserves the total number operator
\[
N = \sum_{j=1}^{N} a_j^\dagger a_j + \sum_{k=1}^{M} b_k^\dagger b_k , \tag{22}
\]
in other words, the total number of molecules accreted onto the surface or remaining in the environment is constant.

In typical measurement situations, the environment density matrix will be diagonal in the number operator \( \sum_{k=1}^{M} b_k^\dagger b_k \) of the molecules being accreted. In this case, which we term “incoherent”, the environmental expectation of \( \Delta H \) vanishes,
\[
\text{Tr}_2 \rho_2 \Delta H = 0 \tag{23}
\]
and the reduction process is governed, according to Eqs. (19a, b), by the measurement system Hamiltonian \( H_1 \) alone. The Hamiltonian \( \Delta H \) still plays a role, since in order \( \Delta H \) in probability amplitudes [corresponding to order \( (\Delta H)^2 \) in probabilities or transition rates] it leads to a sticking probability and an evaporation probability per unit time, respectively, for a molecule in the environment to accrete to the surface of the apparatus, and for a molecule already accreted to evaporate. As a result of these nonvanishing transition probabilities, the number of molecules accreted to the surface is constantly fluctuating. Assuming a simple colloid statistics model [12] in which each accretion site can hold only one molecule, the root mean square fluctuation in the number of accreted molecules can be shown to be the square root of the mean number of accreted molecules.

These fluctuations play an important role in the measurement process. In general, distinguishable measurement outcomes must involve different configurations of the apparatus.
with respect to its environment, and hence will have different values of the accretion numbers $a_j^\dagger a_j$ associated with the $N$ accretion sites. Thus, the energy eigenvalue $H_1$ of the measurement apparatus will differ for each distinguishable measurement outcome, with the spread of eigenvalues between any two outcomes being typically the mass of the accreted molecules $m$ times the root mean square fluctuation in the number of accreted molecules. This statement assumes that the flux of accreting molecules in the environment is high enough for such a fluctuation to actually occur during the state vector reduction time characterizing the measuring process. Quantitative estimates [7,8] show that for typical laboratory conditions, this condition is satisfied, so that the environmental accretion mechanism gives a large enough spread of apparatus energy eigenvalues for the state vector reduction process to be driven rapidly to completion. Specifically, assuming the parameter $\sigma$ governing the stochastic process to be $\sigma \sim M_{\text{Planck}}^{-\frac{1}{2}}$ (in units with $\hbar = c = 1$), the reduction time $t_R$ in seconds is given by $t_R \sim (2.8\text{MeV}/\Delta E)^2$. Hence an energy variance $\Delta E$ equal to the mass of just a few accreted molecules is enough to lead to very rapid state vector reduction.

Anandan [13] has raised the interesting question of whether state vector reduction can proceed to a coherent state endpoint. Although not relevant for most measurement situations, this “coherent” case can be analyzed in terms of the accretion model formulated above, where it corresponds to the case in which the environment is in a coherent state, so that the environmental expectation of $\Delta H$ is nonzero. Assuming for simplicity that there is only one accretion site, which can be multiply occupied, we have then

$$H_1 + \text{Tr}_2 \rho_2 \Delta H = H_0 + ma_1^\dagger a_1 + \lambda a_1^\dagger + \lambda^* a_1 \quad , \quad (24a)$$
with \( \lambda \) given by
\[
\lambda = \sum_{k=1}^{M} A_{1k} \text{Tr}_{2} \rho_{2} b_{k}.
\]  
\( (24b) \)

Assuming \( H_{0} \) to commute with \( a_{1} \), Eqs. (24a) and (24b) describe the zero forcing frequency limit of the forced harmonic oscillator, which has been extensively studied [14,16], and can be succinctly solved by coherent state methods [15,16]. Defining \( z \) and \( c_{1} \) by
\[
z \equiv -\lambda/m, \quad c_{1} \equiv a_{1} - z = a_{1} + \lambda/m,
\]  
\( (25a) \)
we have
\[
H_{1} = H_{0} + mc_{1}^{\dagger} c_{1} + \text{constant},
\]  
\( (25b) \)
which in its \( c_{1} \) dependence is a standard harmonic oscillator. The \( c_{1} \) oscillator ground state \( |0\rangle \) obeys
\[
c_{1}|0\rangle = 0 \Rightarrow a_{1}|0\rangle = z|0\rangle,
\]  
\( (25c) \)
in other words, \( |0\rangle \) is a coherent state in terms of the original operators \( a_{1} \).

Ignoring an overall constant arising from terms in Eq. (25b) that commute with \( a_{1} \), the general eigenstate of Eq. (25b) is \( |n\rangle \), with \( n \) the number of \( c_{1} \) quanta, and has energy eigenvalue \( mn \). This state is a coherent superposition of states with different numbers of molecules on the accretion site. For energy eigenvalue \( n \), the probability \( P(n|k) \) of finding \( n - k \) molecules on the site can be exactly expressed [14,16] as a Laguerre polynomial, and for \( |z| \ll 1 \) and \( n \) large can be approximated [17] as
\[
P(n|k) \approx [J_{|k|}(2n^{\frac{1}{4}}|z|)]^{2},
\]  
\( (26a) \)
with \( J_{k} \) the order \( k \) Bessel function; the Bessel function addition formula [18]
\[
1 = J_{0}(w)^{2} + 2 \sum_{n=1}^{\infty} J_{n}(w)^{2}
\]  
\( (26b) \)
implies that the probabilities of Eq. (26a) sum to unity,
\[ \sum_{k=-\infty}^{\infty} P(n|k) = 1 \quad \text{(26c)} \]

Equation (26a) is rapidly oscillating as a function of \( k \), but using the asymptotic estimate [19]
\[ J_\nu(\nu \sec \beta) \simeq \left( \frac{2}{\pi \nu \tan \beta} \right)^{\frac{1}{2}} \cos(\nu \tan \beta - \nu \beta - \frac{1}{4} \pi) \quad \text{(27a)} \]
it is easily seen that the averaged envelope of \( P(n|k) \) is given by
\[ \overline{P(n|k)} \simeq \frac{1}{\pi} \frac{1}{(4n|z|^2 - k^2)^{\frac{1}{4}}} \quad \text{(27b)} \]
showing that the values of \( k \) are mainly distributed (apart from an exponentially decaying tail) between \(-2n^{\frac{1}{2}}|z|\) and \(2n^{\frac{1}{2}}|z|\).

### 6. Discussion and Conclusions

Throughout the above analysis, we have assumed that the Hamiltonian that is relevant for the stochastic Schrödinger equation is the total Hamiltonian
\[ H = \int d^3x T_{00}(x) \quad \text{(28)} \]
defined by gravitational couplings to the stress-energy tensor \( T_{\mu\nu}(x) \), which includes rest mass terms. This assumption plays a crucial role in our analysis of the accretion model in Sec. 5, in which the apparatus energy fluctuations produced by environmental interactions arise entirely from rest mass terms. Although in non-relativistic quantum mechanics one often drops rest mass terms when they lead to irrelevant constant energy shifts, there is no reason in principle to do so. In fact, in the standard model of elementary particles, all fermion
rest masses arise from the Yukawa couplings of the fermions to the Higgs particle, so that from this point of view rest masses are not an additive constant term in the Hamiltonian, but are a dynamical product of interactions.

The analysis we have given of the decoupling of disjoint systems, and of the effect of the environment on the measurement process, supports the view that the energy-driven stochastic Schrödinger equation gives a viable phenomenology of state vector reduction. According to this picture, a measurement takes place when the different outcomes are characterized by sufficiently large environmentally induced energy fluctuations in the apparatus for the state vector reduction process, which is driven by the energy variance, to proceed rapidly to completion. This requirement meshes in a natural way with the obvious requirement that in a measurement, different experimental outcomes should be macroscopically distinguishable.

A celebrated example of a measurement system in which the different outcomes have distinctly different environmental interactions is provided by the Schrödinger cat paradox, since live and dead cat states differ appreciably in their accretion of environmental molecules (live cats breathe; dead cats don’t), and so the energy variance condition is amply satisfied. Thus the energy-driven stochastic Schrödinger equation resolves the cat paradox. An important question for future work is whether one can find systems for which a measurement is expected according to the Copenhagen interpretation, but for which the conditions for energy-driven reduction are only marginally satisfied, leading to possible experimental tests of the energy-driven stochastic Schrödinger equation as a description of the measurement process.
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