Spherically symmetric, null dust clouds, like their time-like counterparts, may collapse classically into black holes or naked singularities depending on their initial conditions. We consider the Hamiltonian dynamics of the collapse of an arbitrary distribution of null dust, expressed in terms of the physical radius, $R$, the null coordinates, $V$ for a collapsing cloud or $U$ for an expanding cloud, the mass function, $m$, of the null matter, and their conjugate momenta. This description is obtained from the ADM description by a Kuchar-type canonical transformation. Dirac’s constraint quantization program is implemented and solutions are obtained for both expanding and contracting null dust clouds with arbitrary mass functions. We propose that the correct description should be given as a linear superposition of these solutions, with amplitudes that are determined from model dependent boundary conditions. The boundary conditions for the special case of a thin shell are then examined. We require that the initial data are the same for both parts of the superposition. Again, a shell arriving at the center will form a strong curvature singularity. When this happens, the space-time cannot be extended through the singularity and an outgoing shell has nowhere to propagate, so a superposition of incoming and outgoing states makes no sense. Therefore, we also require the shell to avoid the central singularity at all times, i.e., that its wave-function vanishes at the center. Thus we obtain a description of the shell quantum mechanics that is similar to Hájíček’s. A semi-classical picture of the space-time geometry is suggested.

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1. INTRODUCTION

Spherically symmetric dust clouds, depending on their initial matter and velocity distributions, will collapse in classical general relativity to form either black holes or naked singularities. Black holes are better understood than naked singularities. They are generally expected to evaporate via their associated Hawking radiation [1], although no agreement has yet been achieved regarding the end state of collapse, i.e., whether a remnant survives or whether all the matter contained in the original cloud is thermally radiated away. If a portion of the collapsing matter does manage to form a stable black hole, it is expected that the total mass of the remnant will be quantized. On the other hand, if all the matter is radiated away before a stable end state can form then one must explain what happens to the information that was contained in the initial matter distribution. The formation of black holes therefore presents a number of deep puzzles and various approaches to quantum gravity are being employed to address these at the present time [2–5]. On the contrary, naked singularities have received comparatively little attention. Yet, the formation of naked singularities (singularities that are visible either locally or asymptotically) is much more difficult to understand and for an entirely different reason: their existence implies the absence of a well defined Cauchy problem to the future of some light-like surface (the Cauchy horizon), therefore any attempt to describe the system to the future of this surface fails for lack of initial conditions. It seems that space-time must be terminated at the Cauchy horizon. In order to avoid the associated problems, Penrose proposed a Cosmic Censor [6], whose function is essentially to ensure that naked singularities never form. The mechanism by which the Cosmic Censor operates, however, is still shrouded in mystery. The Censor is most likely not classical because most models of classical collapse lead to the formation of both black holes and naked singularities in different domains of the initial phase space [7]. In fact very little is currently understood about the final stages of a collapse that leads to the formation of a classical naked singularity.

There are indications from the semi-classical treatment of naked singularities, in which the gravitational degrees of freedom are considered to be classical, that Penrose’s Cosmic Censor may, in fact, be the quantum theory itself [8]. However, at the very final stages of collapse it is not possible to treat the gravitational degrees of freedom classically and a full blown quantum theory of the gravitational field becomes necessary to establish this possibility [9] firmly. Singularities in general relativity signal a breakdown of the classical theory, a regime in which the classical equations are meaningless. Cosmic Censorship probably points to the need for quantum gravity in the same way as, more than eighty years ago, the electromagnetic instability of atoms pointed to the need for
The mass function is generally taken to vanish when \( U > U_1 \) in terms of a retarded part of Minkowski space-time, where \( U \) is described by the metric \( R \) is a part of Minkowski space-time.

This is what we propose to do in this paper. Our objective is to consider the midisuperspace quantization of a spherically symmetric cloud of null matter specified by an arbitrary mass distribution and collapsing in its own gravitational field. The model we are concerned with therefore is a solution of Einstein’s equations with pressureless, null dust \([10]\) described by the stress energy tensor \( T_{µν} = ε(x)U_µU_ν \), where \( ε(x) \) is the energy density of the cloud and \( U^2 = 0 \). When the cloud is contracting, the solution is characterized by an arbitrary function, \( m(V) \), of the advanced null coordinate, \( V \in (-∞, +∞) \). The mass function is generally taken to be vanishing for \( V < V_0 \) and constant, \( M \), when \( V > V_1 \). The space-time is described by the metric

\[
\text{ds}^2 = \left( 1 - \frac{2m(V)}{R} \right) dV^2 - 2dRdV - R^2dΩ^2, \quad (1.1)
\]

where \( R \in [0, +∞) \) is the area radius. The region \( V > V_1 \) is a part of the Schwarzschild space-time. In this region the metric may be written in terms of the Eddington-Finkelstein coordinates, \( U \) and \( V \) as

\[
\text{ds}^2 = \left( 1 - \frac{2M}{R} \right) d UdV - R^2dΩ^2, \quad (1.2)
\]

where \( U = \tilde{V} - 2R^* \) and \( R^* \) is the tortoise coordinate. The region \( V < V_0 \) is a part of Minkowski space-time, with metric

\[
\text{ds}^2 = du dv - R^2dΩ^2, \quad (1.3)
\]

where \( u \) and \( v \) are the ordinary retarded and advanced times, respectively \( T = R \).

In the time reversed situation the null dust cloud is expanding instead of contracting and the solution is written in an analogous fashion,

\[
\text{ds}^2 = \left( 1 - \frac{2m(U)}{R} \right) dU^2 + 2dRdU - R^2dΩ^2, \quad (1.4)
\]

in terms of a retarded null coordinate, \( U \). Again, the mass function is generally taken to vanish when \( U > U_1 \), having some constant value, \( M \) before some earlier retarded time, \( U < U_0 \). The region \( U > U_1 \) is then a part of Minkowski space-time,

\[
\text{ds}^2 = du dv - R^2dΩ^2, \quad (1.5)
\]

while the region \( U < U_0 \) is a part of the Schwarzschild spacetime.

Depending on the distribution \( m(V) \) of matter in the null cloud, either black holes or naked singularities may develop as the classical final state of the collapse. For example, in the self similar model in which the mass is a linear function of the advanced null coordinate, \( m(V) = λV \), one finds that both outcomes described by the Penrose diagrams in figure 1 are possible, depending on whether \( λ > 1/16 \) (black hole) or \( λ ≤ 1/16 \) (naked singularity) \([11]\).

**Fig.1:** Black hole (left) or naked singularity (right) formation in null dust collapse

When the collapse evolves toward a naked singularity, spatial hypersurfaces in the future of the initial singularity cross the Cauchy horizon and collide with the central singularity but, because no sensible boundary conditions can be specified on a singularity, the evolution in the future of the initial singularity is arbitrary. The Cosmic Censor \([6]\) should come into play before the Cauchy horizon has a chance to form. It is of interest, therefore, to understand how the system behaves close to, but in the past of, the putative Cauchy horizon, where spatial hypersurfaces are well defined and the quantum evolution of the system may be studied. In the present paper both expanding and contracting solutions are quantized and it is proposed that the quantum description of a null cloud is the linear superposition of the infalling and outgoing solutions. The central problem is the evaluation of the amplitudes of the superposition, which requires interpreting it. We are able to solve this problem for a single shell.

In section II we summarize the canonical formulation of the action in ADM variables. The null dust action appropriate to the models being considered is also analyzed in this section. In section III we explicitly perform a transformation of the phase-space to Kuchař variables \([12]\). These variables are useful because their physical meaning is transparent and the constraints, expressed in terms of them, when imposed as operator conditions on state functionals lead to simple, solvable, linear functional differential equations. In section IV we apply the Dirac quantization program to these models. Solutions to the constraints are obtained for collapsing and expanding clouds and the quantum wave-functional is written as a superposition of the two. We next consider a matter distribution representing a single shell and solve the problem of determining the amplitudes. Our solution is finally summarized in a tidy geometric picture, but its extension to arbitrary matter distributions remains a problem for the future.
II. CANONICAL FORMULATION IN ADM VARIABLES

Consider the line element, $d\sigma$ on a spherically symmetric three dimensional Riemann surface, $\Sigma$. It is completely characterized by two functions, $L(r)$ and $R(r)$ of the radial label coordinate

$$ds^2_{(3)} = L^2(r)dr^2 + R^2(r)d\Omega^2,$$

where $\Omega$ is the solid angle. The angular coordinates play no role and will be integrated over. We take both $L(r)$ and $R(r)$ to be positive definite except, possibly, at the center. $R(r)$ represents the physical radius of a shell labeled by $r$ on the surface. It behaves as a scalar under transformations of $r$, whereas $L(r)$ behaves as a scalar density. The corresponding four dimensional line element may be written in terms of two additional functions, the lapse, $N(t, r)$, and the shift, $N'(t, r)$, as

$$ds^2 = N^2dt^2 - L^2(dr + N'dt)^2 - R^2d\Omega^2.$$  

In this spherically symmetric space-time, we will consider the Einstein-Dust system described by the action

$$S = -\frac{1}{16\pi} \int d^4\sqrt{-g}\, \mathcal{R}$$

$$-\frac{1}{8\pi} \int d^4\sqrt{-g}\, \epsilon(x)g^{\alpha\beta}U_\alpha U_\beta$$

where $\mathcal{R}$ is the scalar curvature. As is well known, the gravitational part of this action can be cast into the form

$$S^g = \int dt \int_0^\infty dr \left[ P_L \dot{L} + P_R \dot{R} - NH^g - N'H^g \right]$$

$$+ S^g_{\partial^2}$$

with the momenta conjugate to $L$ and $R$ respectively given by

$$P_L = N \left[ -\dot{R} + N'R' \right]$$

$$P_R = \frac{N}{2} \left[ -L\dot{R} - \dot{L}R + (N'L)R' \right]$$

and where the overdot and the prime refer respectively to partial derivatives with respect to the label time, $t$, and coordinate, $r$. The lapse, shift and phase-space variables are required to be continuous functions of the label coordinates. The boundary action, $S^g_{\partial^2}$, is required to cancel unwanted boundary terms in the hypersurface action, ensuring that the hypersurface evolution is not frozen on the frontiers. It is determined after fall-off conditions appropriate to the models under consideration are specified. The super-Hamiltonian and super-momentum constraints are given by

$$\mathcal{H}^g = -\left[ \frac{P_LP_R}{R} - \frac{LP_R^2}{2R^2} \right]$$

$$+ \left[ \frac{L}{2} - \frac{R'^2}{2L} + \left( \frac{RR'}{L} \right) \right]$$

$$\mathcal{H}^g_{\partial^2} = R'P_R - LP_R'$$

We will assume that the matter distribution is such that at infinity Kuchař’s fall-off conditions [12] are suitable and we will adopt them here. These conditions would be applicable, for example, in models in which the collapsing metric asymptotically approaches or is smoothly matched to an exterior Schwarzschild background at some boundary. They read

$$L(t, r) = 1 + M_+(t)r^{-1} + O^\infty(r^{-1-\epsilon})$$

$$R(t, r) = r + O^\infty(r^{-\epsilon})$$

$$P_L(t, r) = O^\infty(r^{-\epsilon})$$

$$P_R(t, r) = O^\infty(r^{-\epsilon})$$

$$N(t, r) = N_+(t) + O^\infty(r^{-\epsilon})$$

$$N'(t, r) = O^\infty(r^{-\epsilon})$$

and imply that the asymptotic regions are flat with the spatial hypersurfaces asymptotic to surfaces of constant Minkowski time. Again, as $r \to 0$ we require that [13]

$$L(t, r) = L_0(t) + O(r^2)$$

$$R(t, r) = R_1(t)r + O(r^3)$$

$$P_L(t, r) = P_L_0(t)r + O(r^3)$$

$$P_R(t, r) = P_R_0(t)r + O(r^3)$$

$$N(t, r) = N_0(t) + O(r^2)$$

$$N'(t, r) = N'_0(t)r + O(r^3)$$

With these conditions, it is easy to see that the appropriate choice of surface action involves only the contribution,

$$S^g_{\partial^2} = -\int \left( \frac{dt}{\partial^2} \right) N_+(t)M_+(t) f$$

at the boundary at infinity.

Let us now consider the null dust action in (2.3). We note first that the energy density, $\epsilon(x)$, plays the role of a Lagrange multiplier enforcing null dust, i.e., $U^2 = 0$, and variation w.r.t. $g_{\alpha\beta}$ yields the standard dust stress tensor, $T_{\alpha\beta} = \epsilon(x)U^\alpha U^\beta$. The canonical form of the null dust action in various forms has been studied by Kuchař and Bičák [14]. In particular, for the action in the form given in (2.3) one may expand $U_\alpha$ as a Pfaff form of six scalar fields, the three comoving coordinates of the null dust particles, $Z^k$, and three scalars (velocities), $w_k$,

$$U_\alpha = w_k Z^k_{,\alpha}.$$  

This representation is redundant because, by Pfaff’s theorem, only four scalars are required to describe an arbitrary covector in a four dimensional space. Suppose we require one of the scalars, say $w_3$ to be unity and drop the index from the associated comoving coordinate, $Z^3 := Z$,

$$U_\alpha = Z_{,\alpha} + w_k Z^k_{,\alpha}, \quad k \in \{1, 2\}.$$  

Consider the independent variations,

$$0 = \frac{\delta S}{\delta \epsilon} = -\sqrt{-g}g^{\alpha\beta}U_\alpha U_\beta$$

$$0 = \frac{\delta S}{\delta Z} = (\sqrt{-g}\epsilon(x)U^\alpha)_{,\alpha} = \nabla_\alpha(\epsilon(x)U^\alpha)$$
The conservation of the stress energy tensor in the last equation implies that
\[ \epsilon(x) U^\alpha \nabla_\alpha U^\beta + U^\beta \nabla_\alpha (\epsilon(x) U^\alpha) = 0, \] (2.13)
which says that the particles follow geodesic curves. Using the second equation above we find \( U^\alpha \nabla_\alpha U^\beta = 0 \), implying affine parameterization. The third equation says that \( \mathcal{L}_U Z^k = Z^\alpha_{\alpha} U^\alpha = 0 \), i.e., all of the \( Z^k \) are constant along flow lines and none of them are time-like. And finally, multiplying the third equation by \( w_k \) we find
\[ Z^\alpha_{\alpha} U^\alpha = 0, \] (2.14)
saying that \( Z^\alpha_{\alpha} \) may be space-like or null. If the twist, \( U_{[\alpha;\beta]} \), also vanishes, then \( Z^\alpha_{\alpha} \) is null, which would imply that \( w_k Z^\alpha_{\alpha} = 0 \), or \( w_k = 0 \) \( \forall \ k \in \{1, 2\} \), because the \( Z^\alpha_{\alpha} \) are taken to form a (linearly independent) cobasis.

Substituting the decomposition (2.11) into the dust action in (2.3), using (2.2) and integrating over the angular coordinates the action may be put in the form
\[ S^d = \int dt \int_0^\infty dr \left[ \mathcal{P}_Z \dot{Z} + \mathcal{P}_k \dot{Z}^k - N \mathcal{H}^d - N^r \mathcal{H}^d_r \right], \] (2.15)
where the momenta conjugate to \( \{ Z, \dot{Z}^k \} \) are, respectively,
\[ P_Z = \frac{L R^2}{N} \epsilon(r) \left[ (\dot{Z} + w_k \dot{Z}^k) - N^r (Z^r + w_k Z^k) \right], \]
\[ P_k = w_k P_z, \] (2.16)
and the constraints, \( \mathcal{H}^d \) and \( \mathcal{H}^d_r \) are
\[ \mathcal{H}^d = \left[ \frac{P_Z^2}{2 L R^2} + \frac{\epsilon(r)^2 (Z^r + w_k Z^k)^2}{2L} \right], \]
\[ \mathcal{H}^d_r = P_Z (Z^r + w_k Z^k). \] (2.17)

Setting \( \delta \mathcal{L} / \delta \epsilon = 0 \) gives the final form of the dust hamiltonian and momentum constraints
\[ \mathcal{H}^d = \frac{P_Z (Z^r + w_k Z^k)}{L}, \]
\[ \mathcal{H}^d_r = P_Z (Z^r + w_k Z^k), \] (2.18)
where the positive (respectively negative) signs in the dust hamiltonian density represent infalling (respectively outgoing) dust. In the spherically symmetric collapse we are considering, we take \( w_k = 0 = P_k \). Thus we have arrived at the canonical form of our theory, which we will write as
\[ S = \int dt \int_0^{\infty} dr \left[ \dot{Z} P_Z + L P_L + \dot{R} F_R - N \mathcal{H} - N^r \mathcal{H}_r \right] \]
\[ \mathcal{H} = \left[ \frac{P_L P_R}{R} - L P^2_L \right] \]
\[ + \left[ \frac{L}{2} - \frac{R^2}{2L} + \left( \frac{R^r}{L} \right)^2 \right], \]
\[ \mathcal{H}^d_r = \left[ Z^r P_R + R^r P_L - L P^2_L \right] \]
\[ S_{\mathcal{H}^d} = - \int \partial_{\mathcal{H}^d} N_m (t) M_+ (t), \] (2.19)
where \( \eta = \text{sgn}(Z^r) \). In the following section we will show that the comoving coordinate \( Z \) may be identified with the null coordinates according to \( Z = -U \) for an expanding solution and \( Z = -V \) for a collapsing one. \( P_Z \leq 0 \) and the dust hamiltonian density is chosen to be always non-negative. When \( P_Z \) is non-vanishing the phase-space is made up of two disconnected sectors, labeled by \( \eta \). An initial data set with \( P_Z = 0 \) cannot evolve into a set with \( P_Z \neq 0 \) and we will assume from now on that \( P_Z \neq 0 \).

### III. CANONICAL TRANSFORMATION

The description of contracting and expanding clouds is seen to be related by time reversal. The two descriptions may be formally unified in the following way. Introduce a null coordinate \( W^\eta_r \), which may be the “advanced” time or the “retarded” time, satisfying only the requirement that \( W^\eta_r > 0 \) (primes denote differentiation w.r.t. the ADM label coordinate \( r \)) then it represents the retarded coordinate, \( U \), and if, on the contrary, \( W^\eta_r > 0 \), it represents the advanced coordinate, \( V \). Let us write both solutions in terms of a parameter \( \eta \) that represents the behavior of the matter (whether it is expanding or collapsing)
\[ d s^2 = \left( 1 - \frac{2m}{R} \right) d W^\eta_r \,^2 + 2 \eta d W^\eta_r dR - R^2 d\Omega^2. \] (3.1)
The metric (3.1) is appropriate for either expansion or contraction of the dust cloud depending on whether \( \eta = -\text{sgn}(W^\eta_r) \) is +1 or -1. \( \eta \) is the same as appears in (2.19), as we argue below. The null coordinate, \( W^\eta_r \), whose spatial direction is opposite to \( W^\eta_r \) is obtained by integrating
\[ d W^\eta_r = \sigma(W^\eta_r, R) \left[ d W^\eta_r + 2 \eta dR \right], \] (3.2)
where \( \sigma(W^\eta_r, R) \) is an integrating factor. This coordinate must also be always increasing toward the future.

The hypersurfaces (2.1) from which (2.2) is constructed must be embedded in the space-time described by the metric (3.1). Substituting the foliation \( W^\eta_r (t, r) \) and \( R(t, r) \) in (3.1) gives the density \( L(t, r) \) and the lapse and shift functions, \( N(t, r) \) and \( N^r(t, r) \), as
\[ F W^\eta_r + 2 \eta W^\eta_r R = N^2 - L^2 N^r \]
\[-F\Omega_{t}^{2} - 2\eta W_{\eta} R' = L^{2}\]
\[-\eta(\dot{W}_{\eta} R' + W_{\eta} \dot{R}) - F\dot{W}_{\eta} = N_{r} L^{2},\]  \hspace{1cm} (3.3)
where we have set \( F = 1 - 2m/R \). These relations can be used to determine,
\[
N' = \frac{FW_{\eta} W_{\eta} + \eta(W_{\eta} R' + W_{\eta} \dot{R})}{FW_{\eta} R' + 2\eta W_{\eta} R'},
\]
\[
N = \frac{W_{\eta} R' - W_{\eta} \dot{R}}{L}, \hspace{1cm} (3.4)
\]
where we have chosen the sign of the square root so that \( N \) is positive. Inserting these expressions for \( N \) and \( N' \) into equation (2.5) for \( P_{L} \), we find
\[
\frac{LP_{L}}{R} = -\eta R' - F\dot{W}_{\eta}, \hspace{1cm} (3.5)
\]
which, when substituted into the expression for \( L^{2} \) in (3.3), gives
\[
F = \frac{R^{2}}{L^{2}} - \frac{P_{L}^{2}}{R^{2}}, \hspace{1cm} (3.6)
\]
or, equivalently, the mass function in terms of the canonical data
\[
m = \frac{R}{2} \left[ 1 - \frac{R^{2}}{L^{2}} + \frac{P_{L}^{2}}{R^{2}} \right]. \hspace{1cm} (3.7)
\]
By directly taking Poisson brackets, the momentum conjugate to \( m \) can now be shown to be simply
\[
P_{m} = \frac{LP_{L}}{RF}. \hspace{1cm} (3.8)
\]
Kuchař [12] proposed that \( (R, m, T_{R}, P_{m}) \) should form a canonical chart whose coordinates are spatial scalars, whose momenta are scalar densities and which is such that \( H_{r}(r) \) generates Diff \( \mathbb{R} \). This means that
\[
\mathcal{H}_{\eta} = R' P_{R} - LP_{L}' = R T_{R} + m P_{m} = 0. \hspace{1cm} (3.9)
\]
Substituting the expressions derived for \( m \) and \( P_{m} \) into the above constraint one arrives at
\[
T_{R} = P_{R} - \frac{LP_{L}}{2R} - \frac{LP_{L}}{RF} - \frac{\Delta}{RL^{2} F}, \hspace{1cm} (3.10)
\]
where \( \Delta = (RR')(LP_{L}') - (RR')(LP_{L}) \). One can then show that the transformation,
\[
(R, L, P_{R}, P_{L}) \rightarrow (R, m, T_{R}, P_{m}), \hspace{1cm} (3.11)
\]
is canonical, and generated by
\[
\mathcal{G} = \int_{0}^{\infty} dr \left[ LP_{L} - \frac{1}{2} RR' \ln \frac{RR' + LP_{L}}{RR' - LP_{L}} \right]. \hspace{1cm} (3.12)
\]
By computing the difference between the old and the new Liouville forms and using the fall off conditions in (2.7) and (2.8), one can show that the transformation has introduced no fresh boundary terms.

There are (infinite) boundary terms at the horizon, when \( F = 0 \). It can be shown, however, that the contribution from the interior and the exterior cancel each other. There will also be contributions at the boundary between the interior of the star and its exterior or more generally at any frontier between two regions described by different mass functions. Again, if the mass function is continuous across the boundary and regions are consistently matched by equating both the first and second fundamental forms, then the contribution from one side will cancel the contribution from the other.

Before rearranging the action, we will consider the coordinates \( \{ Z, Z^{k} \} \) for the collapsing Vaidya null congruence. Returning to the metric in (3.1) with \( \eta = -1 \) we find that the ingoing null congruence is given by \( V = \text{const.}, \theta = \text{const} \), and \( \phi = \text{const} \). The coordinates \( Z = -V, Z^{1} = \theta \) and \( Z^{2} = \phi \) are co-moving. Let us form the basis
\[
Z_{\mu} = (-1, 0, 0, 0) \quad Z^{1}_{\mu} = (0, 0, 1, 0) \quad Z^{2}_{\mu} = (0, 0, 0, 1) \hspace{1cm} (3.13)
\]
It is easily shown that \( \xi = -R \) is an affine parameter and the covariant components of the velocity \( dx^{\mu}/d\xi \) are \( U_{\mu} = (-1, 0, 0, 0) \), whose decomposition in the co-basis \( Z^{k}_{\alpha} \) yields \( W_{1} = 0 = W_{2} \). Similarly treating the outgoing null congruence shows that the affine parameter is \( \xi = +R \) and that \( Z \) is to be identified with \(-U\). Both cases may be treated simultaneously by letting \( Z = -W_{\eta}, Z_{2} = -P_{W_{\eta}} \) and \( \xi = \eta R \). This identification shows that the \( \eta \) used in the section is identical to that used in the previous section.

Note also that taking the spatial derivative of \( m \) in (3.7) yields
\[
m' = -\frac{R'}{L} \mathcal{H}^{\eta} - \frac{P_{l}}{RL} \mathcal{H}_{\eta}. \hspace{1cm} (3.14)
\]
This may be used to write the action in (2.3) as
\[
S = \int dt \int_{0}^{\infty} dr \left[ P_{W_{\eta}} W_{\eta} + T_{R} R' + P_{m} m' - N \mathcal{H} - N' \mathcal{H}_{r} \right] + S_{\partial \Sigma}, \hspace{1cm} (3.15)
\]
with
\[
\mathcal{H} = -\left[ \frac{m' F^{-1} R' + F P_{m} T_{R}}{L} \right] - \frac{P_{W_{\eta}} W_{\eta}'}{L} \hspace{1cm} (3.16)
\]
\[
S_{\partial \Sigma} = -\int_{\partial \Sigma_{\infty}} N_{+}(t) M_{+}(t). \hspace{1cm} (3.16)
\]
The surface term contains the mass at spatial infinity and may be re-expressed in a more convenient form. Use the fall-off conditions (2.7) at infinity and the expression for \( N \) in (3.4) to write \( N_{+} = W_{\eta} \) and
\[
S_{\partial \Sigma} = -\int_{\partial \Sigma_{\infty}} W_{\eta}(t) M_{+}(t). \hspace{1cm} (3.17)
\]
Then define \( \Gamma(r) \) by
where \( \delta\omega = \delta(M_+W_\eta+) \) is an exact form. The second term in the expression for \( \Omega \) continues to be inconvenient, but may be cast into a more appropriate form using the identity [12]

\[
\left( \int_0^r dr' P_m(r') \times \int_r^\infty dr' \delta\Gamma(r') \right)' = P_m(r) \int_0^\infty dr' \delta\Gamma(r') - \left( \int_0^r dr' P_m(r') \right)' \delta\Gamma(r).
\]

Integrating from \( r = 0 \) to \( r = \infty \), the left hand side of the above equation vanishes identically and one finds

\[
\int_0^\infty dr P_m(r) \int_r^\infty dr' \delta\Gamma(r') = \int_0^\infty dr \delta\Gamma(r) \int_0^r dr' P_m(r'),
\]

so that \( \Omega \) can be cast into the form

\[
\Omega = p_+ \delta M_+ + \int_0^\infty dr P_\Gamma \delta\Gamma - \delta\omega,
\]

where we have defined

\[
p_+ = W_\eta+ + \int_0^\infty dr P_m(r),
P_\Gamma = - \int_0^r dr' P_m(r').
\]

Eliminating a total time derivative turns the action in (3.15) into

\[
S = \int dt [p_+ \dot{M}_+ + \int_0^\infty dr (p W_\eta + \mathbf{P}_R \dot{R} + P_\Gamma \dot{\Gamma} - N^W (P W_\eta - \Gamma) - N^R (\mathbf{F} R - \eta \Gamma))],
\]

where

\[
\mathcal{H} = - \left[ \mathcal{F}^{-1} R' - \mathcal{F} P_\Gamma \mathbf{P}_R \right] - \eta P W_\eta',
\]

\[
\mathcal{H}_r = R \mathbf{P}_R - \Gamma P_\Gamma' + W_\eta' P W_\eta.
\]

Furthermore, it follows from (3.5) that

\[
P_\Gamma' = W_\eta' + \eta \mathbf{R}' / \mathbf{F}.
\]

The constraints, \( \mathcal{H} \approx 0 \approx \mathcal{H}_r \), can be further simplified by using \( \mathcal{H}_r \approx 0 \) to eliminate \( W_\eta' P W_\eta \) from the hamiltonian constraint. This gives

\[
(\mathcal{F} P_\Gamma + \eta R') \left( \frac{\mathbf{P}_R - \eta \mathbf{F}}{\mathbf{F}} \right) \approx 0.
\]

Consider the first of the two factors above. Using (3.27) to substitute for \( P_\Gamma' \), we find

\[
\mathcal{F} P_\Gamma' + \eta R' = \mathcal{F} W_\eta' + 2 \eta R' / \mathbf{F} = \mathcal{F} \frac{W_\eta'}{\sigma},
\]

where \( \sigma (R, W_\eta) \) is the integrating factor introduced in the previous section. But \( W_\eta' \neq 0 \) because it is a null coordinate and is required to increase in time, therefore,

\[
\mathbf{P}_R - \eta \mathbf{F} / \mathbf{F} \approx 0
\]

is equivalent to the Hamiltonian constraint, \( \mathcal{H} \approx 0 \). Inserting this into either of the two constraints then gives \( P W_\eta \approx \eta \mathbf{F} \mathbf{P}_R \approx \Gamma \).

The configuration space consists of the set of variables \( \{ W_\eta, R, M_+, \Gamma \} \), whose physical significance is transparent. This is an advantage of Kuchař variables. \( W_\eta \) is a null coordinate, \( R \) is the area radius of a point labeled \( (r, t) \), \( \Gamma \) is the energy density of the collapsing cloud and \( M_+ \) is the mass measured at infinity. \( M_+ \) is a constant of the motion and may be viewed as part of the initial data. Our gravity-matter system may be re-written in the form

\[
S = \int dt \left[ p_+ \dot{M}_+ + \int_0^\infty dr (P W_\eta \dot{W} + \mathbf{P}_R \dot{R} + P_\Gamma \dot{\Gamma} - N^W (P W_\eta - \Gamma) - N^R (\mathbf{F} R - \eta \Gamma)) \right].
\]

The canonical action (3.31) will be our starting point for Dirac quantization. The configuration space coordinates \( W_\eta \) and \( R \) locate the hypersurface and \( \Gamma \) (along with \( M_+ \)) determines the matter distribution. Below we obtain a solution of the constraints for any matter distribution.

### IV. QUANTIZATION

In Dirac’s approach, the canonical momenta \( P_X \), for \( X \in \{ W_\eta, R, M_+, \Gamma \} \), are raised to operator status,

\[
P_X \rightarrow -i \frac{\delta}{\delta X}
\]

and the constraints are considered as operator restrictions on the state functional, \( \Psi \{ W_\eta, R_*, M_+, \Gamma \} \), i.e., \( \Psi \) obeys

\[
-i \frac{\delta \Psi}{\delta W_\eta} = \frac{\Gamma}{\mathbf{F}} \Psi,
\]

\[
-i \frac{\delta \Psi}{\delta \Gamma} = \mathbf{F} \Psi.
\]
Either one of the above could be replaced by the full form of the momentum constraint
\[ R \frac{\delta \Psi}{\delta R} + W_\eta \frac{\delta \Psi}{\delta W_\eta} - \Gamma \left( \frac{\delta \Psi}{\delta t} \right)' = 0, \] (4.3)
which is solved by any wave-functional that is a spatial scalar. Consider a solution of this constraint that is of the form
\[ \Psi = C_\eta (M_+) \exp \left[ i \int_0^\infty dr \Gamma(r) \cdot K(\eta, W_\eta, R, m) \right], \] (4.4)
where \( C_\eta \) is a constant depending only on \( \eta \) and \( M_+ \), and \( K \) is an arbitrary complex valued function of its arguments (and not their derivatives) that is to be evaluated so that \( \Psi \) satisfies the other constraints. The wave-functional \( \Psi \) in (4.4) is evidently a spatial scalar because \( \Gamma(r) \) is a spatial density and \( K \) is a spatial scalar. It is therefore a solution of the momentum constraint providing that \( K \) has no explicit dependence on \( r \).

The solution, which agrees with all the constraints considered as operator restrictions on the state functional, is given by
\[ \Psi = C_\eta \exp \left[ i \int_0^\infty dr \Gamma(r)(W_\eta(r) + \eta R_\eta(r)) \right], \] (4.5)
where \( R_\eta(R, M_+, \Gamma) \) is a “tortoise”-like coordinate defined by
\[ R_\eta = R + 2m \ln \frac{R}{2m} - 1. \] (4.6)
It is not, of course, the tortoise coordinate \( R^* \) except in a Schwarzschild region when \( m(M_+, \Gamma) = M_+ \) is constant.

The parameter \( \eta \) represents the direction of the flow, being +1 for outgoing null matter and -1 for infalling matter. The combination \( \eta R(r) \) represents the affine parameter, \( \xi(r) \). Let us re-express the wave-functional in (4.4), making the dependence on the affine parameter explicit,
\[ \Psi = C_\eta e^{i \int_0^\infty dr \Gamma(r) \xi(r)} \exp \left[ i \int_0^\infty dr \Gamma(r) \cdot W_\eta(r) \right], \] (4.7)
where \( \xi = \eta R_\eta \).

For \( \eta = -1 \), \( \xi \in (-\infty, 0] \) and \( W_\eta = V \), the wave-functionals are of the form
\[ \Psi_{-1} = A(M_+) e^{-i \int_0^\infty dr \Gamma(r) R_\eta(r)} e^{i \int_0^\infty dr \Gamma(r)V(r)}, \] (4.8)
describing collapsing null matter. Likewise, for \( \eta = +1 \), \( \xi \in [0, \infty) \) and \( W_\eta = U \), the functionals are
\[ \Psi_{+1} = B(M_+) e^{i \int_0^\infty dr \Gamma(r) R_\eta(r)} e^{i \int_0^\infty dr \Gamma(r)U(r)}, \] (4.9)
describing expanding null matter. The natural description of the cloud in the quantum theory is as a sum over the two geometries i.e., we construct the appropriate wave-functional by superposing an infalling and an outgoing solution with arbitrary amplitudes
\[ \Psi = A(M_+) e^{i \int_0^\infty dr \Gamma_{in}(r) R_\eta(r)} e^{i \int_0^\infty dr \Gamma_{in}(r)V(r)} + B(M_+) e^{i \int_0^\infty dr \Gamma_{out}(r) R_\eta(r)} e^{i \int_0^\infty dr \Gamma_{out}(r)U(r)}, \] (4.10)
where \( A(M_+) \) and \( B(M_+) \) are interpreted as the amplitudes to observe the matter as “contracting” and “expanding” respectively. The central problem then is to evaluate these amplitudes and determine the relationship between the outgoing matter density, \( \Gamma_{out} \), and the given infalling one, \( \Gamma_{in} \). It is not clear to us at the present time how they can be determined in the general case, but we suggest that they should follow from some boundary conditions suitable to the particular model being considered.

Such a superposition of ingoing and outgoing states was suggested by Hájíček [15] in connection with the gravitational collapse of a thin shell and we will now illustrate how the amplitudes are determined in that model system.

A given classical collapse problem is specified by a choice of mass function, \( m(W_\eta) \), which determines an initial energy distribution, thus a collapse “model”. We shall consider the function,
\[ m(W_\eta) = M_+ \theta(W_\eta - w), \] (4.11)
where \( \theta \) is the Heaviside unit step-function and \( w \) is constant. The matter energy vanishes when \( W_\eta < w \) and is \( M_+ \) when \( W_\eta > w \). The mass function evidently makes sense only as a thin shell that is collapsing toward the center and we must have \( W_\eta = V \) (\( \eta = -1 \)). We find the energy density by differentiating w.r.t. the ADM label coordinate, \( r \),
\[ \Gamma_{in}(r) = M_+ V'(r) \delta(V - v), \] (4.12)
where \( \tau(t) \) is the solution of \( V(r, t) = v \). Likewise, a thin shell that expands out of the center is represented by the mass function
\[ m(W_\eta) = M_+ \theta(w - W_\eta), \] (4.13)
for \( W_\eta = U \) (\( \eta = +1 \)). The energy density is
\[ \Gamma_{out}(r) = -M_+ U'(r) \delta(u - U), \] (4.14)
where \( \tau(t) \) is the solution of \( U(r, t) = u \). The energy density is always positive and in either case we find
\[ \Gamma = M_+ \delta(r - \tau). \] (4.15)
Using these expressions, let us relate the constraints in (2.19) to the constraints that have been used by others to describe thin shells. The dust hamiltonian and momentum density turn out to be
\[ H^d = \eta \frac{M_+ Z'}{L} \delta(r - \tau) = \eta \frac{p}{L} \delta(r - \tau) \]
\[ H^m = -M_+ Z' \delta(r - \tau) = -p \delta(r - \tau), \] (4.16)
where we have defined \( p = M_z' \) (r) and therefore \( \eta = \text{sgn}(Z') = \text{sgn}(p) \). These expressions were used as a starting point in [13,15,17]. The gravitational contributions to the constraints are, of course, the same.

The corresponding classical solutions are represented in the Penrose diagram of figure 2, where \( AA' \) represents the event horizon in (a) and the Cauchy horizon in (b).

![Figure 2: A thin shell (a) collapsing and (b) expanding](image)

Inserting (4.15) into (4.8) and (4.9) one finds that the quantum mechanics of a single shell is described by the wave-functions

\[
\Psi_{-1} = A(M_{+}^{\text{in}}, u) e^{-iM_{+}^{\text{in}} R^{*}(v)},
\]

for an infalling shell and

\[
\Psi_{+1} = B(M_{+}^{\text{out}}, u) e^{iM_{+}^{\text{out}} R^{*}(v)},
\]

for an expanding one, where \( R^{*} \) as given by (4.6) is, in this case, the usual tortoise coordinate. If we “sum over geometries” as proposed, we are considering the wave function

\[
\Psi = A e^{-iM_{+}^{\text{in}} R^{*}} + B e^{iM_{+}^{\text{out}} R^{*}},
\]

where \( A = (M_{+}^{\text{in}}, M_{+}^{\text{out}}, u, v) \) and \( B(M_{+}^{\text{in}}, M_{+}^{\text{out}}, u, v) \) are the amplitudes for observing the shell as incoming and outgoing respectively. Imagine that in the infinite retarded past we had prepared a thin incoming shell characterized by its mass, \( M_{+} \), and by the parameter \( v \). A first requirement will be that the evolution preserves the initial data, so that \( u = v \) and \( M_{+}^{\text{in}} = M_{+}^{\text{out}} = M_{+} \). The wave function depends only on two parameters, the advanced time of collapse, \( v \), and the mass at spatial infinity, \( M_{+} \). The next condition is clear if we note that, for the superposition of incoming and outgoing states in (4.19) to make sense within the mini-superspace program, the shells cannot reach (or originate from) the center, for then a classical singularity would be present, invalidating our starting metric. Our second requirement is therefore that the wave-function vanishes at the center. At the center, \( R = 0 \), we find \( B(M_{+}, u = v) = -A(M_{+}, v) \). The probability for observing a collapsing shell is therefore the same as the probability for observing an outgoing one. We arrive at

\[
\Psi = A(M_{+}, v) \sin(M_{+} R^{*}).
\]

Furthermore, if we define an inner product by

\[
\langle \Psi_1, \Psi_2 \rangle = \int dR^* \bar{\Psi}_1 \Psi_2,
\]

where the integration is over the entire range of the physical radius, \( R \in [0, \infty) \), then the wave-function (4.20) is a \( \delta \)-function normalized by taking \( A = \sqrt{2} \).

A similar wave-function was obtained by Hájíček [15]. The difference is in our definition of the position of the shell. In our construction it is given by the tortoise coordinate, \( R^{*} \), which approaches minus infinity as the shell approaches its horizon. The wave-function oscillates infinitely rapidly as the shell crosses this radius, but it continues to be well behaved inside of it. Passing into the interior, \( R < 2M_{+} \), our conditions require that it vanishes at the center, which leads to an avoidance of the central singularity by the shell. Other conditions permitting a singularity development may be possible but a superposition would then not be realistic.

The behaviour of wave packets constructed from the wave-function in (4.20) has been analyzed carefully by Hájíček and Kiefer [15,16] and leads to a number of interesting conclusions. Wave-packets beginning in the asymptotic region keep away from the center. Their average motion is time reversal invariant and the shell wave-packet always bounces at the center and re-expands.

V. DISCUSSION

A geometric picture of the boundary conditions applied with the superposition of infalling and outgoing states is presented in the Penrose diagram of figure 3.

![Figure 3: Penrose diagram representing the superposition](image)

In the figure, \( \Sigma_1 \) is a space-like hypersurface in the collapsing Vaidya space-time that crosses the shell before it reaches the center and forms a singularity. Let it intersect the contracting null shell, \( Y_1 \) (\( v=\text{constant} \)) at a radial distance \( R \) from the center. The hypersurface \( T \Sigma_1 \) is a
space-like hypersurface in the outgoing space-time that intersects the expanding null shell $Y_2$ (defined by $u = v$) at the same radial distance $R$ from the center. The two space-times are cut, the region to the future of $\Sigma^1$ and the region to the past of $\Sigma^1$ being discarded. They are then glued along the hypersurfaces. This can be done in a continuous way because the region to the advanced future of the collapsing shell and the region to the retarded past of the expanding shell are both Schwarzschild, of mass $M_+$ (by our condition that the initial data is preserved), while the region to the advanced past of the collapsing shell and the region to the retarded future of the expanding shell are both Minkowski. (Although the resulting space-time is not differentiable, only piecewise differentiable in, for example, the path integral approach to quantization.) A more complete picture would be that the cutting and pasting occurs in principle on a succession of space-like hypersurfaces so that the semi-classical trajectory of the shell is continuous and piecewise differentiable.

In summary, we have studied the collapse and expansion of a null dust cloud of arbitrary mass distribution and shown that there exists a canonical transformation that brings a general Vaidya system to the Kuchař form, in which the dynamics is expressed in terms of embedding variables whose physical meaning is transparent. In these variables, the constraints are simple and when they are realized as operator restrictions on a wave-functional, solutions for collapse and re-expansion can be easily obtained. We have proposed that the quantum description of the null cloud should be given via a superposition of the two solutions. Suitable boundary conditions have to be imposed in order to implement this program consistently and we have shown how this may be accomplished for a single shell. We then provided a geometric picture of the superposition for one shell.

One readily sees, however, that the picture does not translate straightforwardly even to the case of two shells. Further, there is the issue of defining a suitable inner product in the general case. We have suggested a natural inner product only for the case of a single shell. In considering time-like dust, a quadratic form of the Hamiltonian constraint was used to extract a physical measure on the Hilbert space [18], but such a quadratic constraint does not occur here in a natural way. These are topics for further investigation, which will be reported upon in a forthcoming publication.

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