Breaking Eight–fold Degeneracies in Neutrino 

$CP$ Violation, Mixing, and Mass Hierarchy

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Abstract

We identify three independent two-fold parameter degeneracies ($\delta, \theta_{13}$), $\text{sgn}(\delta m_{31}^2)$ and ($\theta_{23}, \pi/2 - \theta_{23}$) inherent in the usual three–neutrino analysis of long–baseline neutrino experiments, which can lead to as much as an eight–fold degeneracy in the determination of the oscillation parameters. We discuss the implications these degeneracies have for detecting $CP$ violation and present criteria for breaking them. A superbeam facility with a baseline at least as long as the distance between Fermilab and Homestake (1290 km) and a narrow band beam with energy tuned so that the measurements are performed at the first oscillation peak can resolve all the ambiguities other than the ($\theta_{23}, \pi/2 - \theta_{23}$) ambiguity (which can be resolved at a neutrino factory) and a residual ($\delta, \pi - \delta$) ambiguity. However, whether or not $CP$ violation occurs in the neutrino sector can be ascertained independently of the latter two ambiguities. The ($\delta, \pi - \delta$) ambiguity can be eliminated by performing a second measurement to which only the $\cos \delta$ terms contribute. The hierarchy of mass eigenstates can be determined at other oscillation peaks only in the most optimistic conditions, making it necessary to use the first oscillation maximum. We show that the degeneracies may severely compromise the ability of the proposed SuperJHF–HyperKamiokande experiment to establish $CP$ violation. In our calculations we use approximate analytic expressions for oscillation probabilities that agree with numerical solutions with a realistic Earth density profile.
The up/down asymmetry of the neutrino flux (originating from cosmic ray interactions with the atmosphere) at SuperKamiokande is now a $10\sigma$ effect. A compelling interpretation of this result is that neutrinos have mass and oscillate from one flavor to another. The atmospheric neutrino deficit is explained as a consequence of $\nu_\mu \rightarrow \nu_\tau$ oscillations with almost maximal amplitude and mass-squared difference, $\delta m^2_{31} \sim 3 \times 10^{-3}$ eV$^2$ [1]. The K2K experiment [2] with a baseline of 250 km has preliminary results that are in agreement with this interpretation. Oscillations of $\nu_\mu$ to $\nu_\tau$ as an explanation of the atmospheric anomaly are ruled out by the CHOOZ [3] and Palo Verde [4] reactor experiments, which place a bound on the amplitude smaller than 0.1 at the 95% C.L. in the $\delta m^2_{31}$ region of interest. The MINOS [5], ICARUS [6] and OPERA [7] experiments are expected to come online in 2005 and study aspects of the oscillations at the atmospheric scale [8]. The low energy beam at MINOS will allow a very accurate determination of the leading oscillation parameters. ICARUS and OPERA should provide concrete evidence that $\nu_\mu \rightarrow \nu_\tau$ oscillations are responsible for the atmospheric neutrino deficit by identifying tau neutrino events.

Measurements of electron neutrinos from the Sun also provide strong evidence for neutrino oscillations. The flux of electron neutrinos from the Sun observed in several different experiments is smaller than the Standard Solar Model [9] (SSM) prediction by a factor of $1/3$–$1/2$. The recent SNO charged-current measurements show that $\nu_e \rightarrow \nu_{\mu,\tau}$ oscillations explain the $\nu_e$ flux suppression relative to the SSM [10]. The solution with a large mixing angle (LMA) and small matter effects ($\delta m^2_{21} \sim 5 \times 10^{-5}$ eV$^2$ and amplitude close to 0.8) has emerged as the most likely solution to the solar neutrino problem [11]. This solution will be tested decisively by the KamLAND reactor neutrino experiment [12].

There are several parameter degeneracies that enter the determination of the neutrino mixing matrix which can be removed only with future oscillation studies with superbeams or neutrino factories. See Table I for a sample of proposed baselines. A notable example is the $U_{e3}$ ($=\sin\theta_{13} e^{-i\delta}$) element. Only an upper bound exists on $\theta_{13}$, nothing is presently known about the $CP$ phase $\delta$, and the two always appear in combination in the mixing matrix. It is the breaking of such degeneracies that will be of concern to us in this work.

In Section II we identify all the potential parameter degeneracies in the mixing matrix. We restrict our attention to the $3 \times 3$ matrix that describes the mixing of active neutrinos, setting aside the possibility that the atmospheric, solar and LSND [13] data may require the existence of a fourth neutrino that is sterile. The parameter ambiguities are connected with
not only neutrino mixing but also the neutrino mass pattern; we pay particular attention to
the implication of these ambiguities for the detection of CP violation. In Section III, within
the context of a superbeam experiment [14], we present methods by which all but one of
these degeneracies can be resolved, and argue that the remaining ambiguity can be settled at
a neutrino factory [15]. We also discuss the implications of the degeneracies on the proposed
SuperJHF–HyperKamiokande experiment [16], which would have a 4 MW proton driver and
a 1 Mt water cerenkov detector (40 times larger than SuperKamiokande). We summarize
our results in Section IV. In an appendix we provide a complete set of approximate analytic
expressions for the oscillation probabilities that are useful for superbeams and neutrino
factories, and define their domain of validity by making comparisons with numerical solutions
of the evolution equations.

II. PARAMETER DEGENERACIES

In this section we identify the three types of parameter degeneracies that can occur
in the three–neutrino framework when $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillation probabilities are
used to extract the neutrino parameters. We use approximate formulas [17,18] for neutrino
propagation in matter of constant density to illustrate the degeneracies. In each case we
discuss the implications for detecting CP violation.

A. Oscillation probabilities in matter

The neutrino flavor eigenstates $\nu_\alpha$ ($\alpha = e, \mu, \tau$) are related to the mass eigenstates
$\nu_j$ ($j = 1, 2, 3$) in vacuum by

$$\nu_\alpha = \sum_j U_{\alpha j}^* \nu_j ,$$

where $U$ is a unitary $3 \times 3$ mixing matrix. The propagation of neutrinos through matter is
described by the evolution equation [19,20]

$$i \frac{d\nu_\alpha}{dx} = \sum_\beta \left( \sum_j U_{\alpha j} U_{\beta j}^* \frac{m_j^2}{2E_\nu} + \frac{A}{2E_\nu} \delta_{\alpha e} \delta_{\beta e} \right) \nu_\beta ,$$

where $x = ct$ and $A/2E_\nu$ is the amplitude for coherent forward charged-current $\nu_e$ scattering
on electrons,

$$A = 2\sqrt{2} G_F N_e E_\nu = 1.52 \times 10^{-4} \text{eV}^2 Y_e \rho (\text{g/cm}^3) E_\nu (\text{GeV}) ,$$

Here $N_e$ is the electron number density, which is the product of the electron fraction $Y_e(x)$
and matter density $\rho(x)$. In the Earth’s crust and mantle the average matter density is
typically 3–5 g/cm$^3$ and $Y_e \simeq 0.5$. The propagation equations can be re-expressed in terms
of mass-squared differences

$$i \frac{d\nu_\alpha}{dx} = \sum_\beta \frac{1}{2E_\nu} \left( \delta m_{31}^2 U_{\alpha 3} U_{\beta 3}^* + \delta m_{21}^2 U_{\alpha 2} U_{\beta 2}^* + A \delta_{\alpha e} \delta_{\beta e} \right) ,$$

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where $\delta m^2_{jk} \equiv m_j^2 - m_k^2$. The neutrino mixing matrix $U$ can be specified by 3 mixing angles $(\theta_{23}, \theta_{12}, \theta_{13})$ and a $CP$-violating phase $\delta$. We adopt the parameterization

$$U = \begin{pmatrix}
c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\
-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & c_{13}s_{23}
\end{pmatrix},$$

where $c_{jk} \equiv \cos \theta_{jk}$ and $s_{jk} \equiv \sin \theta_{jk}$. In the most general $U$, the $\theta_{ij}$ are restricted to the first quadrant, $0 \leq \theta_{ij} \leq \pi/2$, with $\delta$ in the range $0 \leq \delta < 2\pi$. We assume that $\nu_3$ is the neutrino eigenstate that is separated from the other two, and that the sign of $\delta m^2_{31}$ can be either positive or negative, corresponding to the case where $\nu_3$ is either above or below, respectively, the other two mass eigenstates. The magnitude of $\delta m^2_{31}$ determines the oscillation length of atmospheric neutrinos, while the magnitude of $\delta m^2_{21}$ determines the oscillation length of solar neutrinos, and thus $|\delta m^2_{31}| \ll |\delta m^2_{21}|$. If we accept the likely conclusion that the solar solution is LMA [11], then $\delta m^2_{21} > 0$ and we can restrict $\theta_{12}$ to the range $[0, \pi/4]$. It is known from reactor neutrino data that $\theta_{13}$ is small, with $\sin^2 2\theta_{13} \leq 0.1$ at the 95% C.L. [3]. Thus a set of parameters that unambiguously spans the space is $\delta m^2_{31}$ (magnitude and sign), $\delta m^2_{21}$, $\sin^2 2\theta_{12}$, $\sin \theta_{23}$, and $\sin^2 2\theta_{13}$; only the $\theta_{23}$ angle can be below or above $\pi/4$.

In the context of three-neutrino models the usual method proposed for detecting $CP$ violation in long-baseline experiments with a conventional neutrino beam is to measure the oscillation channels $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ (or $\nu_\mu \rightarrow \nu_\mu$ and $\bar{\nu}_e \rightarrow \bar{\nu}_e$ for a neutrino factory). Both leading and subleading oscillation contributions must be involved and the oscillations must be non–averaging for $CP$–violation effects [21]. For illustrative purposes we use the constant density matter approximation, although in an exact study variations of the density along the neutrino path should be implemented. Approximate formulas for the oscillation probabilities in matter of constant density in the limit $|\delta m^2_{21}| \ll A, |\delta m^2_{31}|$ already exist in the literature [17,18]. We adopt the form in Ref. [18], where $\theta_{13}$ is also treated as a small parameter and the mixing angles in matter are found in terms of an expansion in the small parameters $\theta_{13}$ and $\delta m^2_{21}$. We introduce the notation

$$\Delta \equiv |\delta m^2_{31}|L/4E_\nu = 1.27|\delta m^2_{31}/eV^2|(L/\text{km})/(E_\nu/\text{GeV}),$$

$$\hat{A} \equiv |A/\delta m^2_{31}|,$$

$$\alpha \equiv |\delta m^2_{21}/\delta m^2_{31}|.$$  

Up to second order in $\alpha$ and $\theta_{13}$, the oscillation probabilities for $\delta m^2_{31} > 0$ and $\delta m^2_{21} > 0$ are

$$P(\nu_\mu \rightarrow \nu_e) = x^2 f^2 + 2xyfg(\cos \delta \cos \Delta - \sin \delta \sin \Delta) + y^2 g^2,$$

$$\bar{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = x^2 f^2 + 2xyfg(\cos \delta \cos \Delta + \sin \delta \sin \Delta) + y^2 g^2,$$

respectively, where

$$x \equiv \sin \theta_{23} \sin 2\theta_{13},$$

$$y \equiv \alpha \cos \theta_{23} \sin 2\theta_{12},$$

$$f, \tilde{f} \equiv \sin((1 \mp \hat{A})\Delta)/(1 \mp \hat{A}),$$

$$g \equiv \sin(\hat{A}\Delta)/\hat{A}.$$
The coefficients $f$ and $\bar{f}$ differ due to matter effects ($\hat{A} \neq 0$). To obtain the probabilities for $\delta m_{21}^2 < 0$, the transformations $\hat{A} \rightarrow -\hat{A}$, $y \rightarrow -y$ and $\Delta \rightarrow -\Delta$ (implying $f \leftrightarrow -\bar{f}$ and $g \rightarrow -g$) can be applied to the probabilities in Eqs. (9) and (10) to give

$$P(\nu_\mu \rightarrow \nu_e) = x^2\bar{f}^2 - 2xy\bar{f}g(\cos \delta \cos \Delta + \sin \delta \sin \Delta) + y^2g^2,$$

$$P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = x^2f^2 - 2xyfg(\cos \delta \cos \Delta - \sin \delta \sin \Delta) + y^2g^2. \quad (15)$$

For a $T$-reversed channel, the corresponding probabilities are found by changing the sign of the $\sin \delta$ term. In Eqs. (9), (10), (15), and (16) we have assumed $\delta m_{21}^2 > 0$, which is what one expects for the LMA solar solution; for $\delta m_{21}^2 < 0$, the corresponding formulae are obtained by $y \rightarrow -y$. These expressions are accurate as long as $\theta_{13}$ is not too large, and they are valid at $E_\nu > 0$. $\mu eV (\hat{A} \sim 0.04(3 \times 10^{-3} eV^2/|\delta m_{21}^2|))$ for all values of $\delta m_{21}^2$ currently favored by solar neutrino experiments. We expand on the domain of validity of these equations in the Appendix. The corresponding expansion in $\alpha$ and $\beta_{13}$ in a vacuum can be found by the substitutions $f, \bar{f}, g \rightarrow \sin \Delta$.

For reference, the conversion from $\hat{A}$ and $\Delta$ to $L$ and $E_\nu$ is shown in Fig. 1. For neutrinos with $\delta m_{31}^2 > 0$ or anti-neutrinos with $\delta m_{31}^2 < 0$, $\hat{A} = 1$ corresponds to an MSW resonance. For neutrinos, it can be shown that the choice $\hat{A} = 1/2$ maximizes both the $\sin \delta$ and $\cos \delta$ terms for a given $\Delta$; for anti-neutrinos the $\hat{A}$ that maximizes the $\sin \delta$ and $\cos \delta$ terms varies with $\Delta$.

We make two observations regarding the approximate probability formulas above, the consequences of which are discussed below:

(i) Both terms that depend on the $CP$ phase $\delta$ vanish when $g = 0$, i.e., at $\hat{A}\Delta = n\pi$, where $n$ is an integer. The $y^2$ term also vanishes in this case, so that only the $x^2$ term survives.

(ii) The $\cos \delta$ term vanishes when $\Delta = (n - \frac{1}{2})\pi$, while the $\sin \delta$ term vanishes when $\Delta = n\pi$.

The above statements are true for both neutrinos and anti-neutrinos.

The first observation implies that there is no sensitivity to the $CP$–violating phase $\delta$ when $\langle N_e \rangle L = \int N_e dL$ is an integer multiple of $\sqrt{2}\pi/G_F$, where $\langle N_e \rangle$ is the average value of $N_e$ for the neutrino path. Numerically, for $n = 1$, this condition is

$$\langle N_e \rangle L \simeq 16275 \text{ km}, \quad (17)$$

or, for the Earth’s density profile,

$$L \simeq 7600 \text{ km}. \quad (18)$$

This distance has a simple physical interpretation: it is the characteristic oscillation wavelength due to the matter interaction [19]. Furthermore, the condition in Eq. (17) is independent of all oscillation parameters. It is also independent of $E_\nu$. It has often been noted that $CP$ violation is strongly suppressed in long baseline experiments of order 7300 km (nominally the distance from Fermilab to Gran Sasso); we see that this is a universal effect that occurs because $L$ is close to the oscillation length due to matter. Furthermore, the term proportional to $y^2$ also vanishes, which means that there is also no dependence on $\delta m_{21}^2$ or $\theta_{12}$.
at this distance, at least to second order in the small parameters. Therefore this distance is especially well-suited for measuring $\theta_{13}$ without the complications of disentangling it from $\delta$, $\theta_{12}$, or $\delta m^2_{21}$. For baselines greater than about 4000 km the constant density approximation loses accuracy (see results in the Appendix), so that the critical distance in Eq. (18) is not exact, but does explain semi-quantitatively the weakness of $CP$ violating effects near that distance.

The second observation relates to the relative strength of the $\sin \delta$ and $\cos \delta$ terms in $P(\nu_\mu \to \nu_\tau)$. In short $L$, low $E_\nu$ experiments the matter effects are small and the leading terms of the oscillation probability are given by the vacuum formulas. Then $L$ and $E_\nu$
can be chosen such that only the explicitly \( CP \)-violating \( \sin \delta \) term survives (e.g., when \( \Delta = \pi/2 \)), and \( CP \) violation can be measured directly by comparing \( P(\nu_\mu \to \nu_e) \) and \( P(\bar{\nu}_\mu \to \bar{\nu}_e) \) (although even for \( L \sim \) few 100 km there are small matter corrections that must be considered). However, as is evident from Eqs. (9) and (10), when \( \theta_{13} \) is small the relative strengths of the \( \sin \delta \) and \( \cos \delta \) terms in the presence of large matter corrections at longer \( L \) can be selected by an appropriate choice of \( \Delta \) in exactly the same way as in the short \( L \), vacuum–like case. That is, the \( \delta \) dependence with matter effects included can be made pure \( \sin \delta \) for
\[
L/E_\nu \simeq (2n - 1)(410 \text{ km/GeV }) \left( \frac{3 \times 10^{-3} \text{ eV}^2}{|\delta m_{31}^2|} \right),
\]
where \( n \) is an integer *. The only caveat is that matter corrections are much larger for longer \( L \) and the accuracy of the determination of \( \delta \) may be more subject to knowledge of the electron density.

**B. Orbits in probability space**

We assume that \( \sin^2 2\theta_{23} \) and \( |\delta m_{31}^2| \) are well–determined (perhaps at the few percent level or better) by a \( \nu_\mu \) survival or \( \nu_\mu \to \nu_\tau \) measurement [8], and that \( \theta_{12} \) and \( \delta m_{21}^2 \) are also well–determined (KamLAND should be able to measure the parameters of the solar LMA solution to the few percent level [12]). Then the remaining parameters to be determined are \( \delta, \theta_{13} \), and the sign of \( \delta m_{31}^2 \) (the sign of \( \delta m_{21}^2 \) is positive for LMA).

The usual proposal for testing \( CP \) violation in the neutrino sector is to measure both \( \nu_\mu \to \nu_e \) and \( \bar{\nu}_\mu \to \bar{\nu}_e \) probabilities. As \( \delta \) varies for given \( \theta_{13} \) and \( \text{sgn}(\delta m_{31}^2) \), an elliptical orbit will be traced in \( P–\bar{P} \) space [16,22]. The shape of the ellipse is determined by the relative phases of the terms involving \( \delta \). We identify three possible cases:

(i) \( \Delta \neq n\pi/2 \). In this case, both the \( \sin \delta \) and \( \cos \delta \) terms are nonzero and the orbit for fixed \( \theta_{13} \) is an ellipse. Each value of \( \delta \) gives a distinct point in \( P–\bar{P} \) space for a given \( \theta_{13} \). For \( \Delta = (n - \frac{1}{2})\pi \) the ellipse has the maximum “fatness” [22], i.e., it is as close as possible to a circle given the values of \( f \) and \( \bar{f} \).

(ii) \( \Delta = (n - \frac{1}{2})\pi \), where \( n \) is an integer. In this case the \( \cos \delta \) term vanishes and the orbit ellipse collapses to a line. If \( f \simeq \bar{f} \) (such as at short \( L \) where matter effects are small), \( CP \) violation is measured directly by comparing the \( \nu \) and \( \bar{\nu} \) event rates (after correcting for the differences in the cross sections and initial flux normalization).

(iii) \( \Delta = n\pi \). In this case the \( \sin \delta \) term vanishes, the ellipse collapses to a line, and \( CP \) violation is measured indirectly by parametrically determining the value of \( \delta \) and not by the measurement of a \( CP \)-violating quantity.

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*The misconception that the \( \cos \delta \) term dominates at large \( L \) and \( E_\nu \) comes from extending the large \( E_\nu \) approximation beyond its range of validity, as discussed in Ref. [18].
There will be two ellipses for each $\theta_{13}$, one for each sign of $\delta m_{31}^2$; they both fall into the same class, i.e., if the ellipse for $\delta m_{31}^2 > 0$ is Case (ii), the ellipse for $\delta m_{31}^2 < 0$ will also be Case (ii).

C. CP degeneracy: $(\delta, \theta_{13})$ ambiguity

In many cases the parameters $(\delta, \theta_{13})$ can give the same probabilities as another pair of parameters $(\delta', \theta_{13}')$, for fixed values of the other oscillation parameters; this is known as the "$(\delta, \theta_{13})$ ambiguity" [23]. Using Eqs. (9) and (10), the general formulas for the parameters $(x', \delta')$ that give the same $P$ and $\bar{P}$ as $(x, \delta)$ for $\Delta \neq n\pi/2$ (Case (ii)) are

$$x' \cos \delta' = x \cos \delta + \frac{(f + \bar{f})(x^2 - x'^2)}{4 y g \cos \Delta}, \quad (20)$$

$$x' \sin \delta' = x \sin \delta - \frac{(f - \bar{f})(x^2 - x'^2)}{4 y g \sin \Delta}. \quad (21)$$

Equations (20) and (21) can be used to derive

$$x'^2 - x^2 = \frac{4 y g \sin 2\Delta \left[ y g \sin 2\Delta + x f \sin(\Delta - \delta) + x \bar{f} \sin(\Delta + \delta) \right]}{f^2 + \bar{f}^2 - 2 f \bar{f} \cos 2\Delta}, \quad (22)$$

from which $\delta'$ can then be determined from Eq. (20) or (21). In particular, a set of parameters which violates CP ($\sin \delta' \neq 0$) can be degenerate with another set of parameters, with a different $\theta_{13}$, that conserves CP ($\sin \delta = 0$). It can be shown that in all cases real solutions exist for $x'$, so there will be an ambiguity between two sets of oscillation parameters if $|\sin \delta'| \leq 1$. Therefore we conclude that the use of a monoenergetic beam at a fixed $L$ will necessarily entail parameter ambiguities if only the channels $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ are measured. An example is shown in Fig. 2a for $\Delta = 3\pi/4$.

When $\Delta \neq n\pi/2$, the $(\delta, \theta_{13})$ ambiguity can give a degeneracy between CP violating ($CPV$) and CP conserving ($CPC$) solutions. If $\sin \delta = 0$ in Eq. (21), then $\sin \delta'$ is not zero if $f \neq \bar{f}$; the difference can be large if $f$ and $\bar{f}$ differ substantially due to large matter effects. For example, in Fig. 2a the prediction for $(P, \bar{P})$ for $(\sin^2 2\theta_{13}, \delta) = (0.01, 0)$ is identical to that for $(0.00298, 1.48\pi)$.

For the Cases (ii) and (iii) above, the ellipse collapses to a line. The ambiguities then reduce to $x' = x$ (see Eq. (22)) and $\sin \delta' = \sin \delta$ in Case (ii) and $\cos \delta' = \cos \delta$ in Case (iii). Thus the ambiguity no longer involves $\theta_{13}$ (and hence in principle $\theta_{13}$ is determined, at least as far as the $(\delta, \theta_{13})$ ambiguity is concerned), but instead is a $(\delta, \pi - \delta)$ ambiguity (which does not mix CPC and CPV solutions) in Case (ii) and a $(\delta, 2\pi - \delta)$ ambiguity in Case (iii). Examples for Cases (ii) and (iii) are shown in Figs. 2b and 2c, respectively. For $\Delta \approx n\pi/2$ the orbit ellipse is very skinny and the ambiguous $\theta_{13}$ values are close to each other, which qualitatively is similar to either Case (ii) or Case (iii).

Note that in both Figs. 2b and 2c the orbit line has a negative slope. It can be shown that for $\hat{A} < 1$ (i.e., density less than the critical density for resonance) the orbit lines in $(P, \bar{P})$ space have negative slope for $\Delta = n\pi/2$. For $\hat{A} > 1$, the orbit lines for $\Delta = n\pi/2$ have positive slope.
FIG. 2. Orbit ellipses showing $(\delta, \theta_{13})$ ambiguity for $L = 1290$ km with (a) $E_\nu = 2.09$ GeV ($\Delta = 3\pi/4$), (b) $E_\nu = 3.13$ GeV ($\Delta = \pi/2$), and (c) $E_\nu = 1.565$ GeV ($\Delta = \pi$), for $\sin^2 \theta_{13} = 0.01$ and 0.00298. The other parameters are $\delta m_{31}^2 = 3 \times 10^{-3}$ eV$^2$, $\delta m_{21}^2 = 5 \times 10^{-5}$ eV$^2$, $\sin^2 \theta_{23} = 1$, and $\sin^2 \theta_{12} = 0.8$. The value of $\delta$ varies around the ellipse. In (b) and (c) the ellipse collapses to a line and the ambiguity reduces to a $(\delta, \pi - \delta)$ or $(\delta, 2\pi - \delta)$ ambiguity, respectively, and different values of $\theta_{13}$ do not overlap (for the same sgn$(\delta m_{13}^2)$).
D. Mass hierarchy degeneracy: $\text{sgn}(\delta m_{31}^2)$ ambiguity

In addition to the $(\delta, \theta_{13})$ ambiguity discussed above for a given $\text{sgn}(\delta m_{31}^2)$, in some cases there are also parameters $(\delta', \theta'_{13})$ with $\delta m_{31}^2 < 0$ that give the same $P$ and $\bar{P}$ as $(\delta, \theta_{13})$ with $\delta m_{31}^2 > 0$. Three examples of the $\text{sgn}(\delta m_{31}^2)$ ambiguity are shown in Fig 3. Furthermore, there is also a $(\delta', \theta'_{13})$ ambiguity for $\delta m_{31}^2 < 0$, so in principle there can be a four-fold ambiguity, i.e., four sets of $\delta$ and $\theta_{13}$ (two for $\delta m_{31}^2 > 0$ and two for $\delta m_{31}^2 < 0$) that give the same $P$ and $\bar{P}$.

As with the $(\delta, \theta_{13})$ ambiguity, the $\text{sgn}(\delta m_{31}^2)$ ambiguity can mix CP conserving and CP violating solutions. For example, in Fig. 3a the prediction for $(P, \bar{P})$ for $(\sin^2 2\theta_{13}, \delta) = (0.01, 0)$ with $\delta m_{31}^2 = 3 \times 10^{-3}$ eV$^2$ is identical to that for $(0.0138, 4\pi/3)$ with $\delta m_{31}^2 = -3 \times 10^{-3}$ eV$^2$.

Although the general equations for the $\text{sgn}(\delta m_{31}^2)$ ambiguity are somewhat messy, for the case $\Delta = (n - \frac{1}{2})\pi$ the values of $(x', \delta')$ for $\delta m_{31}^2 < 0$ that give the same $P$ and $\bar{P}$ as $(x, \delta)$ for $\delta m_{31}^2 > 0$ are determined by

$$x'^2 = \frac{x^2(f^2 + \bar{f}^2 - f\bar{f}) - 2yg(f - \bar{f})x \sin \delta \sin \Delta}{ff},$$

$$x' \sin \delta' = x \sin \delta \frac{f^2 + \bar{f}^2 - f\bar{f}}{ff} - \frac{x^2}{\sin \Delta} \frac{f^2 + \bar{f}^2 - f\bar{f}}{2yg}. \quad (24)$$

If $\sin \delta = 0$ then Eq. (24) reduces to

$$\sin \delta' = -x \frac{f^2 + \bar{f}^2 - f\bar{f}}{ff} \frac{f - \bar{f}}{2yg \sin \Delta} \sqrt{\frac{ff}{f^2 + \bar{f}^2 - f\bar{f}}}. \quad (25)$$

which is not zero if $f \neq \bar{f}$, i.e., whenever there are matter effects, so there is a potential CPC/CPV confusion as long as the right-hand side of Eq. (25) has magnitude less than unity. It is possible to have $\delta' = \pi/2$ when $\delta = 0$, i.e., CPC can be confused with maximal CPV (see Fig. 3b).

The ambiguity between parameters with $\delta m_{31}^2 > 0$ and $\delta m_{31}^2 < 0$ occurs only for some values of $\delta$, and does not occur at all if matter effects are large enough (i.e., $L$ and $\theta_{13}$ are large enough) [24,25]. The conditions for the existence of this ambiguity will be discussed further in Sec. III B. Note, however, that the $\text{sgn}(\delta m_{31}^2)$ ambiguity can still confuse different values of $\delta$ and $\theta_{13}$ even for $\Delta = n\pi/2$ (see, e.g., Figs. 3b and 3c), unlike the $(\delta, \theta_{13})$ ambiguity where $\theta_{13}$ is removed from the ambiguity for $\Delta = n\pi/2$.

E. Atmospheric angle degeneracy: $(\theta_{23}, \pi/2 - \theta_{23})$ ambiguity

There is yet another ambiguity in the determination of $\delta$ and $\theta_{13}$, which involves the value of $\theta_{23}$. In practice it is only $\sin^2 2\theta_{23}$ that is determined by a $\nu_\mu$ survival measurement (for now we ignore matter corrections to $\nu_\mu \to \nu_\mu$, which are relatively small for oscillations involving active flavors), so $\theta_{23}$ cannot be distinguished from $\pi/2 - \theta_{23}$. The effect of this degeneracy can be seen by interchanging $\sin \theta_{23}$ and $\cos \theta_{23}$ in Eqs. (11) and (12). For $\theta_{23} \simeq \pi/4$ (the favored solution from atmospheric data) the ambiguity vanishes, but for
FIG. 3. Sgn(δm_{31}^2) ambiguity for L = 730 km with (a) $E_\nu = 3.54$ GeV ($\Delta = \pi/4$), (b) $E_\nu = 1.77$ GeV ($\Delta = \pi/2$), and (c) $E_\nu = 0.885$ GeV ($\Delta = \pi$). The other parameters are $\delta m_{21}^2 = 5 \times 10^{-5}$ eV$^2$, $\sin^2 2\theta_{23} = 1$, and $\sin^2 2\theta_{12} = 0.8$. 
\[ \sin^2 2\theta_{23} \simeq 0.9 \] it can have a sizable effect, since in this case \( \sin^2 \theta_{23} = 0.35 \) and \( \cos^2 \theta_{23} = 0.65 \). Three examples of the \( \theta_{23} \) ambiguity are shown in Fig. 4. The \( \theta_{23} \) ambiguity can also mix CPC and CPV solutions; for example, in Fig. 4a, the prediction for \((P, \bar{P})\) for \((\sin^2 2\theta_{13}, \sin \theta_{23}, \delta) = (0.01, 0.585, 0)\) is identical to that for \((0.00107, 0.811, 4\pi/3)\).

As with the \( \text{sgn}(\delta m_{31}^2) \) ambiguity, the equations for the \( \theta_{23} \) ambiguity are rather messy in the general case. For the special case \( \Delta = (n - \frac{1}{2})\pi \), we have

\[ \sin^2 2\theta'_{13} = \sin^2 2\theta_{13} \tan^2 \theta_{23} + \frac{\alpha^2 g^2 \sin^2 2\theta_{12}}{ff} (1 - \tan^2 \theta_{23}), \tag{26} \]
\[ \sin 2\theta'_{13} \sin \delta' = \sin 2\theta_{13} \sin \delta + \frac{\alpha(g(f - \bar{f}) \sin 2\theta_{12} \cot 2\theta_{23}}{ff} \frac{\sin \Delta}{\sin \Delta}, \tag{27} \]

where \((\delta, \theta_{13})\) are the parameters that give a certain \((P, \bar{P})\) for \(0 < \theta_{23} < \pi/4\) and \((\delta', \theta'_{13})\) are the parameters that give the same \((P, \bar{P})\) for \(\pi/2 - \theta_{23}\). We see that even for \(\Delta = n\pi/2\) the \(\theta_{23}\) ambiguity can mix CPC and CPV solutions, since \(\sin \delta = 0\) does not necessarily imply \(\sin \delta' = 0\). Furthermore, even for \(\Delta = n\pi/2\) the \(\theta_{23}\) ambiguity mixes solutions with different \(\theta_{13}\) (see Eqs. (26) and (27), and Figs. 4b and 4c), unlike the \((\delta, \theta_{13})\) ambiguity where \(\theta_{13}\) is removed from the ambiguity for \(\Delta = n\pi/2\).

Since \(\alpha\) is a small parameter (and possibly even small compared to \(\sin 2\theta_{13}\)), the numerical uncertainty in \(\delta\) due to the \(\theta_{23}\) ambiguity is generally small, of order \(0.07\pi\) or less, when \(\Delta = (n - \frac{1}{2})\pi, L = 2900\) km, \(\delta m_{21}^2 = 10^{-4}\) eV\(^2\), and \(\sin^2 2\theta_{13} = 0.01\). This effect is of order the expected experimental uncertainty in \(\delta\) [16,26]. The size of the \(\delta\) ambiguity decreases with decreasing matter effect (smaller \(L\)) and with decreasing \(\delta m_{21}^2\), so for a wide range of parameters the CPC/CPV confusion from the \(\theta_{23}\) ambiguity is not too severe. On the other hand, the \(\sin^2 2\theta_{13}\) confusion is approximately a factor \(\tan^2 \theta_{23}\) (see Eq. (26)), which lies roughly in the range \(\frac{1}{2}\) to 2 for \(\sin^2 2\theta_{23} \geq 0.9\).

F. Comments on parameter degeneracies

In the preceding three sections we have shown that in principle there can be as much as an eight–fold ambiguity in determining \(\delta\) and \(\theta_{13}\) from \(P(\nu_\mu \rightarrow \nu_e)\) and \(P(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)\) at a single \(L\) and \(E_\nu\), which comes from the presence of three independent two–fold ambiguities: \((\delta, \theta_{13})\), \(\text{sgn}(\delta m_{31}^2)\), and \((\theta_{23}, \pi/2 - \theta_{23})\). For each type of ambiguity there is the possibility of being unable to distinguish between \(CP\) violating and \(CP\) conserving parameters. Measurements at multiple \(L\) and \(E_\nu\) can be used to help discriminate the different degenerate solutions, but that would involve extra detectors or a much longer total running time, and probably reduced statistics for each \((L, E_\nu)\) combination. In the next section we will explore what \(L\) and \(E_\nu\) values do best at resolving these potential degeneracies without resorting to measurements at different \(L\) and/or \(E_\nu\). We then will examine what \(L\) and \(E_\nu\) for a second measurement can remove the remaining degeneracies.

In the Appendix we demonstrate that the analytic expressions are accurate for \(E_\nu > 0.5\) GeV for baselines up to 4000-5000 km. For much lower \(E_\nu\) (as low as 0.05 GeV) they are still accurate at shorter distances \((L \lesssim 350\) km\) if \(\alpha\) and \(\theta_{13}\) are not too large (see the discussion in the Appendix). Therefore we expect that the qualitative aspects of the three ambiguities are unchanged for short \(L\), low \(E_\nu\) experiments such as CERN–Frejus.
FIG. 4. ($\theta_{23}, \pi/2 - \theta_{23}$) ambiguity for $L = 1290$ km with (a) $E_\nu = 2.09$ GeV ($\Delta = 3\pi/4$), (b) $E_\nu = 3.13$ GeV ($\Delta = \pi/2$), and (c) $E_\nu = 1.565$ GeV ($\Delta = \pi$). The other parameters are $\delta m^2_{21} = 5 \times 10^{-5}$ eV$^2$, and $\sin^2 2\theta_{12} = 0.8$. 

$L = 1290$ km, $\delta m^2_{31} = 3 \times 10^{-3}$ eV$^2$, $\delta m^2_{21} = 5 \times 10^{-5}$ eV$^2$
TABLE II. Possible neutrino beam energies $E_{\nu}$ (in GeV) versus baseline (in km) and $\Delta$ that will convert the $(\delta, \theta_{13})$ ambiguity to a simple $(\delta, \pi - \delta)$ ambiguity, for $\delta m_{31}^2 = 3 \times 10^{-3}$ eV$^2$. For other values of $\delta m_{31}^2$, $E_{\nu}$ scales proportionately with $\delta m_{31}^2$. Only values of $E_{\nu} > 0.5$ GeV are considered.

<table>
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<tr>
<th>$\Delta$</th>
<th>300 km</th>
<th>730 km</th>
<th>1290 km</th>
<th>1770 km</th>
<th>2100 km</th>
<th>2600 km</th>
<th>2900 km</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/2$</td>
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<td>1.77</td>
<td>3.13</td>
<td>4.29</td>
<td>5.12</td>
<td>6.34</td>
<td>7.03</td>
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<tr>
<td>$3\pi/2$</td>
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<td>1.43</td>
<td>1.71</td>
<td>2.11</td>
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</tr>
<tr>
<td>$5\pi/2$</td>
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<td>0.86</td>
<td>1.02</td>
<td>1.27</td>
<td>1.41</td>
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</tr>
<tr>
<td>$7\pi/2$</td>
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<td>0.73</td>
<td>0.91</td>
<td>1.00</td>
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</tr>
</tbody>
</table>

III. RESOLVING PARAMETER DEGENERACIES

A. Resolving the $(\delta, \theta_{13})$ ambiguity

As discussed in Secs. II B and II C, the choice $\Delta = n\pi/2$ causes the orbit ellipse in $(P, \bar{P})$ space to collapse to a line and the $(\delta, \theta_{13})$ ambiguity reduces to one involving only $\delta$, i.e., the combination of $P$ and $\bar{P}$ gives a unique value of $\theta_{13}$ (at least for one sign of $\delta m_{31}^2$). Furthermore, since $\delta$ only becomes confused with $\pi - \delta$ (in Case (ii) of Sec. II C) or $2\pi - \delta$ (in Case (iii)), $CP$ conserving solutions never become mixed with $CP$ violating ones. Case (ii) (with $\Delta = (n - 1/2)\pi$) has another advantage in that the $\nu_{\mu} \to \nu_{\tau}$ oscillation is approximately maximal (see Eq. (A3)), which would facilitate a better measurement of $\sin^2 2\theta_{23}$ and $\delta m_{31}^2$. Therefore the choice $\Delta = (n - 1/2)\pi$ is the best for resolving the $(\delta, \theta_{13})$ ambiguity. Some representative beam energies for particular baselines are given in Table II.

B. Resolving the sgn($\delta m_{31}^2$) ambiguity

The parameter degeneracy associated with the sign of $\delta m_{31}^2$ can be overcome if there is a large matter effect that splits $P$ and $\bar{P}$, e.g., if $L$ is sufficiently long and $\theta_{13}$ is not too small [24,25]. To determine the minimum value of $\theta_{13}$ that avoids the sgn($\delta m_{31}^2$) ambiguity, we must first find the region in $(P, \bar{P})$ space covered by each sgn($\delta m_{31}^2$), and then determine the condition on $\theta_{13}$ that ensures the regions for different sgn($\delta m_{31}^2$) do not overlap.

The orbit ellipse for a given sgn($\delta m_{31}^2$) moves as $\theta_{13}$ changes, sweeping out a region in $(P, \bar{P})$ space. All points on each orbit ellipse (for a given $\theta_{13}$) that lie inside the region will overlap an orbit ellipse for a different $\theta_{13}$ (this is what leads to the $(\delta, \theta_{13})$ ambiguity). However, the points on the orbit ellipse that lie on the boundaries of the region do not have a $(\delta, \theta_{13})$ ambiguity, i.e., there are unique values of $\theta_{13}$ and $\delta$ for that point. This implies that for points on the boundary of the region, $x = x'$ and $\delta = \delta'$ in Eqs. (20)–(22). For $\Delta \neq n\pi/2$, the condition becomes $xf \sin(\Delta - \delta) + x\bar{f} \sin(\Delta + \delta) + yg \sin 2\Delta = 0$. Solving for $\delta$ and substituting into Eqs. (9) and (10) we find the coordinates of the $\delta m_{31}^2 > 0$ envelope in $(P, \bar{P})$ space are given by

$$P = x^2 f^2 + y^2 g^2 - \frac{2y^2 g^2 f^2 \sin^2 2\Delta \pm 2yg f \sqrt{z(f \cos 2\Delta - \bar{f})}}{f^2 + f^2 - 2f \bar{f} \cos 2\Delta},$$

(28)
where
\[ z = x^2(f^2 + \bar{f}^2 - 2f\bar{f}\cos 2\Delta) - y^2g^2\sin^2 2\Delta, \tag{29} \]
and \( \bar{P} \) is found by interchanging \( f \leftrightarrow \bar{f} \) and letting \( g \to -g \). For \( \delta m^2_{31} < 0 \), the values of \( P \) and \( \bar{P} \) on the envelope can be found by interchanging \( P \) and \( \bar{P} \). Although the general solution is complicated, for the special case \( \Delta = (n - \frac{1}{2})\pi \) it is not hard to show that the two \( \text{sgn}(\delta m^2_{31}) \) regions do not overlap if
\[ x > \frac{2yg}{f - \bar{f}}. \tag{30} \]

Note that matter effects split \( f \) and \( \bar{f} \), which decreases the minimum value of value of \( x \) (and hence of \( \sin^2 2\theta_{13} \)) needed to avoid any \( \text{sgn}(\delta m^2_{31}) \) ambiguity. Because \( x \propto \sin 2\theta_{13} \) and \( y \propto \delta m^2_{21} \), the corresponding minimum value of \( \sin^2 2\theta_{13} \) increases as the square of \( \delta m^2_{21} \).

The minimum values of \( \sin^2 2\theta_{13}/(\delta m^2_{21})^2 \) are plotted versus \( E_\nu \) for various values of \( L \) in Fig. 5. For \( \Delta = \pi/2 \), the \( \text{sgn}(\delta m^2_{31}) \) ambiguity would be resolved for \( \sin^2 2\theta_{13} > 0.01 (0.04) \) at \( L = 1290 \) km, the distance from Fermilab to Homestake, if \( \delta m^2_{21} = 5 \times 10^{-5} \) (10^{-4}) eV^2. For \( L \simeq 2600 \) km (Brookhaven–Homestake or Fermilab–San Jacinto), \( \text{sgn}(\delta m^2_{31}) \) can be determined for values of \( \sin^2 2\theta_{13} \) as low as 0.002 (0.008) for \( \delta m^2_{21} = 5 \times 10^{-5} \) (10^{-4}) eV^2.

![Figure 5](image)

**FIG. 5.** Minimum value of \( \sin^2 2\theta_{13}/(\delta m^2_{31})^2 \) that avoids the \( \text{sgn}(\delta m^2_{31}) \) ambiguity, plotted versus \( L \) for \( \Delta = \pi/2 \) (solid curve) and \( 3\pi/2 \) (dashed), with \( \delta m^2_{31} = 3 \times 10^{-3} \) eV^2. The corresponding values of \( E_\nu \) are marked on the curves.

Figure 5 shows that \( \Delta = 3\pi/2 \) would be unsatisfactory in distinguishing \( \text{sgn}(\delta m^2_{31}) \); in fact, measurements at \( \Delta = (n - \frac{1}{2})\pi \) do increasingly worse as \( n \) increases, as can be shown using Eq. (13). For \( \Delta = (n - \frac{1}{2})\pi \), we have \( |f/\bar{f}| = (1 + \hat{A})/(1 - \hat{A}) \); since \( \hat{A} \) is proportional to \( E_\nu \), and \( E_\nu \) decreases with \( n \) for fixed \( L \), larger values of \( n \) will have smaller \( \hat{A} \). Thus, the values of \( f \) and \( \bar{f} \) will be closer for larger \( n \), reducing the size of the matter effect (at least as far as splitting \( P \) and \( \bar{P} \) is concerned). For \( \Delta = 3\pi/2 \) and
\( \delta m_{32}^2 = 5 \times 10^{-5} \text{ eV}^2 \), the value of \( \sin^2 2\theta_{13} \) must be greater than about 0.25 which is excluded by CHOOZ [3]. Even the most optimistic case for \( \Delta = 3\pi/2 \) (which occurs for the highest value of \( \delta m_{31}^2 \) \( \simeq 10^{-4} \text{ eV}^2 \) allowed in the LMA region) requires \( \sin^2 2\theta_{13} \gtrsim 0.06 \). Thus, the proposal of Ref. [27] to perform experiments at higher \( n \) suffers from an inability to determine \( \text{sgn}(\delta m_{31}^2) \). Practically speaking, only \( n = 1 \) will provide sufficient discrimination for \( \text{sgn}(\delta m_{31}^2) \) if \( \Delta \) is restricted to the values \( (n - \frac{1}{2})\pi \). We henceforth restrict ourselves to this case.

By combining \( \Delta = \pi/2 \) with a sufficiently long \( L \), the combined four-fold ambiguity involving \( \delta, \theta_{13} \), and \( \text{sgn}(\delta m_{31}^2) \) can be reduced to a simple \((\delta, \pi - \delta)\) ambiguity that in principle determines whether \( CP \) is conserved or violated. Some possibilities are shown in Figs. 6 and 7. As Fig. 6 shows, for \( \delta m_{21}^2 = 5 \times 10^{-5} \text{ eV}^2 \), \( L = 1290 \text{ km} \) is (barely) sufficient to distinguish \( \text{sgn}(\delta m_{31}^2) \) for \( \sin^2 2\theta_{13} \) as low as 0.01. However, for \( \delta m_{21}^2 = 10^{-4} \text{ eV}^2 \), \( L > 2000 \text{ km} \) is needed. In practice, experimental uncertainties and uncertainties on the matter distribution [28] increase the likelihood of having a \( \text{sgn}(\delta m_{31}^2) \) ambiguity, so that a separation of the two regions greater than the size of the experimental uncertainties is required.

C. Resolving the \((\theta_{23}, \pi/2 - \theta_{23})\) ambiguity

Even if \( \Delta \) is chosen to mitigate the effects of the \((\delta, \theta_{13})\) ambiguity, and \( L \) is chosen long enough to eliminate the \( \text{sgn}(\delta m_{31}^2) \) ambiguity, there still remains the \((\theta_{23}, \pi/2 - \theta_{23})\) ambiguity. If \( \theta_{23} \approx \pi/4 \) this ambiguity disappears, and choosing \( L \) and \( E_\nu \) such that \( \Delta = \pi/2 \) leaves a simple \((\delta, \pi - \delta)\) ambiguity. If \( \theta_{23} \) deviates from \( \pi/4 \), then there does not appear to be a judicious choice of a single \( L \) and \( E_\nu \) that can resolve the \( \theta_{23} \) ambiguity.

The problem in resolving the \( \theta_{23} \) ambiguity lies in the fact that in the leading term in \( P(\nu_\mu \rightarrow \nu_e) \) and \( \bar{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) \), \( \sin 2\theta_{13} \) is always paired with \( \sin \theta_{23} \) (see Eqs. (9) and (10)), and so if there are two values of \( \theta_{23} \) derived from the measured value of \( \sin^2 2\theta_{13} \), there will be two corresponding values of \( \sin^2 \theta_{23} \). Since \( \sin^2 2\theta_{23} \) can be as low as 0.9, the two values of \( \sin^2 \theta_{23} \) can be as far apart as 0.35 and 0.65, and therefore the ambiguity in \( \sin^2 \theta_{23} \) can be as large as a factor 1.86 at leading order (Eq. (26) in the limit that \( \alpha \) is small). The next-to-leading term in the probabilities in Eqs. (9) and (10) is proportional to \( \sin 2\theta_{23} \), and therefore cannot resolve the ambiguity. The last term in \( P(\nu_\mu \rightarrow \nu_e) \) is proportional to \( \cos^2 \theta_{23} \), so that the relative weighting of the last term compared to the leading term is affected by the value of \( \sin \theta_{23} \). However, the last term is suppressed by \( \alpha^2 \), and is generally much smaller than the leading term (at least for \( \sin^2 2\theta_{13} \geq 0.01 \), the approximate region where superbeam experiments will be able to probe). Hence, even measurements at a second \( L \) and \( E_\nu \) would likely be unable to resolve the \( \theta_{23} \) ambiguity if it exists (i.e., \( \theta_{23} \) not close to \( \pi/4 \)).

If one could also measure \( P(\nu_e \rightarrow \nu_\tau) \) (see Appendix A for an approximate analytic expression), then a comparison with \( P(\nu_e \rightarrow \nu_\mu) \) should determine whether \( \theta_{23} \) is above or below \( \pi/4 \); the leading term in \( P(\nu_e \rightarrow \nu_\tau) \) can be obtained from the leading term in \( P(\nu_\mu \rightarrow \nu_e) \) by the replacement of \( \sin \theta_{23} \) by \( \cos \theta_{23} \). A \( \nu_e \rightarrow \nu_\tau \) measurement could be done in a neutrino factory; in fact, a neutrino factory may be the only practical way to resolve the \( \theta_{23} \) ambiguity, if it exists. A neutrino factory experiment also provides energy spectrum information that could be helpful in resolving parameter ambiguities [23,29].
FIG. 6. Resolution of combined $(\delta, \theta_{13})$ and $\text{sgn}(\delta m^2_{31})$ ambiguities when $\Delta = \pi/2$, for (a) $L = 1290$ km, (b) $L = 1770$ km, and (c) $L = 2600$ km, with $|\delta m^2_{31}| = 3 \times 10^{-3} \text{ eV}^2$, $|\delta m^2_{21}| = 5 \times 10^{-5} \text{ eV}^2$, and $\sin^22\theta_{23} = 1$. 

0.001 0.01 0.1
$\sin^22\theta_{13} = 0.005$

$\delta m^2 = -3 \times 10^{-3} \text{ eV}^2$

$\delta = 0, \pi, 3\pi/2$

$\delta = 3\pi/2, 0, \pi$

$\sin^22\theta_{13} = 0.01$

$P(\nu_{\mu} \rightarrow \nu_e)$

0.001 0.01 0.1
$P(\nu_{\mu} \rightarrow \nu_e)$

$\delta m^2 = -3 \times 10^{-3} \text{ eV}^2$

$\delta = 0, \pi, 3\pi/2$

$\delta = 3\pi/2, 0, \pi$

$\sin^22\theta_{13} = 0.01$

$P(\nu_{\mu} \rightarrow \nu_e)$
FIG. 7. Same as Fig. 6, except for $|\delta m_{31}^2| = 10^{-3} \text{ eV}^2$. 

(a) $L = 1290 \text{ km}$
$E_\nu = 3.13 \text{ GeV}$

(b) $L = 1770 \text{ km}$
$E_\nu = 4.29 \text{ GeV}$

(c) $L = 2600 \text{ km}$
$E_\nu = 6.34 \text{ GeV}$
D. Measurements at a second $L$ and/or $E_\nu$

As we have demonstrated, measurements at a single $L$ and $E_\nu$ cannot resolve all parameter ambiguities. A second experiment at a different $L$ and/or $E_\nu$, with a different value of $\Delta$, is required for this purpose. The best sets of $L$ and $E_\nu$ are those that are complementary, i.e., the second experiment should provide the clearest distinction between the parameter ambiguities of the first experiment. In this section we discuss three possible scenarios, each with measurements at two $L$ and $E_\nu$ combinations.

1. Scenario A

In this scenario, the first measurement would be done at $\Delta_1 = \pi/2$ (with $L/E_\nu$ given by Eq. (19)). As discussed earlier, this choice isolates the $\sin \delta$ term, removes $\theta_{13}$ from the $(\delta, \theta_{13})$ ambiguity, and the remaining $(\delta, \pi - \delta)$ ambiguity does not mix CPC and CPV solutions. These $L/E_\nu$ values also give a large $\nu_\mu$ disappearance, which facilitates the precision measurement of $\delta m^2_{31}$ and $\sin^2 2\theta_{23}$. The baseline $L$ should be large enough to avoid the $\text{sgn}(\delta m^2_{31})$ ambiguity ($L \gtrsim 2000$ km assures this for $\sin^2 2\theta_{13} \gtrsim 0.01$). Representative values of $L$ and $E_\nu$ are given in Table II. Measuring $P(\nu_\mu \to \nu_e)$ and $\bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e)$ at one such $L$ and $E_\nu$ should determine $\text{sgn}(\delta m^2_{31})$, $\theta_{13}$ (modulo the $\theta_{23}$ ambiguity, if present), and whether or not $CP$ is violated (as discussed in Sec. II E, the existence of a $\theta_{23}$ ambiguity will not give a large amount of $\text{CPC}/\text{CPV}$ confusion).

The second measurement should be one that best resolves the $(\delta, \pi - \delta)$ ambiguity. In principle, $\Delta_2 = \pi$, which eliminates the $\sin \delta$ terms in the probabilities and leaves only $\cos \delta$ terms, gives the maximal separation of $\delta$ and $\pi - \delta$. Thus, the first measurement gives $\sin \delta$, the second gives $\cos \delta$, from which the value of $\delta$ may be inferred. Furthermore, if $\theta_{13}$ is determined from the first measurement, then both $P$ and $\bar{P}$ would not have to be measured in the second measurement; one is sufficient to determine $\delta$. Whether one used neutrinos or antineutrinos in the second measurement would be determined by which gave the larger event rate, taking into account neutrino fluxes, cross sections, and oscillation probabilities. If $\delta m^2_{31} > 0$, then neutrinos would be best for the second measurement, due to the larger flux and cross section; for $\delta m^2_{31} < 0$, antineutrinos may be the better choice if the matter enhancement is enough to overcome the lower flux and cross section for antineutrinos. If both the first and second measurements are done at the same $L$, then $\Delta_2 = \pi$ means that the appropriate energy in the second experiment is $E_2 = E_1/2$.

In practice, there are other values $\Delta_2$ that are not close to $\pi/2$ that could potentially work for the second measurement. The optimal $\Delta_2$ also depends on the particular values of $f$, $\bar{f}$ and $g$ at the various $L$ and $E_\nu$, as well as on neutrino parameters that are currently unknown ($\delta m^2_{21}$, $\theta_{12}$, and $\theta_{13}$). We do not pursue the optimization here.

2. Scenario B

If the first measurement is done at an $L$ that is not large enough to resolve the $\text{sgn}(\delta m^2_{31})$ ambiguity, then the second measurement must be tailored to both determine $\text{sgn}(\delta m^2_{31})$ and resolve the $(\delta, \pi - \delta)$ ambiguity, i.e., it must break a four-fold degeneracy. As discussed
TABLE III. Possible sets of neutrino beam energies and baselines that will resolve the four–fold parameter ambiguity when the measurements are done at shorter \( L \) (such that the \( \text{sgn}(\delta m_{31}^2) \) ambiguity is not resolved in either experiment).

<table>
<thead>
<tr>
<th>Fixed ( L ) (km)</th>
<th>( E_1 ) (GeV)</th>
<th>( E_2 ) (GeV)</th>
</tr>
</thead>
<tbody>
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<td>300</td>
<td>0.73</td>
<td>0.355, 0.375</td>
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<tr>
<td>730</td>
<td>1.77</td>
<td>0.835, 0.940</td>
</tr>
<tr>
<td>Fixed ( E ) (GeV)</td>
<td>( L_1 ) (km)</td>
<td>( L_2 ) (km)</td>
</tr>
<tr>
<td>0.73</td>
<td>300</td>
<td>575, 630</td>
</tr>
<tr>
<td>1.77</td>
<td>730</td>
<td>1295, 1700</td>
</tr>
</tbody>
</table>

above, the choice \( \Delta_2 = \pi \) determines \( \delta \), but it can be shown that at shorter \( L \) an approximate degeneracy with parameters of the opposite \( \text{sgn}(\delta m_{31}^2) \) remains (e.g., see Fig. 3c). Another example is shown in Fig. 8a, where the near degeneracy of parameters with the opposite \( \text{sgn}(\delta m_{31}^2) \) remains for some values of \( \delta \) (while the crosses in Fig. 8a are well–separated in the second measurement, the boxes are not). However, at \( \Delta_2 = \pi/(1 \pm \hat{A}) \), either \( \bar{f} \) or \( f \) vanishes (depending on the sign of \( \delta m_{31}^2 \)), and the four ambiguous solutions occupy four separate regions in \((P, \bar{P})\) space, as shown in Fig. 8b. Although the four regions in Fig. 8b overlap somewhat, when the point of one degenerate solution is in the overlap region the points of the other three degenerate solutions are not (the crosses and boxes are always well–separated). Thus the four–fold ambiguity involving \((\delta, \pi - \delta)\) and \(\text{sgn}(\delta m_{31}^2)\) will always be resolved. A disadvantage of Scenario B is that because either \( f \) or \( \bar{f} \) is zero in the second measurement, \( P \) and \( \bar{P} \) tend to be smaller, so that event rates may be somewhat lower than for other values of \( \Delta \).

Some examples are given in Table III. For instance, if the first baseline is \( L_1 = 730 \) km (Fermilab to Soudan), then \( \Delta_1 = \pi/2 \) for \( E = 1.77 \) GeV, and two possibilities for a second experiment with the same beam energy are \( L_2 = 1295 \) and 1700 km, which serendipitously are very close to the distances from Fermilab to Homestake and from Fermilab to Carlsbad (or Brookhaven to Soudan).

In practice, narrow band beams are not monoenergetic. However, values of \( \Delta_2 \) close to \( \pi/(1 \pm \hat{A}) \) also give reasonably good separation of the ambiguities, as long as \( \Delta_2 \) is not close to \( \pi \). If the fractional beam spread is more than \( |\hat{A}| \), a slightly different average value of \( \Delta_2 \) might be preferable, to ensure that no significant part of the beam has \( \Delta_2 \) too close to \( \pi \).

Fig. 9 summarizes the possibilities for Scenarios A and B, showing \( E_\nu \) versus \( L \) for the first measurement done at \( \Delta_1 = \pi/2 \) (solid curve) and a possible second measurement at \( \Delta_2 = \pi \) (Scenario A, dotted curve) or \( \pi/(1 \pm \hat{A}) \) (Scenario B, dashed curves).

3. Scenario C

This scenario uses the fact that the probabilities are insensitive to the parameters of the \( \delta m_{21}^2 \) scale at \( L \approx 7600 \) km, as noted in Sec. II B. If the first measurement of \( P \) and \( \bar{P} \) were done at \( L \approx 7600 \) km, \( \theta_{13} \) would be determined (modulo the \( \theta_{23} \) ambiguity), and because the distance is large enough \( \text{sgn}(\delta m_{31}^2) \) would also be determined from the large matter effect. A second measurement of \( P \) and \( \bar{P} \) could then be done at an \( L \) and \( E_\nu \) such that \( \Delta = (2n - 1)\pi/4 \), which gives the maximum “fatness” of the orbit ellipse [22] (see Sec. II B), which in turn should best distinguish different values of \( \delta \). One disadvantage of
FIG. 8. Values of $P$ and $\bar{P}$ in a second measurement when there is a four-fold degeneracy in the first measurement for (a) $E_2 = 1.18$ GeV ($\Delta_2 = 3\pi/4$) and (b) $E_2 = 0.94$ GeV ($\Delta_2 = \pi/(1 + \hat{A}) = 0.94\pi$), with $L_1 = L_2 = 730$ km and $E_1 = 1.77$ GeV ($\Delta_1 = \pi/2$). Each curve (solid, dashed, dotted, dash–dotted) represents one of the four solutions that are degenerate. Points labelled by the same symbol (crosses or boxes) correspond to solutions that are degenerate with each other in the measurement at $L_1$ and $E_1$.

Scenarios B and C compared to Scenario A is that both $P$ and $\bar{P}$ must be determined in both measurements.

4. Discussion of scenarios

Although the three scenarios discussed above are not necessarily the only solutions to the ambiguities, in each case one measurement is chosen to eliminate one or more of the
FIG. 9. Values of $L$ and $E_\nu$ for a first measurement at $\Delta_1 = \pi/2$ (solid curve) and a second measurement at $\Delta_2 = \pi$ (Scenario A, dotted) or $\pi/(1 \pm \hat{A})$ (Scenario B, dashed), which breaks the parameter degeneracy in each case.

parameters from the ambiguities, leaving the second measurement to resolve only the remaining ambiguities. In this sense, they are cleaner measurements. Scenario A would appear to be more favorable, since in principle the first measurement alone could determine $\sin \delta$, $\text{sgn}(\delta m_{31}^2)$, and $\theta_{13}$ (modulo the $\theta_{23}$ ambiguity), and thus determine whether or not $CP$ is violated. Also, as discussed in Sec. II E, even if there is a $\theta_{23}$ ambiguity, the magnitude of the $CPC/CPV$ confusion appears to be relatively small for the usual range of neutrino parameters considered.

5. Implication for JHF experiments

The proposed SuperJHF–HyperKamiokande experiment [16] satisfies the requirements for the first experiment of Scenario B. The plan is to have a neutrino energy such that $\Delta$ is at the first peak of the oscillation for $L = 300$ km; if $\Delta$ is not exactly on the peak (e.g., if $\delta m_{31}^2 = 3 \times 10^{-3}$ eV$^2$), a long narrow ellipse results instead of a straight line (see Fig. 10). Because the distance is relatively short, the $\text{sgn}(\delta m_{31}^2)$ ambiguity is not likely to be resolved since there is considerable overlap of the two $\text{sgn}(\delta m_{31}^2)$ ellipses (see Fig. 10b). For example, for $\sin^2 2\theta_{13} = 0.01$ the point for $\delta = 0$ with $\delta m_{31}^2 > 0$ is nearly the same as the point for $\delta = 1.18\pi$ for $\delta m_{31}^2 < 0$ (Eq. 25, which measures the size of the $CPC/CPV$ confusion for the $\text{sgn}(\delta m_{31}^2)$ ambiguity, gives the about the same numerical result). The expected 90% C.L. uncertainty in $\delta$ in this case is about $0.07\pi$ near $\delta = 0$ [16], so we see that the $\text{sgn}(\delta m_{31}^2)$ ambiguity caused by the matter effect would seriously impede a proper measurement of $\delta$, although there is the possibility that the SuperJHF–HyperK experiment might measure a point $(P, \bar{P})$ that was sufficiently outside the overlap region,
thereby determining \(\text{sgn}(\delta m_{31}^2)\) [22]. A possible \((\theta_{23}, \pi/2 - \theta_{23})\) ambiguity also remains, which could lead to a corresponding ambiguity in \(\theta_{13}\), as shown in Fig. 10c.

Even though JHF may not sit exactly on the peak of the oscillation (i.e., \(\Delta = \pi/2\)), Fig. 10 shows that the three ambiguities discussed here are present. Also, Fig. 10a shows that the ambiguity in \(\theta_{13}\) is relatively small (of order 10% or less) if \(\Delta\) is close, but not exactly equal, to \(\pi/2\). Thus as long as \(L/E\nu\) is chosen so that the oscillation is close to the first peak, we expect that the scenarios discussed here for determining the neutrino mass and mixing parameters will be valid.

### IV. SUMMARY

There is an eight-fold, \((\delta, \theta_{13})\)-\(\text{sgn}(\delta m_{31}^2)-(\theta_{23}, \pi/2 - \theta_{23})\) degeneracy affecting the neutrino mixing matrix determined in long-baseline neutrino oscillation experiments. If \(\sin^2 2\theta_{23}\) is almost unity as is favored by current Super-K and K2K data, this is reduced to a four-fold ambiguity. To break this four-fold ambiguity to a simple \((\delta, \pi - \delta)\) ambiguity which does not interfere with a determination of whether or not \(CP\) is violated in the neutrino sector, we find that an experiment should be performed at the first oscillation maximum corresponding to \(\Delta = \pi/2\) and at a baseline of at least about 1300–2000 km, depending on the value of \(\delta m_{21}^2\). Representative values of \(L\) and \(E\nu\) that yield \(\Delta = \pi/2\) are given in Table II.

The obvious advantages of choosing \(\Delta = \pi/2\) are that \(\nu_\mu \rightarrow \nu_\tau\) transitions are nearly maximal even when matter effects are accounted for and the \(\nu_\mu \rightarrow \nu_\tau\) oscillation (which has small matter effects) is maximal, allowing a precise measurement of \(\sin^2 2\theta_{23}\) and \(\delta m_{31}^2\). By choosing \(\Delta = \pi/2\), the \((\delta, \theta_{13})\) degeneracy represented by the \(P-\bar{P}\) ellipse collapses to a line leaving a \((\delta, \pi - \delta)\) ambiguity (see Fig. 2) which unambiguously determines whether or not \(CP\) is violated. The central reason for the choice of the first oscillation peak over other peaks is that practically speaking, the \(\text{sgn}(\delta m_{31}^2)\) ambiguity can be resolved only for this peak (see Fig. 5). The other peaks succeed in eliminating this ambiguity only for the smallest values of \(\delta m_{21}^2\) in the LMA region and for \(\sin^2 2\theta_{13}\) close to the CHOOZ bound. As shown in Figs. 6-7 to remove this ambiguity simultaneously with the \((\delta, \theta_{13})\) ambiguity requires that the baseline be at least 1300 km for \(\sin^2 2\theta_{13} > 0.01\) and \(\delta m_{21}^2 = 5 \times 10^{-5}\) eV\(^2\). For lower values of \(\sin^2 2\theta_{13}\) and/or higher values of \(\delta m_{21}^2\), longer baselines than this are needed.

The exciting aspect of an experiment at \(\Delta = \pi/2\) and a sufficiently long baseline is that all degeneracies other than the \((\theta_{23}, \pi/2 - \theta_{23})\) degeneracy can be broken to a harmless \((\delta, \pi - \delta)\) ambiguity with only a single baseline and energy. The remaining \((\delta, \pi - \delta)\) ambiguity can be removed by making a second measurement at \(\Delta = \pi\) which leaves only \(\cos \delta\) terms in the probabilities and provides the maximal separation between \(\delta\) and \(\pi - \delta\). The \((\theta_{23}, \pi/2 - \theta_{23})\) degeneracy cannot be eliminated even with measurements at a second baseline and energy because in the leading term in \(P(\nu_\mu \rightarrow \nu_\tau)\) and \(\bar{P}(\nu_\mu \rightarrow \bar{\nu}_\tau)\), \(\sin 2\theta_{13}\) is paired with \(\sin \theta_{23}\) (see Eqs. (9) and (10)). Fortunately, the mixing of the \(CPC\) and \(CPV\) solutions arising from this degeneracy are of order or smaller than the experimental uncertainty in \(\delta\), thereby making it less severe. Only a neutrino factory, which offers the unique ability to compare \(P(\nu_\mu \rightarrow \nu_\mu)\) and \(P(\nu_\tau \rightarrow \nu_\tau)\), can disentangle \(\sin 2\theta_{13}\) from \(\sin \theta_{23}\) and find whether \(\theta_{23}\) is less than or greater than \(\pi/4\).

If it is not possible to have an experiment with \(L\) sufficiently large to find \(\text{sgn}(\delta m_{31}^2)\), a second experiment is necessary to simultaneously resolve the \((\delta, \pi - \delta)\) ambiguity and
FIG. 10. Examples of the three types of ambiguities for the proposed SuperJHF–HyperK experiment [16] with $L = 300$ km and $E_{\nu} = 0.7$ GeV: (a) ($\delta, \theta_{13}$) ambiguity, (b) $\text{sgn}(\delta m_{31}^2)$ ambiguity, and (c) ($\theta_{23}, \pi/2 - \theta_{23}$) ambiguity. In each case $\delta m_{31}^2 = 5 \times 10^{-5}$ eV$^2$, $\sin^2 2\theta_{23} = 1$, and $\sin^2 2\theta_{12} = 0.8$, unless otherwise stated in the figure. The circle in (b) indicates the size of the expected experimental uncertainties [16].
determine \(\text{sgn}(\delta m_{31}^2)\). One possibility is to choose \(\Delta = \pi\), but as suggested by Fig. 3c, a \(\text{sgn}(\delta m_{31}^2)\) ambiguity may still remain when experimental errors are included. If \(\Delta = \pi/(1 \pm \hat{A})\), either \(f\) or \(\bar{f}\) vanishes, depending on \(\text{sgn}(\delta m_{31}^2)\), and the four-fold degeneracy breaks into four separate regions as in Fig. 8b. See Table III for some examples of how this scenario can be implemented. The proposed SuperJHF–HyperK experiment would satisfy the requirements for the first measurement of this type; it has the limitation of a possible \(\text{sgn}(\delta m_{31}^2)\) confusion that leads to an ambiguity in the value of \(\delta\), which may compromise its ability to unambiguously establish CP violation.

In Fig. 9 we summarize the baselines and energies for two measurements, one at \(\Delta_1 = \pi/2\) and another at either \(\Delta_2 = \pi\) (if the first measurement can determine \(\text{sgn}(\delta m_{31}^2)\) and only the \((\delta, \pi - \delta)\) ambiguity needs resolution) or \(\Delta_2 = \pi/(1 \pm \hat{A})\) (if the first measurement can not be performed at a long enough baseline and the four-fold degeneracy \((\delta, \pi - \delta)\text{-}\text{sgn}(\delta m_{31}^2)\) needs to be broken). Another possibility is to have one measurement at \(L \simeq 7600\) km with \(\hat{A}\Delta_1 = \pi\) and a second measurement with \(\Delta_2 \simeq (2n - 1)\pi/4\). If K2K, MINOS, ICARUS and OPERA find that \(\theta_{23}\) is not very close to \(\pi/4\), a neutrino factory will be needed to resolve the \((\theta_{23}, \pi/2 - \theta_{23})\) ambiguity.

In our analysis we have assumed that \(\delta m_{31}^2\) and \(\sin^2 2\theta_{13}\) are known when the experiments described here are done. In fact, they will likely be determined only to 10% or so. Once these uncertainties are included the minimum value of \(L\) required to resolve the \(\text{sgn}(\delta m_{31}^2)\) ambiguity, e.g., in Scenario A, could be slightly longer than indicated here. Also, because \(\delta m_{31}^2\) is not precisely known, the average neutrino energy will not necessarily be exactly at the peak defined by \(\Delta = \pi/2\). However, as our analysis of the proposed SuperJHF–HyperK experiment shows, only minimal uncertainties in \(\delta\) and \(\sin^2 2\theta_{13}\) are introduced by these factors, and the three principal ambiguities discussed in this paper will be qualitatively unchanged.

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**APPENDIX A**

We provide a complete set of analytic expressions for the off-diagonal probabilities that are valid in the regime \(|\hat{A}| > |\alpha|\), which roughly translates to \(E_\nu > 0.5\) GeV. The diagonal probabilities can easily be found from them. The off-diagonal probabilities are for a normal hierarchy (in addition to Eqs. (9) and (10))

\[
P(\nu_e \rightarrow \nu_\tau) = \cot^2 \theta_{23} x^2 f^2 - 2xyfg(\cos \delta \cos \Delta + \sin \delta \sin \Delta) + \tan^2 \theta_{23}y^2 g^2, \quad (A1)
\]

\[
\bar{P}(\bar{\nu}_e \rightarrow \bar{\nu}_\tau) = \cot^2 \theta_{23} x^2 f^2 - 2xy\bar{g}(\cos \delta \cos \Delta - \sin \delta \sin \Delta) + \tan^2 \theta_{23}y^2 g^2, \quad (A2)
\]

and
\[ P(\nu_\mu \to \nu_\tau) = \sin^2 2\theta_{23} \sin^2 \Delta \\
+ \alpha \sin 2\theta_{23} \sin 2\Delta \left( \frac{\hat{A}}{1 - \hat{A}} \sin \theta_{13} \sin 2\theta_{12} \cos 2\theta_{23} \sin \Delta - \Delta \cos^2 \theta_{12} \sin 2\theta_{23} \right). \tag{A3} \]

For \( \bar{P}(\bar{\nu}_\mu \to \bar{\nu}_\tau) \), replace \( \hat{A} \) by \( -\hat{A} \) in Eq. (A3). Note that \( P(\nu_\mu \to \nu_\tau) \) is independent of \( \delta \) to \( O(\alpha) \). To obtain the probabilities for an inverted hierarchy, the transformations \( \hat{A} \to -\hat{A} \), \( \alpha \to -\alpha \) and \( \Delta \to -\Delta \) must be made (implying \( f \leftrightarrow -f \) and \( g \to -g \) in Eqs. (A1-A2), and for the \( T \)-reversed channels the sign of the sin \( \delta \) term must be changed.

We now compare the results of the analytic expressions with the numerical integration of the evolution equations of neutrinos through the Earth. We integrate the equations along a neutrino path using a Runge-Kutta method. The step size at each point along the path is 0.1% of the shortest oscillation wavelength given by the scales \( \delta m^2_{21} \) and \( A \). We account for the dependence of the density on depth by using the Preliminary Reference Earth Model (PREM) [30]. To calculate the analytic probability, we use the average value of the electron density along the neutrino path. We provide some values in Table IV for the reader’s use; they are not indicative of the precision with which the electron density is known. We include subleading \( \theta_{13} \) effects, which however are not relevant for \( \sin^2 2\theta_{13} \) of \( O(0.01) \) or smaller. They are of importance at \( \theta_{13} \) for which the CHOOZ limit \( \sin^2 2\theta_{13} < 0.1 \) (at 95\% C.L.) is saturated. The parameters chosen to make this comparison are \( \delta m^2_{21} = 3.5 \times 10^{-3} \, \text{eV}^2 \), \( \delta m^2_{31} = 5 \times 10^{-5} \, \text{eV}^2 \), \( \theta_{23} = \pi/4 \), \( \theta_{12} = \pi/6 \), \( \sin^2 2\theta_{13} = 0.01 \) and \( E_\nu = 5 \, \text{GeV} \). Thus, the ensuing comparison is not affected by dropping subleading \( \theta_{13} \) effects. A normal mass hierarchy (\( \delta m^2_{21} > 0 \)) is assumed. We will comment on the comparison involving an inverted mass hierarchy.

Figure 11 shows \( P(\nu_\mu \to \nu_e) \) and \( \bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e) \) versus distance for \( \delta = 0, \pi/2, \pi, \) and \( 7\pi/4 \). The agreement between the analytic formulae (solid lines) and the numerical results (dashed lines) is excellent for distances up to about 4000 km. Beyond that, the overlap between the lines degrades and for \( L \gtrsim 5000 \) km, the analytic equation completely breaks down. The analytic expression for \( P(\bar{\nu}_\mu \to \bar{\nu}_e) \) works for much longer distances than that for \( P(\nu_\mu \to \nu_e) \). Analogously, for the inverted mass hierarchy, \( P(\nu_\mu \to \nu_e) \) is valid to longer distances than \( \bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e) \).

Reference [18] claims good agreement between the analytic and numerical results for \( L \) even larger than 10000 km when a constant density is assumed for the Earth’s density profile. The use of a realistic density profile as in the PREM model shows that the agreement deteriorates at much smaller distances.

For the sake of completeness we display the corresponding comparisons for \( P(\nu_e \to \nu_\tau) \), \( P(\bar{\nu}_e \to \bar{\nu}_\tau) \) and \( P(\nu_\mu \to \nu_\tau) \) in Figs. 12 and 13. The parameter values chosen are the same as for Fig. 11. The range of validity of Eqs. (A1) and (A2) is the same as for Eqs. (9) and (10). However, Eq. (A3) agrees almost exactly with the numerical result for the entire range considered. This is because matter effects are very small in comparison to the leading contribution. For the same reason, \( \bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e) \) is almost identical to \( P(\nu_\mu \to \nu_\tau) \).

In Fig. 14, we show how well the analytic probabilities \( P(\nu_\mu \to \nu_e) \) and \( \bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e) \) agree with the numerical integration for \( L = 2900 \) km (the longest baseline emphasized in this work) as a function of neutrino energy. The oscillation parameters used are the same as for Fig. 11. The precision is remarkable for the spectrum of energies of interest. For shorter baselines, the agreement gets even better.
We now make some cautionary remarks. Our comparisons were made for \( \alpha = 0.0143 \), the parameter in which the series was expanded, and \( \sin^2 2\theta_{13} = 0.01 \) which is assumed to be no greater than of \( \mathcal{O}(\alpha) \). These values are motivated by the existing reactor bounds and global fits to the atmospheric and solar data. However, as either of these parameters gets larger, the agreement between the analytic equations and the numerical results deteriorates at long baselines even if subleading \( \theta_{13} \) effects are included. Conversely, the agreement improves with smaller values of \( \alpha \) and \( \theta_{13} \). As a rule of thumb, we recommend that the constant density approximation to the probabilities be used only for distances less than 4000 km. As can be seen from Fig. 15, for \( L < 4000 \text{ km} \), the density profile is nearly constant for most of the neutrino path, thereby satisfying the implicit assumption (of a constant density profile) under which analytic probabilities are valid.

We have stated that the analytic expressions are accurate for \( E_\nu > 0.5 \text{ GeV} \) for baselines of 4000-5000 km. This robust bound can be relaxed for \( L \lesssim 350 \text{ km} \) to \( E_\nu \) as low as 0.05 GeV. However, for such low values of \( E_\nu \), the sensitivity of the analytic probabilities to \( \alpha \) and \( \theta_{13} \) is high and care must be taken in their use. For example, a comparison with a numerical integration is desirable if \( \alpha \) and \( \theta_{13} \) are relatively large.
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**TABLE IV.** The average values of the electron density $\langle N_e \rangle$ at baselines for which the analytic approximations of the probabilities accurately represent the numerical integration of the evolution equations.
FIG. 11. $P(\nu_\mu \to \nu_e)$ and $\bar{P}(\bar{\nu}_\mu \to \bar{\nu}_e)$ versus $L$ for $\delta = 0, \pi/2, \pi, 7\pi/4$. The agreement between the analytic formulae (solid lines) and the numerical results (dashed lines) is excellent for distances up to about 4000 km. The parameters chosen to make this comparison are $\delta m^2_{31} = 3.5 \times 10^{-3}\text{eV}^2$, $\delta m^2_{21} = 5 \times 10^{-5}\text{eV}^2$, $\theta_{23} = \pi/4$, $\theta_{12} = \pi/6$, $\sin^2 2\theta_{13} = 0.01$ and $E_\nu = 5\text{ GeV}$. 

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FIG. 12. $P(\nu_e \to \nu_\tau)$ and $\bar{P}(\bar{\nu}_e \to \bar{\nu}_\tau)$ versus $L$ for the same set of parameters as in Fig. 11.
FIG. 13. $P(\nu_\mu \rightarrow \nu_\tau)$ versus $L$. The analytic expression (solid line) and the numerical calculation (dashed line) agree almost exactly for the entire range in $L$. $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_\tau)$ is almost identical to $P(\nu_\mu \rightarrow \nu_\tau)$ because of insignificant matter effects.
FIG. 14. $P(\nu_\mu \rightarrow \nu_e)$ and $\bar{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)$ vs $E_\nu$ for $L = 2900$ km. The oscillation parameters are the same as in Fig. 11. The solid lines (analytic equations) and dashed lines (numerical evaluation) are almost undistinguishable.
FIG. 15. Density profiles along a selection of chords of length $L$ passing through the Earth; the horizontal axis is the fraction of the total path length.
REFERENCES