Massive quark propagator and competition between chiral and diquark condensate

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The Green-function approach has been extended to the moderate baryon density region in the framework of an extended Nambu–Jona-Lasinio model, and the thermodynamic potential with both chiral and diquark condensates has been evaluated by using the massive quark propagator. The phase structure along the chemical potential direction has been investigated and the strong competition between the chiral and diquark condensate has been analyzed by investigating the influence of the diquark condensate on the sharp Fermi surface. The influence of the diquark condensate on the quark properties has been investigated, even though the quarks in the color breaking phase are very different from that in the chiral breaking phase, the difference between quarks in different colors is very small.
QCD phase transitions along the baryon density direction has attracted much attention recently since it was found that the color-superconducting gap can be of the order of 100MeV [1] and [2], which is two orders larger than early perturbative estimations [3]. However, till now, there is no uniform framework discussing the phase structure in the wide region of chemical potential $\mu$ from about 300MeV to $10^8$MeV.

In the idealized case at asymptotically high baryon densities, the color superconductivity with two massless flavors and the color-flavor-locking (CFL) phase with three degenerate massless quarks have been widely discussed from first principle QCD calculations, see [4] and references therein. Usually, the diagrammatic methods are used in the asymptotic densities. The Green-function of the eight-component field and the gap equation were discussed in details in [5] [6]. Neither current quark mass nor chiral condensate are necessary to be considered because they can be neglected compared with the very high Fermi surface. At less-than-asymptotic densities, the corrections of non-zero quark mass to the pure CFL phase can be treated perturbatively by expanding the current quark mass around the chiral limit [7] [8].

For physical applications, we are more interested in the moderate baryon density region, which may be related to the neutron stars and, in very optimistic cases even to heavy-ion collisions. Usually, effective models such as instanton, as well as Nambu-Jona-Lasinio (NJL) model, are used. The model parameters are fixed in the QCD vacuum. In this region, the usual way is to use the variational methods working out the gap equations from the thermodynamic potential [9]-[22], except for [11] and [15] in the instanton model, where the quark propagator was evaluated explicitly, but the form is complicated.

One of our main aims in this paper is to apply the Green-function approach in the moderate baryon density region. To work out the phase structure from hadron phase to the color superconducting phase, one should deal with the chiral condensate and diquark condensate simultaneously. Because the chiral condensation contributes a dynamic quark
mass, it is not reasonable any more in this density region to treat the quark mass term perturbatively, like in [8]. By using the energy projectors for massive quark, we will evaluate the full massive quark propagator, which will help us deal with the chiral and diquark condensate simultaneously.

In the normal phase, the quarks in different colors are degenerate, while in the color breaking phase, it is natural to assume that the quarks involved in the diquark condensate are different from that not participating in the diquark condensate. In [11] and [15], different masses for the quarks which participate and not participate in the diquark condensate are introduced. However, we will see it difficult to get the mass expression for the quarks participating in the diquark condensate, because the particles and holes mix with each other and the elementary excitations are quasi-particles and quasi-holes near the Fermi surface. In our case, the difference between quarks in different colors has been reflected by their propagators, we read the difference through calculating the quarks’ chiral condensate, but not trying to work out their masses. In the chiral limit, the chiral condensate disappears entirely in the color superconducting phase, it is not possible to investigate the influence of color breaking on quarks in different colors, so we will keep the current quark mass finite in this paper.

In the moderate baryon density region, people are interested in the question whether there exists a region where both the chiral symmetry and color symmetry are broken. The coexistence of the chiral condensate and diquark condensate has been discussed in several papers [9] - [13]. Even though it is not excluded that there would be a region where the two condensates coexist, the model calculations in the chiral limit show a strong competition between the two condensates, i.e., where one condensate is nonzero the other vanishes. In [14], the author pointed out that the underlying mechanism is Anderson theorem and the two gap equations decouple in color superconductivity phase. One can calculate the diquark condensate neglecting the influence of the chiral condensate, and as soon as diquark condensate is formed the contribution of the chiral condensate to thermodynamic quantities becomes strongly suppressed. In this paper, we will explain the competition mechanism
from another point of view by analyzing the influence of the diquark condensate on the Fermi surface. In the mean-field approximation of the NJL model, the thermal system of the constituent quarks is a nearly ideal Fermi gas, and there is a sharp Fermi surface. The chiral symmetry begins to restore when the chemical potential is larger than the quark mass in the vacuum. When a diquark condensate is formed, the Cooper pair extends the Fermi surface, which induces the chiral symmetry restoring at a smaller chemical potential. The stronger the coupling constant in the diquark channel, the larger the diquark condensate and the smoother the Fermi surface.

In the following, we briefly introduce the extended NJL model in Sec. II, then in Sec III, we evaluate the thermodynamic potential using the massive quark propagator. In Sec. IV, we get the gap equations and condensates. The numerical results and conclusions are given at the end.

II. THE EXTENDED NJL MODEL

The choice of the NJL model [23] is motivated by the fact that this model displays the same symmetries as QCD and that it describes well the spontaneous breakdown of chiral symmetry in the vacuum and its restoration at high temperature and density. The model we used in this paper is an extended version of the two-flavor NJL model including interactions in the color singlet quark-antiquark channel and color anti-triplet diquark channel, which is not directly extended from the NJL model, but from the QCD Lagrangian [24]- [26].

The importance of color 3 diquark degree of freedom is related to the fact that one can construct a color-singlet nucleon current based on it. Because the gluon exchange between two quarks in the color 3 channel is attractive, one can view a color singlet baryon as a quark-diquark bound state. And experimental data from pp collisions indicate the existence of this quark-diquark component in nucleons.

The first attempt to investigate the diquark properties in the NJL model was taken in [27]. Starting from an NJL model for scalar, pseudoscalar, vector and axial-vector interactions of
the $(\bar{q}q) \times (\bar{q}q)$ type and Fierz-transforming away the vector and axial-vector interactions, the scalar and pseudoscalar mesons, and diquarks can be obtained. However, this method could not get a consistent treatment of vector and axial-vector particles.

The extended NJL model we used was derived directly from QCD Lagrangian [24] [25]. Integrating out gluon degrees of freedom from QCD Lagrangian, and performing a local approximation for the (nonperturbative) gluon propagator, one gets a contact current-current interaction. By using a special Fierz-rearrangement [28], one can completely decompose the two-quark-current interaction term into "attractive" color singlet $(\bar{q}q)$ and color antitriplet $(qq)$ channels. In this way, a complete simultaneous description of scalar, pseudo-scalar, vector, and axial-vector mesons and diquarks is possible, thus the extended NJL model including $(\bar{q}q) \times (\bar{q}q)$ interactions is completed by a corresponding $(\bar{q}q) \times (qq)$ interaction part.

In our present work, we only consider scalar, pseudoscalar mesons and scalar diquark, and using the following Lagrangian density

$$\mathcal{L} = \bar{q} \left( i\gamma^\mu \partial_\mu - m_0 \right) q + G_S \left[ (\bar{q}q)^2 + (\bar{q}i\gamma_5 \vec{\tau} q)^2 \right] + G_D \left[ (i\bar{q}^C \varepsilon^b \gamma_5 q) (i\bar{q} \varepsilon^b \gamma_5 q^C) \right],$$  \hspace{1cm} (1)

where $q^C = C\bar{q}^T$, $q^C = q^TC$ are charge-conjugate spinors, $C = i\gamma^2\gamma^0$ is the charge conjugation matrix (the superscript $T$ denotes the transposition operation), $m_0$ is the current quark mass, the quark field $q \equiv q_{i\alpha}$ with $i = 1, 2$ and $\alpha = 1, 2, 3$ is a flavor doublet and color triplet, as well as a four-component Dirac spinor, $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ are Pauli matrices in the flavor space, where $\tau^2$ is antisymmetric, and $(\varepsilon)^{ik} \equiv \varepsilon^{ik}$, $(\varepsilon^b)^{\alpha\beta} \equiv \varepsilon^{\alpha\beta b}$ are totally antisymmetric tensors in the flavor and color spaces.

In (1), $G_S$ and $G_D$ are independent effective coupling constants in the scalar quark-antiquark and scalar diquark channel. The former is responsible for the meson excitations, and the latter for the diquark excitations, which in principle can be determined by fitting mesons’ and baryons’ properties in the vacuum. The attractive interaction in different channels in this Lagrangian will give rise to a very rich structure of the phase diagram. At zero temperature and density, the attractive interaction in the color singlet channel is
responsible for the appearance of a quark anti-quark condensate and for the spontaneous breakdown of the chiral symmetry, and the interaction in the $qq$ channel binds quarks into diquarks (and baryons), but is not strong enough to induce diquark condensation. As the density increases, Pauli blocking suppresses the $\bar{q}q$ interaction, while the attractive interaction in the color anti-triplet diquark channel will induce the quark-quark condensate around the Fermi surface which can be identified as a superconducting phase.

After bosonization [24] [25], one can obtain the linearized version of the model (1)

$$
\tilde{\mathcal{L}} = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m_0)q - \bar{q}(\sigma + i\gamma^5\vec{\tau}\vec{\pi})q - \frac{1}{2}\Delta^b(i\bar{q}^C\varepsilon^b\gamma_5q) - \frac{1}{2}\Delta^{*b}(i\bar{q}\varepsilon^b\gamma_5q^C) - \frac{\sigma^2 + \vec{\pi}^2}{4G_S} - \frac{\Delta^b\Delta^{*b}}{4G_D},
$$

with the bosonic fields

$$
\Delta^b \sim i\bar{q}^C\varepsilon^b\gamma_5q, \quad \Delta^{*b} \sim i\bar{q}\varepsilon^b\gamma_5q^C, \quad \sigma \sim \bar{q}q, \quad \vec{\pi} \sim i\bar{q}\gamma^5\vec{\tau}q.
$$

Clearly, the $\sigma$ and $\vec{\pi}$ fields are color singlets, and the diquark fields $\Delta^b$ and $\Delta^{*b}$ are color antitriplet and (isoscalar) singlet under the chiral $SU(2)_L \times SU(2)_R$ group. $\sigma \neq 0$ and $\Delta^b \neq 0$ indicate that chiral symmetry and color symmetry are spontaneously broken.

We will first assume that the two condensates coexist, i.e.,

$$
\sigma \neq 0, \quad \vec{\pi} = 0;
\quad \Delta^1 = \Delta^2 = 0, \Delta^3 \neq 0,
$$

Here it has been regarded that only the first two colors participating in the condensate, while the third one does not. In the later expressions, we will simply use $\Delta \equiv \Delta^3$.

The real vacuum will be determined by the minimum of the thermodynamic potential at $T = 0$ and $\mu = 0$, and the minimum of the thermodynamic potential at any $T, \mu$ determines the stable state at that point.

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III. PARTITION FUNCTION AND THERMODYNAMIC POTENTIAL

A. Nambu-Gorkov formalism

The partition function of the grand canonical ensemble can be evaluated by using standard method [29] [30],

\[ Z = N' \int [d\vec{q}] dq \exp\left\{ \int_0^\beta d\tau \int d^3 \vec{x} \left( \tilde{L} + \mu \bar{q} \gamma_0 q \right) \right\}, \] (5)

where \( \mu \) is the chemical potential, and \( \beta = 1/T \) is the inverse of temperature \( T \).

According to the mean field approximation Eq. (4), we can write the partition function as a product of three parts,

\[ Z = Z_{const} Z_{q_1,2} Z_{q_3}. \] (6)

The constant part is

\[ Z_{const} = N' \exp\left\{ - \int_0^\beta d\tau \int d^3 \vec{x} \left[ \frac{\sigma^2}{4G_S} + \frac{\Delta^* \Delta}{4G_D} \right] \right\}. \] (7)

For the quarks in the first two colors (named as "the first two quarks" in the following) \( Q = q_{1,2} \) participating in the quark condensate, one has

\[ Z_{q_{1,2}} = \int [d\vec{Q}] [dQ] \exp\left\{ \int_0^\beta d\tau \int d^3 \vec{x} \left[ \frac{1}{2} \tilde{Q}(i \gamma^\mu \partial_\mu - m + \mu)Q + \frac{1}{2} \tilde{Q}^C(i \gamma^\mu \partial_\mu - m - \mu)Q^C + \frac{1}{2} \tilde{Q} \Delta^+ Q^C + \frac{1}{2} \tilde{Q}^C \Delta^- Q \right] \right\}. \] (8)

Here we have introduced the constituent quark mass \( m = m_0 + \sigma \), \( \Delta^+ = -i \Delta^* \tau^2 e^b \gamma_5 \), \( \Delta^- = -i \Delta \tau^2 e^b \gamma_5 \), \( \Delta^+ \) and \( \Delta^- \) satisfy the relation \( \Delta^+ = \gamma^0 (\Delta^-)^t \gamma^0 \). For the quark in the third color (named as "the third quark" in the following), which is not involved in the diquark condensate, one has

\[ Z_{q_3} = \int [d\vec{q}_3] [dq_3] \exp\left\{ \int_0^\beta d\tau \int d^3 \vec{x} \left[ \frac{1}{2} \bar{q}_3 (i \gamma^\mu \partial_\mu - m + \mu)q_3 + \frac{1}{2} \bar{q}_3^C (i \gamma^\mu \partial_\mu - m - \mu)q_3^C \right] \right\}. \] (9)

Introducing the 8-component spinors for the third quark and the first two quarks, respectively
\[ \Psi_3 = \begin{pmatrix} q_3 \\ q_3^c \end{pmatrix}, \quad \bar{\Psi}_3 = (\bar{q}_3 \quad \bar{q}_3^c), \]  
(10)

\[ \Psi = \begin{pmatrix} Q \\ Q^c \end{pmatrix}, \quad \bar{\Psi} = (\bar{Q} \quad \bar{Q}^c), \]  
(11)

and using the Fourier transformation in the momentum space,

\[ q(x) = \frac{1}{\sqrt{V}} \sum_n \sum_{\vec{p}} e^{-i(\omega_n \tau - \vec{p} \cdot \vec{x})} q(\vec{p}), \]  
(12)

where \( V \) is the volume of the thermal system, we can re-write the partition function Eqs. (8) and (9) in the momentum space as

\[ Z_{q_{1,2}} = \int [d\Psi] \exp\left\{ \frac{1}{2} \sum_{n,\vec{p}} \bar{\Psi} G^{-1} \bar{\Psi} \right\} = \text{Det}^{1/2}(\beta G^{-1}), \]  
(13)

and

\[ Z_{q_3} = \int [d\Psi_3] \exp\left\{ \frac{1}{2} \sum_{n,\vec{p}} \bar{\Psi}_3 G_0^{-1} \bar{\Psi}_3 \right\} = \text{Det}^{1/2}(\beta G_0^{-1}), \]  
(14)

where the determinantal operation Det is to be carried out over the Dirac, color, flavor and the momentum-frequency space. In Eqs. (13) and (14), we have defined the quark propagator in the normal phase

\[ G_0^{-1} = \begin{pmatrix} [G_0^+]^{-1} & 0 \\ 0 & [G_0^-]^{-1} \end{pmatrix}, \]  
(15)

with

\[ [G_0^+]^{-1} = (p_0 \pm \mu) \gamma_0 - \vec{\gamma} \cdot \vec{p} - m \]  
(16)

and \( p_0 = i\omega_n \), and the quark propagator in the color-breaking phase

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\[ G^{-1} = \begin{pmatrix} [G^+_0]^{-1} & \Delta^- \\ \Delta^+ & [G^-_0]^{-1} \end{pmatrix}. \] (17)

The full propagator \( G(p) \) is determined from solving \( 1 = G^{-1} G \), resulting in

\[ G = \begin{pmatrix} G^+ & \Xi^- \\ \Xi^+ & G^- \end{pmatrix}, \] (18)

with the components

\[ G^\pm \equiv \left\{ [G_0^\pm]^{-1} - \Sigma^\pm \right\}^{-1}, \quad \Sigma^\pm \equiv \Delta^\mp G_0^\pm \Delta^\pm, \]

\[ \Xi^\pm \equiv -G^\mp \Delta^\pm G_0^\pm. \] (19)

Here all components depend on the 4-momentum \( p^\mu \).

In the case of the mass term \( m = 0 \), i.e., both the current quark mass \( m_0 \) and the chiral condensate \( <\sigma> \) are zero, the full quark propagator has a simple form which could be derived from the energy projectors for massless particles [5] [6], while if there is a small mass term, the quark propagator can be expanded perturbatively around \( m = 0 \), but its form is very complicated [8]. However, in our case the quark mass term cannot be treated perturbatively. We have to find a general way to deal with the massive quark propagator.

Fortunately, we can evaluate a simple form for the massive quark propagator by using the energy projectors for massive particles. The energy projectors onto states of positive and negative energy for free massive particles are defined as

\[ \Lambda_{\pm}(\vec{p}) = \frac{1}{2} \left( 1 \pm \frac{\gamma_0 (\vec{\gamma} \cdot \vec{p} + m)}{E_p} \right), \] (20)

where the quark energy \( E_p = \sqrt{\vec{p}^2 + m^2} \). Under the transformation of \( \gamma_0 \) and \( \gamma_5 \), we can get another two energy projectors \( \tilde{\Lambda}_{\pm} \),

\[ \tilde{\Lambda}_{\pm}(\vec{p}) = \frac{1}{2} \left( 1 \pm \frac{\gamma_0 (\vec{\gamma} \cdot \vec{p} - m)}{E_p} \right), \] (21)

which satisfy
\[ \gamma_0 \Lambda_\pm(\vec{p}) \gamma_0 = \tilde{\Lambda}_\pm(\vec{p}), \quad \gamma_5 \Lambda_\pm(\vec{p}) \gamma_5 = \tilde{\Lambda}_\pm(\vec{p}). \]  

(22)

The normal quark propagator elements can be re-written as

\[ G_0^\pm = \frac{\gamma_0 \tilde{\Lambda}_+}{p_0 + E_p^\pm} + \frac{\gamma_0 \tilde{\Lambda}_-}{p_0 - E_p^\mp}, \]  

(23)

with \( E_p^\pm = E_p \pm \mu \). The propagator has four poles, i.e.,

\[ p_0 = \pm E_p^-, \quad p_0 = \mp E_p^+, \]  

(24)

where the former two correspond to the excitation energies of particles and holes, and the latter two are for antiparticles and antiholes, respectively.

The quark propagator including the diquark condensate’s contribution can be evaluated as

\[ G^\pm = \left( \frac{p_0 - E_p^\pm}{p_0^2 - E_\Delta^2} \gamma_0 \tilde{\Lambda}_+ + \frac{p_0 + E_p^\mp}{p_0^2 - E_\Delta^2} \gamma_0 \tilde{\Lambda}_- \right) (\delta_{\alpha\beta} - \delta_{\alpha3} \delta_{\beta3}) \delta^{ij}, \]  

(25)

and

\[ \Xi^\pm = \left( \frac{\mp \Delta \gamma_5}{p_0^2 - E_\Delta^2} \tilde{\Lambda}_+ + \frac{\mp \Delta \gamma_5}{p_0^2 - E_\Delta^2} \tilde{\Lambda}_- \right) \varepsilon^b, \]  

(26)

with \( E_\Delta^\pm = E_p^\pm^2 + \Delta^2 \). This propagator is very similar to the massless propagator derived in [5].

From the full propagator, it is difficult to obtain the mass of the quark which participates in the diquark condensate. The four poles of the full propagator, i.e.,

\[ p_0 = \pm E_\Delta^-, \quad p_0 = \mp E_\Delta^+, \]  

(27)

correspond to the excitation energies of quasi-particles (quasi-holes) and quasi-antiparticle (quasi-antiholes) in the color breaking phase. These quasiparticles are superpositions of particles and holes.

We plot the excitation spectrum at \( \mu = 500\text{MeV} \) as a function of \( E_p \) with different values of \( \Delta/\mu \) in Fig. 1. \( \Delta/\mu = 0 \) (the circles) correspond to the excitation spectrum in the normal phase, \( \Delta/\mu = 0.2 \) (the squares), 0.5 (the stars) and 1 (the triangles) correspond
to the excitation spectrum in the color superconducting phase, the black points are for the particles and the white points are for the holes. (a) is for the (quasi-)particles $E_p^-(E^\Delta_-)$ and (quasi-)holes $-E_p^-(E^\Delta_-)$, and (b) is for the (quasi-)antiparticles $-E_p^+(E^\Delta_+)$ and (quasi-)antiholes $E_p^+(E^\Delta_+)$. It can be easily seen from (a) that the quasi-particles and the quasi-holes mix particles and holes, which are called ”Bogoliubons”. In the normal phase, to excite a pair of particle and hole on the Fermi surface does not need energy, while in the superconducting phase, to excite a pair of quasi-particle and quasi-hole at least needs the energy $2\Delta$ when $E_p = \mu$. With increasing $\Delta$, it is found that to excite a quasi-particle needs a larger energy, and the difference between the excitation energies at $E_p \neq \mu$ and that at $E_p = \mu$ becomes smaller.

From (b), we see that to excite an antiparticle is much more difficult, and the diquark condensate has little effects on the excitation spectrum of (quasi-)antiparticles and (quasi-)antiholes.

**B. The calculation of $\ln Z_{q3}$**

For the third quark which does not participate in the diquark condensate, from Eq. (14), we have

$$
\ln Z_{q3} = \frac{1}{2} \ln \{ \text{Det}(\beta[G_0]^{-1}) \} = \frac{1}{2} \ln [\text{Det}(\beta[G_0^+]^{-1})\text{Det}(\beta[G_0^-]^{-1})].
$$

(28)

Using the Dirac matrix, we first perform the determinant in the Dirac space,

$$
\text{Det} \beta [G_0^+]^{-1} = \text{Det} \beta \left[ (p_0 + \mu)\gamma_0 - \vec{\gamma} \cdot \vec{p} - m \right] \\
= \text{Det} \beta \left( \begin{array}{cc} (p_0 + \mu) - m & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & -(p_0 + \mu) - m \end{array} \right), \\
= -\beta^2 \left[ (p_0 + \mu)^2 - E^2_p \right], \\
$$

(29)

and in the similar way, we get

$$
\text{Det} \beta [G_0^-]^{-1} = -\beta^2 \left[ (p_0 - \mu)^2 - E^2_p \right].
$$

(30)
After performing the determinant in the Dirac space, we have

$$\text{Det} \beta [G_0^+]^{-1} \text{Det} \beta [G_0^-]^{-1} = \beta^2 [p_0^2 - (E_p + \mu)^2] \beta^2 [p_0^2 - (E_p - \mu)^2].$$

(31)

Considering the determinant in the flavor, color, spin spaces and momentum-frequency space, we get the standard expression

$$\ln Z_{q_3} = \sum_{n} \sum_{\vec{p}} \{ \ln(\beta^2 [p_0^2 - (E_p + \mu)^2]) + \ln(\beta^2 [p_0^2 - (E_p - \mu)^2]) \},$$

(32)

remembering that the color space for the third quark is one-dimensional.

C. The calculation of $\ln Z_{q_{1,2}}$

It is more complicated to evaluate the thermodynamic potential for the quarks participating in the diquark condensate. From Eq. (13), we have

$$\ln Z_{q_{1,2}} = \frac{1}{2} \ln \text{Det}(\beta G^{-1}).$$

(33)

For a $2 \times 2$ matrix with elements $A, B, C$ and $D$, we have the identity

$$\text{Det} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Det}(-CB + CAC^{-1}D) = \text{Det}(-BC + DC^{-1}AC).$$

(34)

To prove the above equation, we have used

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 1 & C^{-1}D \\ B^{-1}A & 1 \end{pmatrix} \equiv \begin{pmatrix} BC^{-1} & AB^{-1} \\ DC^{-1} & CB^{-1} \end{pmatrix} \begin{pmatrix} 0 & C \\ B & 0 \end{pmatrix}.$$ 

(35)

Replacing $A, B, C$ and $D$ with the corresponding elements of $G^{-1}$, we have

$$\text{Det}(\beta G^{-1}) = \beta^2 \text{Det}D_1 = \beta^2 \text{Det}[-\Delta^+ \Delta^- + \Delta^+[G_0^+]^{-1}[\Delta^+]^{-1}[G_0^-]^{-1}]$$

$$= \beta^2 \text{Det}D_2 = \beta^2 \text{Det}[-\Delta^+ \Delta^- + [G_0^-]^{-1}[\Delta^-]^{-1}[G_0^+]^{-1} \Delta^-].$$

(36)

Using the energy projectors $\tilde{\Lambda}_\pm$, we can work out $D_1$ and $D_2$ as
\[ D_1 = \Delta^2 + \gamma_5[\gamma_0(p_0 - E_p^-)\tilde{\Lambda}_+ + \gamma_0(p_0 + E_p^+)\tilde{\Lambda}_-] \gamma_5[\gamma_0(p_0 - E_p^+)\tilde{\Lambda}_+ + \gamma_0(p_0 + E_p^-)\tilde{\Lambda}_-] \]
\[ = -[(p_0^2 - E_p^2 - \Delta^2)\tilde{\Lambda}_- + (p_0^2 - E_p^2 - \Delta^2)\tilde{\Lambda}_+], \]
\[ D_2 = -[(p_0^2 - (E_p^-)^2 - \Delta^2)\tilde{\Lambda}_+ + (p_0^2 - (E_p^+)^2 - \Delta^2)\tilde{\Lambda}_-]. \] (37)

Using the properties of the energy projectors, we can get
\[ D_1 D_2 = [(p_0^2 - (E_p^-)^2 - \Delta^2)] [(p_0^2 - (E_p^+)^2 - \Delta^2)] = [p_0^2 - E^{-\Delta^2}][p_0^2 - E^{+\Delta^2}]. \] (38)

With the above equations, Eq. (33) can be expressed as
\[ \ln Z_{q_1,2} = \frac{1}{2}\ln[\text{Det}\beta G^{-1}] = \frac{1}{4}\text{Tr}\ln[\beta^2 D_1\beta^2 D_2] \]
\[ = \frac{1}{4}\{\text{Tr}\ln[\beta^2(p_0^2 - E^{-\Delta^2})] + \text{Tr}\ln[\beta^2(p_0^2 - E^{+\Delta^2})]\} \]
\[ = 2N_f \sum_n \sum_p \{\ln[\beta^2(p_0^2 - E^{-\Delta^2})] + \ln[\beta^2(p_0^2 - E^{+\Delta^2})]\}. \] (39)

### D. The thermodynamic potential

The frequency summation of the free-energy
\[ \ln Z_f = \sum_n \ln[\beta^2(p_0^2 - E_p^2)] \] (40)
can always be obtained by performing the frequency summation of the propagator \(1/(p_0^2 - E_p^2)\). Differentiate Eq. (40) with respect to \(E_p\):
\[ \frac{\partial \ln Z_f}{\partial E_p} = -2E_p \sum_n \frac{1}{p_0^2 - E_p^2} = \beta[1 - 2\tilde{f}(E_p)], \] (41)
where \(\tilde{f}(x) = 1/(e^{\beta x} + 1)\) is the usual Fermi-Dirac distribution function. Then integrating with respect to \(E_p\), one can get the free-energy
\[ \ln Z_f = \beta[E_p + 2T\ln(1 + e^{-\beta E_p})]. \] (42)

With the help of the above expression, and replacing
\[ \sum_p \rightarrow V \int \frac{d^3 p}{(2\pi)^3}, \] (43)
we get the expressions

\[ \ln Z_{q_3} = N_f \beta V \int \frac{d^3p}{(2\pi)^3} [E_p^+ + 2T \ln(1 + e^{-\beta E_p^+}) + E_p^- + 2T \ln(1 + e^{-\beta E_p^-})], \quad (44) \]

\[ \ln Z_{q_{1,2}} = 2N_f \beta V \int \frac{d^3p}{(2\pi)^3} [E_{p\Delta}^+ + 2T \ln(1 + e^{-\beta E_{p\Delta}^+}) + E_{p\Delta}^- + 2T \ln(1 + e^{-\beta E_{p\Delta}^-})]. \quad (45) \]

Finally, we obtain the familiar expression of the thermodynamic potential

\[ \Omega = -T \frac{\ln Z}{V} = \frac{\sigma^2}{4G_S} + \frac{\Delta^2}{4G_D} - 2N_f \int \frac{d^3p}{(2\pi)^3} [E_p + T \ln(1 + e^{-\beta E_p^+}) + T \ln(1 + e^{-\beta E_p^-})]
+ E_{p\Delta}^- + 2T \ln(1 + e^{-\beta E_{p\Delta}^-}) + E_{p\Delta}^+ + 2T \ln(1 + e^{-\beta E_{p\Delta}^+})]. \quad (46) \]

**IV. CONDENSATES AND GAP EQUATIONS**

**A. Condensates**

With the full quark propagator Eqs. (25) and (26), the diquark condensate is generally expressed as

\[ \langle \bar{q}^C \gamma_5 q \rangle = -iT \sum_n \int \frac{d^3p}{(2\pi)^3} \text{tr} [\Xi^- \gamma_5]. \quad (47) \]

Performing the Matsubara frequency summation and taking the limit \( T \to 0 \), we get the diquark condensate at finite chemical potential

\[ \langle \bar{q}^C \gamma_5 q \rangle = -2\Delta N_c N_f \int \frac{d^3}{(2\pi)^3} \left[ \frac{1}{2E^-} + \frac{1}{2E^+} \right]. \quad (48) \]

For the third quark, its chiral condensate can be evaluated by using the quark propagator in the normal phase,

\[ \langle \bar{q}_3 q \rangle = -iT \sum_n \int \frac{d^3p}{(2\pi)^3} \text{tr} [G^+_0], \quad (49) \]

while for the quarks participating in the diquark condensate, the chiral condensate should be evaluated by using the quark propagator in the color breaking phase,
\[
\langle \bar{q}_1 q^1 \rangle = -iT \sum_n \int \frac{d^3p}{(2\pi)^3} tr[G^+] .
\] (50)

We have no explicit mass expression for the first two quarks which participate in the diquark condensate, the influence of diquark condensate has been reflected in the quark propagator. The difference between the first two quarks which participate in the diquark condensate and the third quark which does not participate in the diquark condensate can be read from their chiral condensates and can be defined as

\[
\delta = \langle \bar{q}_1 q_1 \rangle^{1/3} - \langle \bar{q}_3 q_3 \rangle^{1/3},
\] (51)

where \( \delta \) has the dimension of energy. In the case of chiral limit, the quark mass \( m \) decreases to zero in the color superconducting phase, and the influence of the diquark condensate on quarks in different colors vanishes.

After performing the Matsubara frequency summation and taking the limit \( T \to 0 \), we get the expressions of the condensates at \( \mu \neq 0 \),

\[
\langle \bar{q}_3 q^3 \rangle = 4mN_f \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [\theta(\mu - E_p) - 1],
\]

\[
\langle \bar{q}_1 q^1 \rangle = 4mN_f \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} [n^+_p - n^-_p],
\] (52)

where

\[
n^\pm_p = \frac{1}{2} (1 \pm \frac{E^\pm_p}{E^\Delta})
\] (53)

are the occupation numbers for quasi-particles and quasi-antiparticles at \( T = 0 \). Correspondingly, \( 1 - n^\pm_p \) are the occupation numbers of quasi-holes and quasi-antiholes, respectively.

We plot the occupation numbers for (quasi-)particles \( n^+_p \) and (quasi-)holes \( 1 - n^+_p \) in Fig. 2 (a), and the occupation numbers for (quasi-)antiparticles \( n^-_p \) and (quasi-)antiholes \( 1 - n^-_p \) in (b) as a function of \( E_p \) with respect to \( \Delta/\mu = 0 \) (circles), 0.2 (squares), 0.5 (stars) and 1 (triangles); the black and white points correspond to particles and holes, respectively.

It is seen that the Fermi surface is very sharp in the normal phase \( \Delta/\mu = 0 \), and becomes smooth when diquark condensate appears. The smearing is a consequence of the fact that
the "Bogliubons" are superpositions of particle and hole states. The smearing of the Fermi surface induces the chiral symmetry restoring at a smaller chemical potential. The larger the diquark condensate is, the smoother the Fermi surface will be.

From Fig. 2(b), it can be seen that the occupation numbers for the (quasi-)antiparticles (antiholes) in the normal phase or color breaking phase are not sensitive to the magnitude of the diquark condensate.

**B. Gap equations**

The two gap equations $m$ and $\Delta$ can be derived by minimizing the thermodynamic potential Eq. (46) with respect to $m$ and $\Delta$,

$$\frac{\partial \Omega}{\partial m} = \frac{\partial \Omega}{\partial \Delta} = 0.$$ (54)

Taking into account the general expressions for the diquark condensate and chiral condensates, one can get the relations between the chiral gap $m$ and the chiral condensate $<\bar{q}q>$, i.e.,

$$m = m_0 + \sigma,$$

$$\sigma = -2G_S <\bar{q}q>,$$ (55)

here the chiral condensate should perform summation in the color space

$$<\bar{q}q> = 2 <\bar{q}_1 q^1> + <\bar{q}_3 q^3>;$$ (56)

and the relation between the diquark gap $\Delta$ and the diquark condensate $<q^C\gamma_5 q>$, i.e.,

$$\Delta = -2G_D <q^C\gamma_5 q>,$$ (57)

substituting Eq. (48) into the above equation, the gap equation for the diquark condensate Eq. (57) in the limit of $T \to 0$ can be written as

$$1 = 4N_cN_fG_D \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2E^-_\Delta} + \frac{1}{2E^+_\Delta} \right].$$ (58)
V. NUMERICAL RESULTS

In this section, through numerical calculations, we will investigate the phase structure along the chemical potential direction, analyze the competition mechanism between the chiral condensate and diquark condensate, and discuss the influence of the color breaking on the quarks in different colors.

Before the numerical calculations, we should fix the model parameters. The current quark mass $m_0 = 5.5\text{MeV}$, the Fermion momentum cut-off $\Lambda_f = 0.637\text{GeV}$, and the coupling constant in color singlet channel $G_S = 5.32\text{GeV}^{-2}$ are determined by fitting pion properties. The corresponding constituent quark mass in the vacuum is taken to be $m(\mu = 0) = 330\text{MeV}$. The coupling constant in the color anti-triplet channel $G_D$ can in principle be determined by fitting the nucleon properties. In [25] $G_D/G_S \simeq 2.26/3$ was chosen by fitting the scalar diquark mass of $\simeq 600\text{MeV}$ to obtain a realistic baryon mass in the order of $\simeq 900\text{MeV}$. In our case, to investigate the influence of diquark condensate on the chiral phase transition, we will set $G_D/G_S = 0, 2/3, 1, 1.2, 1.5$, respectively.

A. Phase structure at zero temperature

First, we investigate the phase structure along the chemical potential direction with respect to different magnitude of $G_D$.

The corresponding gaps $m$ (white points) and $\Delta$ (black points) determined by Eqs. (55) and (57) are plotted in Fig. 3 as functions of $\mu$ with respect to different $G_D/G_S = 2/3, 1, 1.2, 1.5$ in (a), (b), (c) and (d), respectively. We see that in the region where the constituent quark mass keeps its value in the vacuum $m(\mu = 0)$, the diquark condensate keeps zero. When the constituent quark mass starts to decrease at $\mu_s$, the diquark condensate begins to increase.

In Fig. 3 (a), i.e., in the case of $G_D/G_S = 2/3$, we see that the quark mass starts to decrease at $\mu_s = 336\text{MeV}$, then jumps down to a small quark mass at $\mu_c = 340.1\text{MeV}$.
Correspondingly, the gap of the diquark condensate begins to increase at $\mu_s$ and jumps up to 65MeV at $\mu_c$. We can see that the chiral symmetry restores and the color superconductivity phase transition occurs at the same critical chemical potential $\mu_c$, and the phase transitions are of the first order.

In (b), the constituent quark mass decreases and the diquark condensate increases fast in the region of chemical potential $\mu_s < \mu < \mu_c$. At $\mu_c = 304.8$MeV, the two phase transitions take place simultaneously. While the phase transitions are still the first order, the jumps become smaller compared with the case of $G_D/G_S = 2/3$.

In (c) and (d), we see that with increasing of $G_D/G_S$, the first order phase transition becomes the second order one, the quark mass decreases and the diquark condensate increases slowly in a wider chemical potential region $\mu_s < \mu < \mu_c$, and the critical chemical potential becomes small, $\mu_c = 266$MeV and 190MeV for $G_D/G_S = 1.2$ and 1.5, respectively. For the second phase transitions, the critical chemical potential $\mu_c$ corresponds to the point at which $m$ and $\Delta$ have maximum changes. It can be seen that the gap of the diquark condensate $\Delta$ at the critical point increases from 162MeV for $G_D/G_S = 1.2$ up to 310MeV for $G_D/G_S = 1.5$.

In Fig. 4, the two gaps $m$ (white points) and $\Delta$ (black points) are plotted as functions of the scaled baryon density $n_b/n_0$, where $n_0$ is the normal nuclear matter density. We see that the chiral and color superconductivity phase transitions occur at $n_b/n_0 = 1.83$ for $G_D/G_S = 2/3$, and the critical density decreases with increasing $G_D/G_S$.

**B. The Competition between the chiral and diquark condensate**

From the above analysis, the strong competition between the chiral condensate and the diquark condensate has been found. When the chiral condensate begins to decrease at $\mu_s$, the diquark condensate starts to increase; and when the chiral condensate jumps down at $\mu_c$, the diquark condensate jumps up; The larger the coupling constant $G_D$ is, the smaller the two jumps are, and the smaller the two chemical potentials $\mu_s$ and $\mu_c$ are. In order to
explicitly show how the diquark condensate influences the chiral phase transition, we plot the constituent quark mass $m$ and the diquark gap $\Delta$ as functions of $\mu$ for different values of $G_D/G_S$ in Fig. 5 and try to understand the competition mechanism.

In the case of $G_D/G_S = 0$, only chiral phase transition occurs, the thermal system in the mean-field approximation is nearly a free Fermi gas made of constituent quarks. In the limit of $T = 0$, there is a very sharp Fermi surface of the constituent quark. When the chemical potential is larger than the constituent quark mass in the vacuum, the chiral symmetry restores, and the system of constituent quarks becomes a system of current quarks.

When a diquark gap $\Delta$ forms in the case of $G_D/G_S \neq 0$, it will smooth the sharp Fermi surface of the constituent quark. In other words, the diquark pair lowers the sharp Fermi surface, and induces the chiral symmetry restoring at a smaller chemical potential.

In TABLE I, we list the chemical potentials $\mu_s$, at which the diquark gap starts appearing, and $\mu_c$, at which the two phase transitions occur, for different values of $G_D/G_S$. $\mu_c^0 = 345.3\text{MeV}$ is the critical chemical potential in the case of $G_D/G_S = 0$. For $G_D/G_S = 2/3$ and 1, the phase transitions are of the first order, the $\Delta(\mu_c)$ listed in TABLE I correspond to the lower value of $\Delta$ at $\mu_c$.

We see that for larger $G_D/G_S$, the diquark condensate appears at a smaller chemical potential $\mu_s$, and the phase transition occurs at a smaller critical chemical potential $\mu_c$, while the chemical potential region $\mu_c - \mu_s$ becomes wider, it goes up from $4\text{MeV}$ for $G_D/G_S = 2/3$ to $120\text{MeV}$ for $G_D/G_S = 1.5$.

From our numerical calculations, we find there is an approximate relation between the diquark gap and the effect of diquark condensate on the critical point of phase transitions. If there is a diquark gap of magnitude $\Delta$, then the critical chemical potential can be estimated by using the formula

$$\mu_c = \mu_c^0 - \frac{\Delta}{2}. \quad (59)$$

The ratio $\Delta(\mu_c)/\mu_c^0$ indicates the magnitude of the diquark condensate’s influence on the sharp Fermi surface. It is seen that in the cases of $G_D/G_S = 2/3$ and 1, which are close to
the real physical case, the ratio is very small; and in the case of $G_D/G_S = 1.2, 1.5$, the ratio is nearly 50% and 90%, which indicates that the diquark condensate influences the sharp Fermi surface strongly.

Now we turn to study how the chiral gap influences the color superconductivity phase transition.

Firstly, we change the constituent quark mass in the vacuum from $m(\mu = 0) = 330\text{MeV}$ to $m(\mu = 0) = 486\text{MeV}$. To fit the pion properties, the coupling constant in the quark-antiquark channel is correspondingly increased from $G_S$ to $1.2G_S$. We plot the diquark gap as a function of $\mu$ in Fig. 6a for the two vacuum masses and for $G_D/G_S = 2/3, 1, 1.2, 1.5$. We find that for the same $G_D/G_S$, the critical chemical potential $\mu_c$ increases very much when $m(\mu = 0)$ increases from $330\text{MeV}$ to $486\text{MeV}$.

Then we withdraw the quark mass, i.e., taking $m = 0$ even in the vacuum. We plot the diquark gap as a function of $\mu$ in Fig. 6b for $m(\mu = 0) = 0$ and $330\text{MeV}$ and for $G_D/G_S = 2/3, 1, 1.2, 1.5$. We find that for any $G_D/G_S$, the phase transition is of second order, and the diquark condensate starts to appear at a much smaller chemical potential $\mu_s$ for $m(\mu = 0) = 0$ compared with $m(\mu = 0) = 330\text{MeV}$.

While the constituent quark mass changes the critical point of color superconductivity, the diquark gaps for two values of constituent quark mass coincide in the overlap region of color superconductivity. This can be seen clearly in Fig. 6a and b.

From the influence of the diquark gap on the chiral phase transition and the influence of the chiral gap on the color superconductivity phase transition, it is found that there does exist a strong competition between the two phases. The competition starts at $\mu_s$ and ends at $\mu_c$. We call the region $\mu_c - \mu_s$ the "competition region". With increasing of $G_D$, the competition region becomes wider. The system is in the chiral breaking phase before $\mu_s$, in the color superconducting phase after $\mu_c$, and in the coexisting phase between $\mu_s$ and $\mu_c$. 
C. The influence of color breaking on quarks’ properties

Finally, we study how the diquark condensate influences the quark properties.

In the normal phase, the quarks in different colors are degenerate. However, in the color breaking phase, the first two quarks are involved in the diquark condensate, while the third one is not.

The quark mass $m$ appeared in the formulae of this paper is the mass for the third quark which does not participate in the diquark condensate. We have seen from Fig. 5 (a) that the diquark condensate influences much the quark mass $m$ in the competition region $\mu_s < \mu < \mu_c$. In the color breaking phase, i.e., when $\mu > \mu_c$, the quark mass $m$ in different cases of $G_D$ decreases slowly with increasing $\mu$, and reaches the same value at about $\mu = 500\text{MeV}$.

The difference of the chiral condensates for quarks in different colors $\delta$ defined in Eq. (51) is shown in Fig. 7 as a function of the chemical potential $\mu$ with respect to $G_D/G_S = 2/3, 1, 1.2, 1.5$. It is found that in any case $\delta$ is zero before $\mu_s$, then begins to increase at $\mu_s$ and reaches its maximum at $\mu_c$, and starts to decrease after $\mu > \mu_c$, and finally approaches to zero at about $\mu = 500\text{MeV}$. When $\mu > 500\text{MeV}$, $\delta$ becomes negative. With increasing $G_D$, $\delta(\mu_c)$ increases from 1MeV for $G_D/G_S = 2/3$ to 13MeV for $G_D/G_S = 1.5$. Comparing with the magnitude of the chiral and diquark condensate, $\delta$ is relatively small.

VI. CONCLUSIONS

In summary, the full massive quark propagator has been evaluated in this paper, which makes it possible to extend the Green-function approach in the moderate baryon density region, and the familiar expression of the thermodynamic potential has been re-evaluated by using the massive quark propagator.

The phase structure along the chemical potential direction has been investigated, and the strong competition between the chiral breaking phase and the color breaking phase has
been found in the chemical potential region \( \mu_s < \mu < \mu_c \). The competition mechanism has been analyzed by investigating the influence of the diquark condensate on the sharp Fermi surface. The diquark condensate smooths the sharp Fermi surface, and induces the chiral phase transition occurring at a smaller chemical potential. The two phases compete with each other in the chemical potential region \( \mu_c - \mu_s \). A large enough diquark condensate can even change the phase transition from the first order to second order. The influence of the diquark condensate on the properties of quarks in different colors has also been investigated. It is found that the difference of the chiral condensates between quarks in different colors induced by the diquark condensate is very small.

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FIG. 1. The excitation spectrum at $\mu = 500\text{MeV}$ for (quasi-)particles and (quasi-)holes in (a) and for (quasi-)antiparticles and (quasi-)antiholes in (b) as a function of $E_p$ with different values of $\Delta/\mu$, $\Delta/\mu = 0$ (circles), 0.2 (squares), 0.5 (stars) and 1 (triangles). The black and white points are for the particles and holes, respectively.
FIG. 2. The occupation numbers for (quasi-)particles $n_p^+$ and (quasi-)holes $1 - n_p^+$ in (a), and for (quasi-)antiparticles $n_p^-$ and (quasi-)antiholes $1 - n_p^-$ in (b) as a function of $E_p$ with respect to $\Delta/\mu = 0$ (circles), 0.2 (squares), 0.5 (stars) and 1 (triangles). The black and white points correspond to particles and holes, respectively.
FIG. 3. The two gaps $m$ (white points) and $\Delta$ (black points) as functions of chemical potential $\mu$ for $G_D/G_S = 0, 2/3, 1, 1.2, 1.5$, respectively.
FIG. 4. The two gaps $m$ (white points) and $\Delta$ (black points) as functions of the scaled baryon density $n_b/n_0$ for $G_D/G_S = 0, 2/3, 1, 1.2, 1.5$, respectively.
FIG. 5. The gaps $m$ in (a) and $\Delta$ in (b) as a function of the chemical potential $\mu$ with respect to $G_D/G_S = 0, 2/3, 1, 1.2, 1.5$, respectively.
FIG. 6. The influence of chiral gap on the color superconductivity phase transition in the case of $m(\mu = 0) = 486\text{MeV}$ in (a) and $m(\mu = 0) = 0$ in (b).
FIG. 7. The difference of the chiral condensates for quarks in different colors $\delta$ as a function of the chemical potential $\mu$ with respect to $G_D/G_S = 0, 2/3, 1, 1.2, 1.5$, respectively.
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TABLE I. The $G_D$ dependence of chemical potentials $\mu_s$ and $\mu_c$, $\mu_c^0 = 345.3$MeV.