In the paper, we will discuss the problem of developing a model for understanding and predicting the behavior of high-energy density systems, such as those found in laser physics and nuclear reactions. The model is based on the principle of causality and the idea that the system can be modeled as a network of interacting components. The model is tested against experimental data and shows good agreement with observations. The results are consistent with the predictions of other models and suggest that the model is a valuable tool for understanding complex systems.
on the incident energy with that of the scaling parameters. Temperature measurements, in principle, also permit a test whether the reaction scenario, and specifically the temperature as an important parameter characterizing it, are indeed independent of the isotopic composition of the system as commonly assumed. For isotope temperatures, however, this property is already implied if isoscaling holds. It is a consequence of Eq. (1) according to which the double yield ratios from which isotope temperatures are derived are identical for the pair of reactions. The observations of approximately identical isotope temperatures in the $^{112}$Sn + $^{113}$Sn and $^{124}$Sn + $^{125}$Sn reactions at 50 MeV per nucleon [15], as well as for the present reactions, are therefore part of the more general phenomenon of isoscaling.

It has been shown that in both, light-ion induced collisions and peripheral heavy-ion collisions at high energy, the fragment production and observed isotopic effects can be explained in the framework of a hybrid approach consisting of dynamical initial stage and a subsequent statistical breakup of a highly excited residual at low density [16, 17, 18, 19]. With the aim to identify reasons for the isoscaling in the present case, an analysis with the statistical multifragmentation model (SMM, Ref. [16]) was carried out. It will be demonstrated that isotopic scaling arises naturally in a statistical fragmentation mechanism. The isoscaling parameter $\alpha$ deduced for hot primary fragments is, furthermore, found to be directly proportional to the symmetry part of the binding energy of the fragments when they are formed at low density. To the extent that the modification of this parameter during secondary deexcitation remains small this opens the possibility of testing components of the nuclear equation of state in fragmentation reactions.

II. EXPERIMENTAL DATA

The experimental data are taken from the literature [8, 9, 10, 12]. They were obtained in the JINR laboratories in Dubna with beams of protons of 660 MeV, 1.0 and 6.7 GeV, of deuterons with 3.1 GeV, and of $\alpha$ particles with 15.3 GeV incident energy. Isotopically resolved cross sections of light fragments were measured with semiconductor telescopes placed at $\theta_{lab} = 90^\circ$ and with thin internal targets made from enriched $^{112}$Sn (81%) and $^{124}$Sn (96%). From the yields, integrated over energy intervals specified in Refs. [8, 10], ratios $R_{12}$ for the production of a particular fragment in the reactions with the two Sn isotopes were determined. It is convenient to introduce a reduced isotopic effect for a fragment species $X$ by normalizing with respect to the ratio observed for $^4$Li, i.e. $R_{12}(X)/R_{12}(^4$Li). Uncertainties of the absolute normalizations of the data sets measured with the two targets are thus eliminated.

The reduced isotopic effects measured for the five pairs of reactions are shown in Fig. 1. The cross section ratios for the most neutron poor and the most neutron rich fragments differ by about one order of magnitude in all cases except for the d(3.1 GeV) reaction for which only a narrow range of isospin is covered by the detected products. A nearly perfect exponential dependence on the third component of the fragment isospin $t_3$ is observed, with slope parameters $\beta_{33}$ (Eq. (3)) that decrease gradually from 1.08 to 0.68 as the projectile energy increases (Table I). This variation of the isotopic effect with the incident energy has been noted in Ref. [14] and tentatively ascribed to a gradual rise of the temperatures of the emitting systems.

Two-parameter fits according to Eq. (1) were also performed, with results that are listed in Table I. The mono-

![Image of graph](https://example.com/graph.png)

FIG. 1: Isotopic effect $R_{12}$, normalized with respect to $R_{12}(^4$Li), versus the third component of the fragment isospin $t_3$. The five reactions are offset from each other by multiple factors of three and are labelled with the total projectile energy, given in units of GeV. H, He, Li, Be, and B fragments are distinguished by different data symbols as indicated. The lines are the results of exponential fits according to Eq. (3). Some of the data symbols are slightly displaced horizontally for reasons of clarity.

<table>
<thead>
<tr>
<th>Projectile</th>
<th>$Z$</th>
<th>$\beta_{33}$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>p 0.66 GeV</td>
<td>1-3</td>
<td>1.08 ± 0.06</td>
<td>0.53 ± 0.04</td>
<td>-0.51 ± 0.05</td>
</tr>
<tr>
<td>p 1.00 GeV</td>
<td>2-4</td>
<td>1.00 ± 0.10</td>
<td>0.52 ± 0.04</td>
<td>-0.65 ± 0.05</td>
</tr>
<tr>
<td>d 3.10 GeV</td>
<td>2-4</td>
<td>0.88 ± 0.04</td>
<td>0.43 ± 0.03</td>
<td>-0.45 ± 0.04</td>
</tr>
<tr>
<td>p 6.70 GeV</td>
<td>1-5</td>
<td>0.81 ± 0.02</td>
<td>0.39 ± 0.01</td>
<td>-0.43 ± 0.02</td>
</tr>
<tr>
<td>$\alpha$ 15.3 GeV</td>
<td>2-3</td>
<td>0.68 ± 0.02</td>
<td>0.34 ± 0.01</td>
<td>-0.32 ± 0.03</td>
</tr>
</tbody>
</table>
tonic trend exhibited by the parameter \( \alpha \) as a function of the incident energy reflects that of \( \beta_3 \). It apparently extends to much lower energies, as evident from the value \( \alpha = 0.66 \) reported for the \( \alpha(200 \text{ MeV}) \) reaction in Ref. [5]. The dependence on \( Z \) is not equally well established for all reactions since only a limited range of elements has been covered in some cases. There is, however, a tendency of the absolute value of \( \beta \) being larger than \( \alpha \). For protons of 6.7 GeV, this is illustrated in Fig. 2 with the results from yet another parameterization in terms of \( A \) and \( t_3 \).

\[
R_{12}(A, t_3) = C \cdot \exp(A \cdot \alpha_4 + t_3 \cdot \beta_3). \tag{4}
\]

The logarithmic slope of the cross section ratios for given \( t_3 \) as a function of \( A \), by its definition equal to half the difference between \( |\beta| \) and \( \alpha \), is finite with a value \( \alpha_4 = (1.8 \pm 1.1) \cdot 10^{-5} \). Weighted over the five reactions \( |\beta| \) is found to be larger than \( \alpha \) by \( 8\% \pm 4\% \).

III. INITIAL DYNAMICAL STAGE

For the simulation of the initial stage of the collision the intranuclear cascade (INC) model developed in Dubna was used [20, 21]. The INC describes the process of the hadron-nucleon collisions inside the target nucleus. High energy protons of these interactions are allowed to escape while low energy protons are assumed to be trapped by the nuclear potential of the target system.
IV. CHEMICAL EQUILIBRIUM

Isotopic effects and isotope yield ratios confront us with the question of chemical equilibrium in the system. Here, the grand-canonical quantum-statistical models (QSM) are useful for extracting relative isotopic abundances that correspond to the thermodynamical limit. The model of Hahn and Stöcker [13], chosen in the present case, assumes thermal and chemical equilibrium at the breakup point where the fragmenting system is characterized by a density \( \rho \), temperature \( T \), and by its overall \( N/Z \) ratio. The model respects fermion and boson statistics which, however, is not crucial at high temperature. It does not take into account the finite size of the nuclear systems nor the Coulomb interactions between fragments but follows the sequential decay of excited fragments according to tabulated branching ratios. It has already been shown that the \( I_3 \) scaling (Eq. (3)) exhibited by the p(6.7 GeV) + \(^{112,124}\text{Sn} \) reactions is well reproduced by the QSM if appropriate parameters are chosen [14]. Even if \( \Delta (N/Z) \) is fixed, e.g. with the aid of the INC model, a continuous set of pairs of \( T = \rho \) parameters can be found that all permit equally good descriptions of the data. By varying either the temperature or the density the observed variation of the scaling parameter with incident energy can be followed.

In the grand-canonical approximation, the scaling parameters \( \alpha \) and \( \beta \) (Eq. (1)) are equal to the difference of the chemical potentials for neutrons and protons in the two systems, \( \alpha = \Delta \mu_n/T \) and \( \beta = \Delta \mu_p/T \), provided a common temperature \( T \) for both systems exists [5, 28]. The observation of \( I_3 \) scaling, consequently, implies that these differences are of different sign and about equal magnitude, or that the sum of \( \mu_n \) and \( \mu_p \) is invariant with the \( N/Z \) ratio of the system. This is illustrated in Fig. 4 in which the chemical potentials extracted from the model calculations are given as a function of \( N/Z \). The chosen parameters are \( T = 6 \) MeV and \( \beta/\rho_0 \) from 0.1 to 0.5 in steps of 0.1 (the p(6.7 GeV) data, e.g., are reproduced with \( T = 6 \) MeV and \( \rho/\rho_0 = 0.1 \) [14]). The sum \( \mu_p + \mu_n \) is approximately independent of \( N/Z \), with a very small tendency of \( \mu_p \) to change more rapidly than \( \mu_n \).

For the Sn isotopes with \( N/Z = 1.24 \) and 1.48 and for a breakup density \( \rho/\rho_0 = 0.3 \) the calculated differences of the chemical potentials are \( \Delta \mu_n = 2.3 \) MeV and \( \Delta \mu_p = -2.9 \) MeV. From these values coefficients \( \alpha = 0.38 \) and \( \beta = -0.48 \) are obtained which are not far from the experimental observation in central Sn + Sn collisions [5] and in some of the present reactions. A more stringent model test will have to include a comparison with fragment yields and an accurate estimation of the temperature. However, as also shown in Ref. [14], the chemical equilibrium hypothesis is quite adequate for the description of isotopic phenomena in these reactions even though the heavy fragments or residues in the final channels are not explicitly taken into account. These degrees of freedom will be included in the SMM analysis presented in Section VI.

V. TEMPERATURES

The reported cross sections for helium and lithium isotopes were used to construct temperature observables from double-isotope ratios [28] for the present set of reactions. The production of \(^{3}\text{He} \) and of \(^{6,7}\text{Li} \) has been measured for incident protons of 600 MeV, and the frequently used \( I_{\text{He-Li}} \) temperature [23] can be determined for this particular case. Cross sections for the production of \(^{3}\text{He} \) are not reported for the reactions at higher energies, so that \( I_{\text{He-Li}} \) cannot be used to follow the evolution of the breakup temperature with incident energy.

A common temperature observable for four out of the set of five reactions can be obtained from the \(^{4}\text{He}^{3}\text{He} \) and \(^{6}\text{Li}^{6}\text{Li} \) yield ratios, and in two cases \(^{4}\text{He} \) and \(^{6}\text{He} \) yields are available which can also be combined with the
lithium ratios $^6\text{Li}/^8\text{Li}$. The corresponding expressions are

$$T_{\text{He}^{6,8},0} = 13.3\,\text{MeV}/\ln(2.2\frac{Y_{^6\text{Li}}/Y_{^8\text{Li}}}{Y_{^4\text{He}}/Y_{^8\text{He}}})$$

(5)

$$T_{\text{He}^{6/8},0} = -8.3\,\text{MeV}/\ln(1.4\frac{Y_{^6\text{Li}}/Y_{^8\text{Li}}}{Y_{^4\text{He}}/Y_{^8\text{He}}})$$

(6)

$$T_{\text{He}^{8/8},0} = 7.2\,\text{MeV}/\ln(1.7\frac{Y_{^6\text{Li}}/Y_{^8\text{Li}}}{Y_{^4\text{He}}/Y_{^8\text{He}}})$$

(7)

The two latter isotopic thermometers do not fulfill the requirement that the double difference of the binding energies of the four isotopes, the prefactor in Eqs. (5) - (7), should be large compared to the anticipated temperatures \cite{23, 29}. They may thus be more strongly influenced by sequential decays. In particular, the contributions from residue evaporation to the inclusive yields of $^4\text{He}$ will have a large effect on $T_{\text{He}^{6/8},0}$. The true breakup temperature is likely to be underestimated by this observable but its trend with incident energy may be preserved. Therefore, at this stage, a potential attempt has been made to derive corrections, and the so-called apparent temperatures, labelled with the subscript 0 in the above expressions, are presented in Table II and Fig. 5. The differences of the energy intervals of the fragment detection \cite{8, 10} and the systematic errors associated with the isotope identification \cite{12} are taken into account.

The deduced values of $T_{\text{He}^{6,8},0}$ and $T_{\text{He}^{6/8},0}$ of 1.4 to 5 MeV are in the range typical for reaction processes near the onset of multi-fragment emissions \cite{23, 29, 30, 31}. The values obtained for $T_{\text{He}^{6/8},0}$ are lower by 1 MeV or more, as expected. Most of the temperatures, within errors, are about equal for the corresponding pairs of reactions. Larger differences, as e.g. of $T_{\text{He}^{6,8},0}$ for protons of 660 MeV, reflect similarly prominent deviations from isoscaling for some of the isotopes involved (cf. Fig. 1).

The trend with incident energy exhibited by $T_{\text{He}^{4/8},0}$ is found to follow very closely that of the inverse of the scaling parameters $\alpha$ and $\beta_3$ (Fig. 5). This suggests that the gradual flattening of the slopes of the isoscaling curves, as the projectile energy increases, is indeed caused by a rising mean temperature. A variation of isotopic observables associated with a temperature change has recently been reported for the fragmentation of $^{25}$Mg projectiles in collisions with $^{112,124}$Sn targets at 30 and 50 MeV per nucleon \cite{32}.

### VI. SMM Interpretation

The statistical multifragmentation model (SMM) is based upon the assumption of statistical equilibrium at a low-density freeze-out stage \cite{16}. All breakup channels (particles) composed of nucleons and excited fragments are considered and the conservation of mass, charge, momentum and energy is taken into account. The formation of a compound nucleus is included as one of the channels. In the microcanonical treatment the statistical weight of decay channel $j$ is given by $W_j \propto \exp S_j$, where $S_j$ is the entropy of the system in channel $j$ which is a function of the excitation energy $E_x$, mass number $A$, charge $Z$, and other parameters of the source. In the standard version of the model, the Coulomb interaction between the fragments is treated in the Wigner–Seitz approximation. Different breakup partitions are sampled according to their statistical weights uniformly in the phase space. After breakup, the fragments propagate independently in their mutual Coulomb field and undergo secondary decays. The decexcitation of the hot primary fragments proceeds via evaporation, fission, or Fermi-breakup \cite{33}.

#### A. Liquid-drop description of primary fragments

An important difference of the SMM from other statistical models, e.g. QSM \cite{13} or the Berlin statistical multifragmentation model \cite{34, 35}, is the treatment of the hot fragments at the freeze-out density. In the SMM light fragments with mass number $A \leq 4$ are considered as stable particles ("nuclear gas") with masses and spins taken from the nuclear tables. Only translational degrees of freedom of these particles contribute to the entropy of the system. Fragments with $A > 4$ are treated as heated
nuclear liquid drops, and their individual free energies $F_{AZ}$ are parameterized as a sum of the bulk, surface, Coulomb and symmetry energy contributions

$$F_{AZ} = F^B_{AZ} + F^S_{AZ} + F^C_{AZ} + F^{sym}_{AZ}.$$  

(8)

The standard expressions [16] for these terms are: $F^B_{AZ} = (-W_0 - T^3/\varepsilon_0)A$, where the parameter $\varepsilon_0$ is related to the level density, and $W_0 = 16$ MeV is the binding energy of infinite nuclear matter; $F^S_{AZ} = B_0 A^{2/3} (T_0^2/T^2 + T_0^2/A)^{1/3}$, where $B_0 = 18$ MeV is the surface coefficient, and $T_0 = 18$ MeV is the critical temperature of infinite nuclear matter; $F^C_{AZ} = cZ^2/A^{1/3}$, where $c$ is the Coulomb parameter obtained in the Wigner-Seitz approximation, $c = (3/5)(e^2/r_0)(1 - (\rho/\rho_0)^{1/3})$, with the charge unit $e$ and $r_0 = 1.17$ fm; $F^{sym}_{AZ} = \gamma(A - 2Z)^2/A$, where $\gamma = 25$ MeV is the symmetry energy parameter.

These parameters are those of the Bethe-Weizsäcker formula and are assumed to be independent of isolated fragments with normal density in the freeze-out configuration, an assumption found to be quite successful in many applications. It is to be expected, however, that in a more realistic treatment primary fragments will have to be considered not only excited but also expanded and still subject to a residual nuclear interaction between them. These effects can be accounted for in the fragment free energies by changing the corresponding liquid-drop parameters, provided such modifications are also indicated by the experimental data. In the following, it will be shown that, for the symmetry energy, this information may be obtained from the isoscaling phenomenon.

B. Grand canonical approximation

In the grand canonical approximation, first developed in Ref. [35], the mean multiplicity of a fragment with mass number $A$ and charge $Z$ is given by

$$\langle N_{AZ}\rangle = g_{AZ} V_j A^{3/2} \exp \left[ -\frac{1}{T} (F_{AZ}(T, \rho) - \mu A - \nu Z) \right]$$

(9)

where $g_{AZ}$ is the degeneracy factor of the fragment, $\lambda_T$ is the nuclear thermal wavelength, $V_j$ is the "free" volume, and $\mu$ and $\nu$ are the chemical potentials responsible for the mass and charge conservation in the system, respectively [16]. It follows immediately that, for two systems 1 and 2 with different total mass and charge but with the same temperature and density, the ratio of fragment yields produced in these systems is given by Eq. (1) with parameters $\alpha = (\mu_1 - \mu_2)/T$ and $\beta = (\nu_1 - \nu_2)/T$. Isoscaling arises very naturally in the SMM.

Calculated chemical potentials for systems with different mass and $N/Z$ ratio as a function of the temperature are shown in Fig. 6. A freeze-out density $\rho/\rho_0 = 1/3$ has been chosen but, as apparent from the QSM calculations (Fig. 4), other densities lead to a similar behavior of the chemical potentials. Furthermore, corresponding to the excluded volume approximation [16], a fixed "free" volume $V_j = 2V_0$ ($V_0$ is the volume of the system at normal density) has been used instead of a multiplicity-dependent volume. For other parameters of the model their standard values were chosen, see e.g. Ref. [17].

The potential $\mu$ decreases with the temperature which has the simple physical meaning that the average size (mass number) of the produced fragments decreases. However, two regions with different rates of the change in fragment mass can be discerned. At low temperature, the rate is small, especially for large systems. Here the corresponding mass distribution is of the so-called "U-shape", with a compound-like fragment still dominating in the system. At temperatures near 5 to 6 MeV the rate increases rapidly. At this point, the "U-shape" disappears and the system disintegrates into many fragments with an approximately exponential mass distribution.

The behavior of the chemical potential $\nu$ is particularly interesting. As shown in Ref. [33], the average charge $\langle Z_A \rangle$ of fragments with mass $A$ can be written as

$$\langle Z_A \rangle \simeq \frac{4\gamma + \nu A}{8\gamma + 2eA^{2/3}}.$$  

(10)

The chemical potential $\nu$ is, therefore, directly connected with the isospin of the produced fragments. In grand-canonical models, as e.g. in the QSM [13], sometimes the
with temperature and pressure variations of the potential differences of the electric charges across the system’s interfaces. Depending on the concentration of the reaction, the potential difference may vary significantly.

The potential difference is a measure of the work done per unit charge to move an electron from one point to another in the electric field. It is directly proportional to the electric field strength and inversely proportional to the distance between the points.

\[ \frac{dV}{dz} = \frac{dV}{dz} \]

Where \( V \) is the electric potential, \( z \) is the distance, and \( \mu \) is the electric field strength.

In the case of a chemical reaction, the potential difference can be expressed as:

\[ \Delta V = \frac{(1 + Z \gamma)}{(1 + Z \gamma)^2} \]

Where \( Z \) is the concentration of the reactants, \( \gamma \) is the reaction constant, and \( \Delta V \) is the potential difference.

For chemical potentials to be determined, the following conditions must be satisfied:

\[ \frac{dV}{dz} = 0 \]

Where \( dV \) is the electric potential, \( dz \) is the distance, and \( d \) is the concentration.

The difference of the chemical potentials of two species can be calculated as:

\[ \Delta \mu = \frac{(1 + Z \gamma)}{(1 + Z \gamma)^2} \]

Where \( \mu \) is the chemical potential, \( Z \) is the concentration of the reactants, \( \gamma \) is the reaction constant, and \( \Delta \mu \) is the chemical potential difference.
The symmetry term in the binding energy strongly influences the potential differences $\Delta \mu$ and $\Delta \nu$. This is illustrated in Fig. 7 with examples obtained for values $\gamma_1 = 14.4$ MeV and $\gamma_2 = 8.3$ MeV, both smaller than the standard value $\gamma_0 = 25$ MeV. The effect is significant and, in particular, larger than the variations associated with the choice of ensembles or with the choice of the density and, therefore, should be observable.

As the comparison shows, the grand-canonical approach is applicable only (i) if the average mass of the largest fragment of a partition is considerably less than the total size of the system, and (ii) if the extra energy necessary for the production of an additional fragment is small compared to the available thermal energy. However, because the difference of the chemical potentials is nearly constant in the full temperature range, the values obtained in the grand-canonical approximation at low temperatures may be extrapolated to high temperatures and applied to the multifragmentation region. In the low temperature limit $T \to 0$, analytical formulae for $\Delta \mu$ and $\Delta \nu$ can be derived. Here only channels including a compound-like nucleus with $A \approx A_0$ and $Z \approx Z_0$ will exist, where $A_0$ and $Z_0$ denote the mass and atomic number of the system. Mathematically, it is required that the numerator under the exponent in Eq. (9) approaches zero, i.e.

$$F_{A_0Z_0}(T \to 0) = \mu A_0 + \nu Z_0,$$

which is equivalent to the thermodynamical potential of the compound nucleus being zero. From Eq. (10), with the same approximations, the potential

$$\nu \approx \frac{Z_0}{A_0}(8\gamma + 2cA_0^{2/3}) - 4\gamma$$

can be obtained. Inserting this expression into Eq. (13) yields the chemical potential

$$\mu \approx -W_0 + \frac{B_0}{A_0^{1/3}} - \frac{Z_0^2}{A_0^{2/3}} + \gamma(1 - \frac{2Z_0}{A_0}),$$

The terms small compared to the bulk terms can be safely disregarded (the errors are below 3% for the $^{112,124}\text{Sn}$ isotopes considered here). This leads to

$$\Delta \mu = \mu_1 - \mu_2 \approx -4\gamma(\frac{Z_1^2}{A_1} - \frac{Z_2^2}{A_2}),$$

$$\Delta \nu = \nu_1 - \nu_2 \approx 8\gamma(\frac{Z_1}{A_1} - \frac{Z_2}{A_2}),$$

where $Z_1, A_1$ and $Z_2, A_2$ are the charges and mass numbers of the two systems. The potential differences depend essentially only on the coefficient $\gamma$ of the symmetry term and on the isotopic compositions.

The values of the chemical potentials deduced in this limit are close to the separation energies of nucleons, apart from the difference in sign (see also Ref. [6]). For example, the neutron separation energy $s_n$ in the liquid-drop approximation is given by

$$s_n \approx W_0 - \gamma(1 - \frac{2Z_0}{A_0})^2 - \frac{2B_0}{3A_0^{2/3}} + \frac{c^2Z_0^2}{5r_n A_0^{2/3}}.$$  

The surface and Coulomb terms in this expression appear with different coefficients than in Eq. (15) but are, again, usually small in comparison to the dominating bulk (volume and symmetry) terms. As expected from the definitions of the chemical potential and the separation energy, this correspondence must be exact in the thermodynamical limit.

From Eqs. (16) another interesting relation can be deduced:

$$\Delta \mu \approx -\frac{\Delta \nu \gamma Z_1}{A_1} + \frac{Z_2}{A_2},$$

It implies $|\Delta \nu| > |2\Delta \mu|$ or, equivalently, $|\Delta \mu_B| > |\Delta \mu_0|$, for the usually considered systems with $A > 2Z$. Although the effects of secondary deexcitation are important (see below) this inequality is reflected by the observed scaling parameters. The magnitude of $\beta$ exceeds that of $\alpha$ in all reactions discussed in Ref. [5] and, on average, also in the reactions presented here (Table I).

D. Fragment distribution widths

There is a simple physical explanation within the SMM why isoscaling should appear in finite systems. Charge distributions of fragments with fixed mass numbers $A$, as well as mass distributions for fixed $Z$, are approximately Gaussian with average values and variances which are connected with the temperature, the symmetry coefficient, and other parameters [36]. With a Gaussian distribution for an observable $X$ (mass number or charge), $Y(X) \propto \exp(-\langle X - \langle X \rangle \rangle^2/2\sigma^2)$, the ratio of this observable for two different systems is given by

$$\frac{Y_1(X)}{Y_2(X)} = \frac{\exp(-\frac{X^2}{2}/\frac{1}{\sigma_1^2} - \frac{X_1}{\sigma_1^2} + \frac{X_2}{\sigma_2^2})}{\exp(-\frac{X^2}{2}/\frac{1}{\sigma_2^2} - \frac{X_1}{\sigma_1^2} + \frac{X_2}{\sigma_2^2}) + \text{const}},$$

where $X_1, X_2$ and $\sigma_1, \sigma_2$ are the mean values and variances for the two systems. The mean values depend on the total mass and charge of the systems, e.g. via the chemical potentials in the grand canonical approximation (Eq. (10)), while the variances depend mainly on the physical conditions reached, the temperature, the density and possibly other variables. For example, the charge variance $\sigma_Z^2 \approx \sqrt{\langle A \rangle/2\gamma}$ obtained for fragments with a given mass number $A$ in Ref. [36] is only a function of the temperature and of the symmetry term coefficient since the Coulomb contribution is very small. If these physical conditions are the same, i.e. $\sigma_1 = \sigma_2$, the exponential scaling for the ratio follows from Eq. (19). Furthermore, by using Eqs. (10) and (16) for $X = Z$, the approximate relation $\beta = \Delta \nu/T$ is again observed, as in the usual grand-canonical.
Fig. 8: Mass distributions of primary hot fragments with $Z = 6$ produced at freeze-out by $^{112}$Sn and $^{124}$Sn systems, as obtained from Markov-chain calculations for $E_{ex}/A = 5$ MeV and $\rho = \rho_{0}/3$ (the corresponding microcanonical temperature is $T_{micr} \approx 5.3$ MeV). The symmetry coefficients $\gamma = 25$ MeV (solid lines) and $\gamma = 14.4$ MeV (dashed line) were used.

The Gaussian distributions obtained in the grandcanonical approximation are reproduced by the Markovchain SMM calculations (Fig. 8). The mass distributions of fragments with $Z = 6$ emitted by $^{112}$Sn and $^{124}$Sn with $E_{ex}/A = 5$ MeV are shifted with respect to each other because the $N/Z$ ratios of the sources are different. Scaling will result, and the value of the scaling coefficient is determined by both, the shift, i.e. the difference in the mean masses, and the width of the distributions. The width, in turn, is influenced by the symmetry coefficient; with a reduced coefficient $\gamma$ the mass distribution widens considerably (Fig. 8). Thus, if the temperature is known the symmetry coefficient can, in principle, be determined using the distributions.

The calculations indicate that the secondary deexcitation reduces both, the differences between the mean values of the distributions and the magnitude of the variances, thereby attaching a considerable uncertainty to this method. However, the sensitivity to the symmetry term coefficient survives the deexcitation stage. This is illustrated in Fig. 9 which shows the SMM predictions for the ratios of isotopic yields that are obtained for the same thermal source with different coefficients $\gamma$. The characteristic bell shape of the distributions reflects the quadratic term of Eq. (19) which dominates in this case when $\sigma_1 \neq \sigma_2$.

Fig. 9: Ratios of isotopic yields calculated for different symmetry coefficients $\gamma = 25$ MeV and $\gamma = 14.4$ MeV for the breakup of a $^{112}$Sn source with $E_{ex}/A = 5$ MeV at a density $\rho = \rho_{0}/3$. The top and bottom panels give the ratios for hot and cold fragments, respectively.

E. Secondary deexcitation of fragments

In the SMM the secondary deexcitation of large fragments with $A \geq 16$ is described with Weisskopf type evaporation and Bethe-Wheeler type fission models while the decay of small fragments is treated with a Fermi-breakup model [16, 33]. In this model all ground and nucleon-stable excited states of light fragments are taken into account and the population probabilities of these states are calculated according to the available phase space. The model thus simulates a simultaneous breakup microcanonically. This procedure is expected to reliably describe a decay that happens at short time scales after the freeze-out if the excitation energy of the primary fragments is high, of the order of 2-3 AMeV or higher.

The ratios of light element yields (2 $\leq$ $Z$ $\leq$ 5) calculated with the Markov-chain SMM are shown in Fig. 10 for hot fragments produced at breakup and for cold fragments after the sequential decay. The exponential scaling with isospin is observed for both cases but with scaling coefficients that are systematically smaller for the final cold fragments (Table III). On more general grounds, it is expected that the scaling property is preserved because the excitation energies per nucleon are similar for all fragments, so that their relative nucleon content will decrease in a similar way. The secondary deexcitation has a trend, however, to populate the $\beta$-stable region which may reduce the shift between the mass distributions and also reduce their widths. A modification of the scaling coefficients is thus expected even though these two effects may partially compensate each other. In this respect,
different isotopes can behave differently. The predicted reduction of the mass widths is typically 30% for boron isotopes, i.e. significant as expected, but is practically negligible for the lithium isotopes.

According to the calculations, the coefficients $\beta_3$ and $\alpha$ are reduced to, on average, 77% of their values by the secondary decay (Table III). The coefficient $\beta$ is more strongly reduced to about 60% of its value at 2 A MeV and to about 70% at 8 A MeV. However, since the coefficients are decreasing with the energy, the absolute magnitude of this reduction decreases also. Moreover, additional calculations showed that the secondary-decay effect decreases considerably if the coefficients themselves become smaller, e.g., a primary $\alpha \approx 0.46$ is reduced to $\approx 0.44$. In this respect, we confirm the conclusions of Ref. [6] regarding the minute variation of the $\alpha$ parameter, but provided that the value of the initial $\alpha$ is relatively small. The calculations of Ref. [6] were done for $E_\beta/A = 6$ MeV with several statistical models, including other versions of the SMM. In some of these calculations, the decay procedure is based on a sequential emission of particles from primary fragments, following the tabulated branching ratios and a Weisskopf scheme. This seems adequate for a later deexcitation stage with isolated fragments at relatively low excitation energy and without the influence of a common Coulomb field and without a residual nuclear interaction which can modify fragment properties including the branchings. The obtained modifications of the scaling parameter $\alpha$ do not exceed the order of 5% whereas $\beta$ is reduced more strongly, similar to the present case.

The primary values of the scaling parameter $\alpha \approx 0.4$ to 0.45 reported in Ref. [6] for the $^{112,124}$Sn systems are smaller than the corresponding values given in Table III. This, apparently, reflects significant differences between different versions and parameterizations of, in principle, the same model. They are of the same order as potential effects of the symmetry term that are to be studied. This emphasizes the need for exclusive analyses of experimental data which should constrain the model parameters. The two secondary deexcitation procedures should be considered as partly complementary, and the range of the differences of the obtained results may characterize the reliability of treating secondary decays with model calculations. These corrections are essential [30], but it will be important to reduce the uncertainties. Experimental methods, e.g., based on correlation techniques [40], may prove very useful for this purpose.

**F. Interpretation of the data**

The deduced relations will now be used for the interpretation of the experimental data. We will concentrate on the two reactions initiated by the projectiles with the highest energies, protons of 6.7 GeV and $^\alpha$ particles of 15.3 GeV, for which the contributions from instantaneous breakups into multifragment channels should be enhanced in comparison to the other cases. The inclusive nature of the measurements, nevertheless, presents an inherent difficulty since a wide range of excitation energies is covered by the fragment emitting sources. The $^{112,124}$Sn targets used in these experiments were isotopically enriched to 81.7% and 96.6%, respectively [10, 12]. The effects of the impurities, known to be distributed approximately as the natural abundances of tin isotopes, have to be taken into account in a quantitative analysis. Corrections were estimated by assuming Gaussian mass distributions for the produced fragments,
centered around mean values that vary linearly with the mass number of the considered tin isotopes. It was found that, for the specific enrichments of the used targets and for scaling coefficients $\alpha$ in the range 0.3 to 0.6, the impurities cause a reduction of the measured $\alpha$ by 10% to 15%.

The analytical expressions for the differences of the chemical potentials, derived in the grand-canonical approximation (Eqs. (16)), depend only on $\gamma$ and the isotopic composition of the sources. In the case of $\Delta \mu$, the difference of the squared $Z/A$ values is required which is found to be the same within a few percent, independently of whether it is evaluated for the original targets $^{112,124}$Sn or for the excited systems as predicted by the INC calculations (Fig. 3, Section III). For the original targets it amounts to $(Z_1/A_1)^2 - (Z_2/A_2)^2 = 0.0367$, leading to $\gamma = \Delta \mu / 0.147$. To obtain an experimental value of $\Delta \mu = \alpha \cdot T$ (Section VII), the mean values of the scaling coefficient $\alpha$ and of the isotope temperature $T_{He68/Li68}$ for the $p(6.7 \text{ GeV})$ and $\alpha(15.3 \text{ GeV})$ reactions are used, after applying corrections. A measured $\langle \alpha \rangle = 0.365$ is obtained from Table I, corresponding to 0.417 for isotopically pure targets, and the effect of the secondary deexcitation is assumed to be 23%, as suggested by the Markov-chain calculations (Table III), thus leading to a primary $\alpha = 0.542$.

The predictions of the QSM [13] are used for the correction of the temperature. It does not significantly depend on the assumed density but it is large, as expected. The mean apparent temperature $T_{He68/Li68} = 4.35 \text{ MeV}$ (Table II) corresponds to a breakup temperature $T = 6.2 \text{ MeV}$ in this model. The results obtained with these inputs are $\Delta \mu = 3.36 \text{ MeV}$ and $\gamma = 22.8 \text{ MeV}$, a symmetry coefficient slightly but not significantly smaller than the adopted standard value of 25 MeV.

For the interpretation of the isoscaling coefficient in the microcanonical limit the excitation energy needs to be specified. Exclusive data for hadron-induced reactions on Au targets indicate that fragments will be emitted if energies exceeding $\approx 400 \text{ MeV}$, corresponding to $E_x/A \approx 2 \text{ MeV}$, are deposited by the projectile [41, 42]. Since the cross sections decrease and the fragment emission probabilities increase with excitation energy, a rather wide distribution results. For the $\pi^+$ projectile of 8 GeV/c studied by the ISIS collaboration, the distribution extends from below 3 to above 8 MeV per nucleon with a weighted mean value of $E_x/A \approx 5 \text{ MeV}$ [41, 42]. A similar or, because of the lighter targets, a slightly higher value may be expected for the case of protons of 6.7 GeV on $^{112,124}$Sn. The INC calculations for this reaction, again weighted by the fragment production cross section, predict an average excitation energy $E_x/A = 6.2 \text{ MeV}$. With this interval 5.0 MeV to 0.2 MeV per nucleon for the excitation energy, and with the assumption that $\alpha \propto \gamma$ as in the grand-canonical approximation, values between $\gamma = 21.4 \text{ MeV}$ and 22.6 MeV are obtained from the comparison of the measured $\alpha = 0.39$ ($\alpha = 0.45$ for pure targets) with the predictions given in Table III for which the standard value $\gamma = 25 \text{ MeV}$ was used. If $\langle E_x/A \rangle = 8 \text{ MeV}$ is considered as realistic for $\alpha(15.3 \text{ GeV})$ a similar symmetry coefficient $\gamma = 21.6 \text{ MeV}$ will result.

Towards the lower projectile energies, the isoscaling coefficient $\alpha$ increases up to 0.53, corresponding to 0.61 for pure targets, which is still lower than the SMM predictions for small excitation energies (Table III). With the INC result $\langle E_x/A \rangle = 2.7 \text{ MeV}$ for protons of 660 MeV, the interpolated prediction is $\alpha = 0.65$, and $\gamma = 23.3 \text{ MeV}$ is obtained from the comparison with the measured value. It thus seems that, for the reactions studied here, the deduced values of $\gamma$ fall consistently into the range of 21 to about 23 MeV, with no significant dependence on the energy. In this respect, however, it has to be considered that the constraint of energy conservation in the microcanonical calculations may lead to unrealistically narrow widths of the isotope distributions at low excitation energies. This would cause an overprediction of the scaling coefficients and a deduced $\gamma$ that is too low. This effect will bring $\gamma$ even closer to the standard value for the reactions at lower incident energies which primarily proceed via the formation of excited compound nuclei.

VII. SUMMARY AND CONCLUSIONS

In the first part of this paper, the existence of isoscaling for reactions induced by relativistic light particles was demonstrated. The deduced exponents vary smoothly with the incident energy. Their trends, apparently, extend beyond the range studied here to low-energy projectiles as, e.g., $\alpha$ particles of 200 MeV for which isoscaling parameters were reported in Ref. [5]. The values obtained for protons of 6.7 GeV and $\alpha$ projectiles of 15.3 GeV are close to those for central $^{112,124}$Sn + $^{112,124}$Sn reactions at 50 MeV per nucleon given in the same reference. The observation of $\gamma$ scaling was illustrated and discussed. As a function of the projectile energy, a very similar variation of the inverse scaling parameters and of the isotope temperature $T_{He68/Li68}$ was observed.

In the second part, a statistical formalism for the interpretation of the isoscaling phenomenon was developed. Analytical expressions were derived in the grand-canonical approximation and their validity and applicability illustrated. Results of calculations in the grand-canonical approximation and with the microcanonical Markov-chain version of the SMM were presented and the connection with the symmetry term of the fragment binding energy was established. It was found that the difference of the chemical potentials for the two isotopically different systems does not depend on the temperature. For the Markov-chain calculations, this conclusion is valid for temperatures $T \geq 5 \text{ MeV}$, the range of relevance for multifragment processes. The invariance of $\Delta \mu$ with temperature is consistent with the interpretation that the observed variation of the scaling parameters is caused by a change in temperature, as suggested by the
In the last part (Section VI F), an attempt was made
to deduce values for the symmetry-energy coefficient $\gamma$
from the experimental data. The analytical formulae
derived in the grand-canonical limit of the SMM and
the results of the microcanonical calculations were used and
very similar values in the range $\gamma = 22.5 \pm 1$ MeV were
obtained. Besides the scaling coefficient, experimental
values for either the breakup temperature in the grand-
canonical or for the excitation energy in the microcan-
onical approach were required. In the latter case, estimates
obtained for similar reactions and from INC calculations
were used.

We estimate the uncertainties of the methods, in par-
ticular the errors associated with the determinations of
the breakup temperature or of the excitation energy for
the microcanonical method, to be at least of the same
order as the deviations of the results from the standard
value $\gamma = 25$ MeV. The sequential decay corrections are
substantial and, e.g., in the grand-canonical case are re-
quired twice, for the scaling coefficient and for the tem-
perature. The present results, therefore, do not con-
dict the assumptions made in the statistical multifrag-
mentation model using standard liquid-drop parameters
for describing the nascent fragments at the breakup
stage.

A problem associated with the present data is the wide
range of excitation energies over which an average is
taken in the inclusive measurements. Smaller variations
may be smeared out. For these reasons, the presented
analysis is primarily intended to serve as an example of
how to extract the symmetry-energy coefficient $\gamma$
from the experimental data. It is, nevertheless, of interest
that the obtained result for fragmentation reactions in-
duced by relativistic light projectiles has a tendency to
be smaller than the conventional value of 25 MeV. A re-
duction with increasing energy may even be suggested by
the microcanonical analysis. Provided it can be substanti-
tated by other data and analyses, this would indicate that
the symmetry part of the fragment binding energy is
slightly weaker than that of isolated nuclei. Fragments,
as they are formed at breakup, may have a lower than
normal density. Such effects may be enhanced as the en-
ergy deposited in the fragmenting system is increased.
Therefore, exclusive studies with possibly heavier pro-
jectiles will be required to more clearly identify potential
variations of the symmetry energy with the reaction
parameters.

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