Classical and Quantum-like approaches to Charged-Particle Fluids in a Quadrupole

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Abstract

A classical description of the dynamics of a dissipative charged-particle fluid in a quadrupole-like device is developed. It is shown that the set of the classical fluid equations contains the same information as a complex function satisfying a Schrödinger-like equation in which Planck’s constant is replaced by the time-varying emittance, which is related to the time-varying temperature of the fluid. The squared modulus and the gradient of the phase of this complex function are proportional to the fluid density and to the current velocity, respectively. Within this framework, the dynamics of an electron bunch in a storage ring in the presence of radiation damping and quantum-excitation is recovered. Furthermore, both standard and generalized (including dissipation) coherent states that may be associated with the classical particle fluids are fully described in terms of the above formalism.

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1 Introduction

The dynamics of a charged-particle system in a quadrupole-like device concerns with a number of problems in accelerator physics [1, 2], in plasma physics [3], and in the physics of electromagnetic (e.m.) traps [4]. In particular, charged-particles execute synchrotron oscillations in radio-frequency (RF) fields; then, the system can be suitably bunched as a result of the bucket formation [5]. When oscillations are sufficiently small, the synchrotron motion can be described as being in a harmonic-like potential well whose strength is proportional to the RF voltage amplitude [5]. Typically, this strength is a slow function of time compared to the RF period.

Sometimes, during the above dynamics in a RF field, the phenomenon of dissipation due to the e.m. radiation emission cannot be neglected. However, it allows the system to reach a steady state.

In the combined traps for positrons (electrons) and antiprotons (protons), such a kind of dissipation enables the system to cool and condense, and consequently to make possible the anti-hydrogen (hydrogen) atom formation [6].

In the circular accelerating machines, the radiation damping, in competition with the quantum excitation (photon noise), leads the system to an asymptotic equilibrium state, which typically corresponds to a Gaussian particle distribution in both configuration space and momentum space, with a given final temperature (or final emittance) [1]. This final steady state can be considered as a sort of coherent state reached by the system which does not depend on its history [1]. In fact, it can be determined a priori in terms of both the bunch and the machine parameters only [1]. It is worth to note that this phenomenon of dissipation does not always lead the bunch to cool. In fact, the combined effect of radiation damping and quantum excitation can make possible the warming up of the bunch. It happens when the initial temperature (initial emittance) of the bunch is smaller than the final one.

In this paper, we develop a fluid description for a charged-particle system in a quadrupole-like potential well in the presence of dissipations in both classical-like and quantum-like domains. In particular, a generalized class of coherent states are shown to be possible, within this fluid framework, in the presence of dissipation. It is shown that coherent states described in the classical domain are equivalent to the ones described in the quantum-like domain.

2 Hydrodynamical description of coherent states for charged-particle beams

In this section, following a recent hydrodynamic approach applied to the radiation fluids [8], we start to consider a classical-like description of coherent states for a dilute charged-particle fluid, for which the ideal-gas-state equation is assumed. For the sake of simplicity, we consider the 1-D motion fluid motion.
equation of a dilute charged-particle beam. Accordingly, we have

\[
\left( \frac{\partial}{\partial s} + P \frac{\partial}{\partial x} \right) P = -\frac{\partial U}{\partial x} - \frac{1}{n} \frac{\partial \Pi}{\partial x},
\]

(1)

\[
\frac{\partial n}{\partial s} + \frac{\partial}{\partial x} (nP) = 0,
\]

(2)

where \( s = ct \) (\( c \), being the speed of light), \( P = P(x, s) \) is the current velocity, \( n = n(x, s) \) is the particle number density, and the quantity \( U = U(x, s) \) is a dimensionless effective potential acting on the system. Assuming the ideal gas state equation

\[
\Pi = \frac{n k_B T}{m c^2},
\]

(3)

where \( \Pi \) is the thermal pressure, normalized with respect to \( mc^2 \), \( k_B \) is the Boltzmann constant, \( m \) is the particle mass, and \( T = T(s) \) is the temperature of the system, we have

\[
\frac{\partial \Pi}{\partial x} = \frac{\partial \Pi}{\partial n} \frac{\partial n}{\partial x} = \eta \frac{\partial n}{\partial x},
\]

(4)

with the definition

\[
\eta \equiv \frac{\partial \Pi}{\partial n}.
\]

(5)

Note that, for an isothermal transformation,

\[
\eta = \frac{k_B T}{m c^2} = \frac{v_{th}^2}{c^2} = \text{const.},
\]

(6)

where \( v_{th} = \sqrt{k_B T/m} \) is the thermal velocity. In general, the explicit expression of \( \eta \) depends on the particular thermodynamical transformation that the system undergoes, but it is related someway to the r.m.s. of momentum-space distribution \( \sigma_P \). Let us assume, in the following, that \( P \) and \( \eta \) are functions of \( s \) only, namely,

\[
P(x, s) = P_0(s)
\]

(7)

and

\[
\eta(x, s) = \eta_0(s).
\]

(8)

Consequently, (1) and (2) become

\[
P_0'(s) = -\frac{\partial U}{\partial x} - \eta_0(s) \frac{\partial}{\partial x} \ln n,
\]

(9)

\[
\frac{\partial n}{\partial s} + P_0(s) \frac{\partial n}{\partial x} = 0,
\]

(10)
where the prime denotes differentiation with respect to \( s \). Thus, (9) can be easily integrated with respect to \( x \), giving

\[
n(x,s) = \exp\left\{-\frac{1}{\eta_0(s)} \left[ U(x,s) + P_0'(s)x + g(s) \right] \right\},
\]

(11)

where \( g(s) \) is an arbitrary function of \( s \). By substituting (11) in (10), we obtain

\[
\frac{\eta_0}{\eta_0} \left( U + P_0' x + g \right) - \left( \frac{\partial U}{\partial s} + P_0'' x + g \right) - P_0 \left( \frac{\partial U}{\partial x} + P_0' \right) = 0.
\]

(12)

In order to consider the special case of coherent state associated with the beam, let us assume that the potential \( U(x,s) \) is given by

\[
U(x,s) = \frac{1}{2} K(s) x^2.
\]

(13)

Equation (13) defines a quadrupole-like potential well with time-varying strength \( K(s) \). Substituting (13) in (12) we get the following conditions for the quantities \( \eta_0, P_0, K, \) and \( g \):

\[
\frac{\eta_0'}{\eta_0} = \frac{K'}{K},
\]

(14)

\[
P_0'' - \frac{\eta_0}{\eta_0} P_0' + K P_0 = 0,
\]

(15)

\[
d\left[ \frac{1}{2} P_0'^2 + g \right] = \frac{\eta_0'}{\eta_0} g.
\]

(16)

From (14), we obtain

\[
\beta = \frac{\eta_0(s)}{K(s)} = \text{const.}
\]

(17)

On the other hand, substituting (13) and (17) into (11), we have

\[
n(x,s) = \exp\left\{-\frac{(x - x_0(s))^2}{2\beta} \right\},
\]

(18)

where

\[
x_0(s) = -\frac{2g(s)}{P_0'(s)},
\]

(19)

and

\[
g(s) = \frac{1}{2} \frac{\eta_0(s)}{\beta} x_0^2(s) = \frac{1}{2} K(s) x_0^2(s).
\]

(20)

By combining (19) and (20), we find

\[
P_0'(s) = -K(s)x_0(s).
\]

(21)
Furthermore, substituting (18) in (10), we obtain
\[ P_0(s) = x'_0(s) \quad . \] (22)

Combining the results given by (14)–(22), we finally obtain
\[ x''_0 + K(s)x_0 = 0 \quad , \] (23)
\[ P''_0 - \Gamma(s)P'_0 + K(s)P_0 = 0 \quad , \] (24)
and
\[ \frac{d}{ds} \left[ \frac{1}{2}P_0^2 + \frac{1}{2}Kx_0^2 \right] = \Gamma(s) \left( \frac{1}{2}Kx_0^2 \right) \quad , \] (25)
where
\[ \Gamma(s) \equiv \frac{\eta'_0}{\eta_0} = \frac{K'}{K} \quad . \] (26)

In conclusion, a normalized density solution of the system of Eqs. (9) and (10) associated with potential (13) is
\[ n(x, s) = \frac{1}{\sqrt{2\pi\beta}} \exp \left\{ -\frac{[x - x_0(s)]^2}{2\beta} \right\} \quad . \] (27)

From (27), it is clear that the beam size (r.m.s.) \( \sigma_0 \) is
\[ \sigma_0 \equiv \langle (x - x_0)^2 \rangle^{1/2} = \sqrt{\beta} = \text{const.} \quad , \] (28)
which implies that \( \beta \) must be positive. It is clear that the Gaussian solution (27) with (23) and the positivity of \( \beta \) may be a coherent state associated with the charged-particle beam in a quadrupole-like potential well with also a time-varying strength \( K(s) \). But, in this case, (24)–(26) show that this coherent state exists for a dissipative system, otherwise \( K \) must be constant. In fact, the \( s \)-variation of \( \eta_0 \) implies that the system, in principle, is not closed and can exchange its energy with the environment (see Eqs. (23)–(25)). This way, the distribution of the particles around the center remains unchanged (see Eqs. (27) and (28)) and corresponds to the coherent-beam configuration, for which the following condition holds:
\[ \eta_0(s) = \sigma_0^2 K(s) \quad . \] (29)

Let us now consider, as a special case, an isothermal transformation of the beam, for which \( \eta_0 \) is constant. This is equivalent to say that the beam emittance \( \epsilon \) [9] is constant too
\[ \frac{\epsilon^2}{4\sigma_0^2} = \frac{\eta_0^2}{\epsilon^2} = \text{const.} \] (30)
According to (14) and (15), it is clear that a coherent state exists, in this case, if and only if, $K$ is constant. Moreover, (23)–(25) become

$$x_0'' + Kx_0 = 0 \ ,$$

$$P_0'' + KP_0 = 0 \ ,$$

and

$$\frac{1}{2}P_0^2 + \frac{1}{2}Kx_0^2 \equiv E_0 = \text{const.}$$

Additionally, (29) becomes

$$K\sigma_0^4 = \frac{\epsilon^2}{4} \ .$$

In this case, (27) with (31)–(34) represent a coherent structure which preserves both the energy and the beam emittance. We may refer to this kind of coherent states as ordinary or isothermal coherent states. Furthermore, (26) can be written as

$$\Gamma(s) = \frac{1}{\epsilon^2} \frac{de^2}{ds} \ .$$

Let us now observe that, on the basis of (22), (23), (27), and (28), one can introduce the following complex function

$$\Psi(x, s) = \sqrt{n(x, s)} \exp i\theta(x, s) \ ,$$

where $\theta(x, s)$ is defined by

$$P_0(s) = \epsilon(s) \frac{\partial \theta(x, s)}{\partial x} \ .$$

Consequently, (36) can be explicitly written as

$$\Psi(x, s) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp \left[ -\frac{[x - x_0(s)]^2}{4\sigma_0^2} + \frac{iP_0(s)x}{2\epsilon(s)} + i\phi_0(s) \right] \ ,$$

where $\phi_0(s)$ is an arbitrary function of $s$. It is worth to note that the above complex function $\Psi$ contains the same information as the system (22), (23), (27), and (28). However, Eq. (38) recovers a coherent state (similar to the ones in the ordinary quantum mechanics [10]), when $\epsilon$ is assumed to be constant (non-dissipative system).

The constant $\epsilon$ plays the similar role as $\hbar$ in quantum mechanics. Consequently, in this case, the complex function defined by Eq. (38) satisfies a Schrödinger-like equation for a harmonic oscillator potential $U = Kx^2/2$, where $K$ is assumed to be constant, as well. We point out that the above isothermal coherent states coincide with the coherent states for charged-particle beam that have been recently described, within a wave-like context, in Ref.s [7].
Additionally, when \( \epsilon \) and \( K \) depend both on \( s \), it has already been proven that Eq. (38) still describes a quantum-like coherent state, provided that Eq. (30) is satisfied [11, 12]. The coherent states of a driven oscillator with dissipation within the framework of ordinary quantum mechanics were constructed in [13].

3 Quantum-like description

On the basis of the results given in the previous section, we extend the above correspondence between the classical and the quantum-like description [14] of the fluid motion, beyond the coherent state assumption. To this regard, we assume, in the absence of collective effects, that the dynamics of our system is governed by the following Schrödinger-like equation:

\[
i\alpha \frac{\partial \psi}{\partial s} = -\frac{\alpha^2}{2} \frac{\partial^2 \psi}{\partial x^2} + U(x, s)\psi ,
\]

where \( \alpha = \alpha(s) \) plays the role of a dispersion parameter and \( U, x \) and \( s \) have the same meaning as in the previous section. We show that the above Schrödinger-like equation is equivalent, under certain conditions, to the classical fluid system of equations (1) and (2). In fact, if we write

\[
\psi(x, s) = M(x, s) \exp i\varphi(x, s) ,
\]

and if we substitute (40) back into (39) we can easily derive the following dissipative Madelung-like fluid equations, namely,

\[
\left( \frac{\partial}{\partial s} + P \frac{\partial}{\partial x} \right) P = -\frac{\partial U}{\partial x} + \frac{\alpha'}{\alpha} P + \frac{\alpha^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{M} \frac{\partial^2 M}{\partial x^2} \right) ,
\]

\[
\frac{\partial M^2}{\partial s} + \frac{\partial}{\partial x} \left( M^2 P \right) = 0 ,
\]

where

\[
P = \alpha \frac{\partial \varphi}{\partial x} .
\]

Note that we can define the fluid density

\[
n(x, s) = |\psi(x, s)|^2 = M^2(x, s) .
\]

Consequently, under the following condition

\[
\frac{\alpha'}{\alpha} P + \frac{\eta_0}{n} \frac{\partial n}{\partial x} + \frac{\alpha^2}{2} \frac{\partial}{\partial x} \left( \frac{1}{M} \frac{\partial^2 M}{\partial x^2} \right) = 0 ,
\]
(41) reduces to the following classical-like form

\[
\left( \frac{\partial}{\partial s} + P \frac{\partial}{\partial x} \right) P = - \frac{\partial U}{\partial x} - \frac{\eta_0}{n} \frac{\partial n}{\partial x},
\]

where \( \eta_0 = \eta_0(s) \) has been already defined above. It is clear that (42) and (46) together with the quantum-like interpretation (44) formally coincide with our starting classical system as given by (1) and (2). Now we show that the classical-like solution for the dissipative Schrödinger-like equation (39) satisfying (45) can effectively be determined in the case of a quadrupole-like potential, i.e., \( U = K(s)x^2/2 \), where \( K(s) \) is the quadrupole strength. Indeed, in this case, Eq. (39) admits the following solution:

\[
\psi = \frac{1}{\sqrt{2\pi \sigma^2(s)}} \exp \left[ - \frac{x^2}{4\sigma^2(s)} + \frac{ix^2}{2\alpha(s)\rho(s)} + i\chi(s) \right].
\]

From (47) and (43), we obtain the following expression for the current velocity:

\[
P(x, s) = - \frac{x}{\rho(s)}. \tag{48}
\]

In (47) and (48), \( \sigma(s), \rho(s), \) and \( \chi(s) \) are real functions satisfying the following set of differential equations:

\[
\frac{1}{\rho} = \frac{1}{\sigma} \frac{d\sigma}{ds}, \tag{49}
\]

\[
\frac{d\chi}{ds} = - \frac{\alpha(s)}{4\sigma^2(s)}, \tag{50}
\]

\[
\frac{d^2\sigma}{ds^2} + K(s)\sigma - \frac{1}{\alpha} \frac{d\alpha}{ds} \frac{d\sigma}{ds} - \frac{\alpha^2}{4\sigma^2} = 0. \tag{51}
\]

Up to this point, the function \( \alpha(s) \) is quite arbitrary within a purely quantum-like context. However, we point out that, by substituting (47) into (45), in view of (40), (43), and (44), the previous equations (48)--(51) are exactly obtained, provided that the following condition for \( \alpha(s) \) is satisfied:

\[
\eta_0(s) = \sigma \frac{d\alpha}{ds} + \frac{\alpha^2}{4\sigma^2}. \tag{52}
\]

This condition clearly shows that \( \alpha(s) \) is essentially determined by the temperature \( T(s) \) of the fluid through \( \eta_0(s) \) (see Eqs. (3), (5), and (8)). On the other hand, within the quantum-like framework, the r.m.s of the momentum distribution \( \sigma_P \) is defined as

\[
\sigma_P(s) = \alpha \left[ \int_{-\infty}^{+\infty} \left| \frac{\partial\psi(x, s)}{\partial x} \right|^2 dx \right]^{1/2} = \left[ \left( \frac{d\sigma}{ds} \right)^2 + \frac{\alpha^2}{4\sigma^2} \right]^{1/2}. \tag{53}
\]
Note that, consistently with the quantum-like formalism, the above definition of the r.m.s. of the momentum distribution, suggests to make the following assumption:

\[ \eta_0(s) = \sigma_P^2(s) \]  (54)

On the other hand, in the classical-like interpretation, \( \sigma_P(s) \) is r.m.s. of a Maxwellian-like (Gaussian) distribution in the momentum space and, consequently, it is proportional to the square root of the temperature of the system (see Eq. (46)). By inserting (52) and (53) into (54), we obtain the following condition:

\[ \frac{1}{\alpha} \frac{d\alpha}{ds} = \frac{1}{\sigma} \frac{d\sigma}{ds} . \]  (55)

### 4 Connection between \( \alpha \) and the beam emittance \( \epsilon \)

Within the classical framework, it is well known that the beam emittance \( \epsilon \) can be obtained by the relation [9]

\[ \epsilon^2 = \langle x^2 \rangle \langle p^2 \rangle - \langle xp \rangle^2 , \]  (56)

where \( \langle x^2 \rangle = \sigma^2 \), \( \langle p^2 \rangle = \sigma_P^2 \), and \( \langle xp \rangle^2 = \sigma^2 (d\sigma/ds)^2 \); the averages are taken with respect to the classical phase-space Gaussian distribution whose configuration projection coincides with \( |\psi|^2 \). Taking into account the above relations, (56) can be written as

\[ \sigma_P^2 = \left( \frac{d\sigma}{ds} \right)^2 + \frac{\epsilon^2}{4\sigma^2} . \]  (57)

Consequently, by comparing (57) and (53), we obtain the following equality:

\[ \epsilon(s) = \alpha(s) , \]  (58)

and the envelope equation (51) becomes

\[ \frac{d^2\sigma}{ds^2} + K(s)\sigma - \left( \frac{1}{\epsilon} \frac{d\epsilon}{ds} \right) \frac{d\sigma}{ds} - \frac{\epsilon^2}{4\sigma^2} , \]  (59)

which coincides with the envelope equation obtained in Ref. [11].

### 5 Conclusions and remarks

We have proven that a dissipative classical fluid, moving in a quadrupole-like focusing device, can be fully described in terms of a Schrödinger-like equation for harmonic oscillator where the Planck’s constant is replaced by the time-varying
beam emittance. This result justifies the main assumption of Refs. [11, 12] where the longitudinal dynamics of an electron bunch in a storage ring in the presence of radiation damping and quantum excitation has been described by Eqs. (39) and (58). We point out that coherent state, in the dissipative case, are recovered from Eq. (59) in the limit of $1/\rho \to 0$, namely, $d\sigma/ds = 0$ for any $s$ ($\sigma = \sigma_0$ for any $s$), to give Eq. (29). Furthermore, in this case, Eq. (57) reduces to the minimum uncertainty relation condition, which is a typical feature of coherent states. Remarkably, note that, in the absence of dissipation ($\epsilon = \text{const.}$), all the results of Ref. [7], concerning with coherent states of charged-particle beams, are fully recovered by the present treatment, provided that $K$ is assumed constant, as well. Finally, we point out that the fluid treatment presented in this paper can be also applied to the e.m. traps [15] with the inclusion of the dissipation.

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References


