Innermost circular orbit of binary black holes
at the third post-Newtonian approximation

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Abstract

The equations of motion of two point masses have recently been derived at the 3PN approximation of general relativity. From that work we determine the location of the innermost circular orbit or ICO, defined by the minimum of the binary’s 3PN energy for circular orbits, as a function of the orbital frequency. We compare the result with a recent numerical calculation of the ICO in the case of two black holes moving on exactly circular orbits (helicoidal symmetry). The agreement is remarkably good, indicating that the 3PN approximation is adequate to locate the ICO of two black holes with comparable masses. This conclusion is reached with the post-Newtonian expansion expressed in the standard Taylor form, without using resummation techniques such as Padé approximants and/or one-body-effective methods.
The aim of this Letter is to compute the innermost circular orbit (ICO) of point-particle binaries in post-Newtonian approximations, and to compare the result with numerical simulations. For the present purpose, the ICO is defined as the minimum, when it exists, of the binary’s energy function $E(\Omega)$ of circular orbits, where $\Omega$ denotes the orbital frequency. The definition is motivated by our comparison with the numerical work, because this is precisely that minimum which is computed numerically\(^1\). The energy function is given by the invariant quantity associated with the conservative part of the post-Newtonian dynamics, i.e. ignoring the radiation reaction effects. It can be argued, because the radiation reaction damping is neglected, that the importance of the ICO does not lie so much on its strong physical significance, but on the fact that it represents a very useful reference point on the definition of which the post-Newtonian and numerical methods agree.

The question of the conservative dynamics of compact binary systems has been resolved in recent years at the third post-Newtonian (3PN) approximation, corresponding to the order $1/c^6$ beyond the Newtonian force. After the previous work of Ref. [1], Jaranowski and Schäfer [2], and Damour, Jaranowski and Schäfer [3], have applied at 3PN order the ADM-Hamiltonian formalism of general relativity. On the other hand, extending the method of Ref. [4], Blanchet and Faye [5,6] performed a 3PN iteration of the equations of motion (instead of a Hamiltonian) in harmonic coordinates. In the latter approaches, the compact objects are modelled by point particles, described solely by two mass parameters $m_1$ and $m_2$. The end results are physically equivalent in the sense that there exists a unique transformation of the particle’s dynamical variables that changes the 3PN harmonic-coordinates Lagrangian of de Andrade, Blanchet and Faye [7] into another Lagrangian, whose Legendre transform is identical with the 3PN ADM-coordinates Hamiltonian of Damour, Jaranowski and Schäfer [3].

Post-Newtonian computations of the motion of point particles face the problem of the reg-

\(^1\)In particular, we do not define the ICO as a point of dynamical unstability.
ularization of the infinite self-field of the particles. The regularization scheme of Hadamard, in “standard” form, was originally adopted in the ADM-Hamiltonian approach [2]. Then an “improved” version of this regularization was defined in Refs. [6] and applied to the computation of the harmonic-coordinates equations of motion [5]. Unfortunately, it has been shown that the Hadamard regularization, either in standard or improved form, leaves unspecified one and only one numerical coefficient in the 3PN equations of motion, $\omega_{\text{static}}$ in the ADM-Hamiltonian approach, $\lambda$ in the harmonic-coordinates formalism. The parameter $\omega_{\text{static}}$ can be seen as due to some “ambiguity” of the standard Hadamard regularization, while $\lambda$ appears rather like a parameter of “incompleteness” in the improved version of this regularization. However, these constants turned out to be equivalent, in the sense that [5,3,7]

$$\lambda = -\frac{3}{11} \omega_{\text{static}} - \frac{1987}{3080}. \quad (1)$$

It has been argued in Ref. [8] that the numerical value of $\omega_{\text{static}}$ could be $\simeq -9$, because for such a value some different “resummation” techniques, when they are implemented at the 3PN order, give approximately the same numerical result for the ICO. Even more, it was suggested [8] that $\omega_{\text{static}}$ might be precisely equal to $\omega^*_{\text{static}} = -\frac{47}{3} + \frac{41}{64} \pi^2 \simeq -9.34$ (corresponding to $\lambda^* \simeq 1.90$). But, more recently, a computation of $\omega_{\text{static}}$ has been performed by means of a dimensional regularization, instead of the Hadamard regularization, within the ADM-Hamiltonian formalism [9], with result

$$\omega_{\text{static}} = 0 \iff \lambda = -\frac{1987}{3080} \simeq -0.64. \quad (2)$$

We adopt in this Letter the latter value as our preferred one, but in fact it is convenient to keep the ambiguity parameter unspecified, and to investigate the behaviour of the solutions for different values of $\lambda$ and $\omega_{\text{static}}$. For instance, we shall keep an eye on the values $\omega^*_{\text{static}} \simeq -9.34$ and also $\lambda = 0 \iff \omega_{\text{static}} \simeq -2.37$. The latter case corresponds to the special instance where certain logarithmic constants associated with the Hadamard regularization
in harmonic coordinates do not depend on the masses [5]. Notice that the result (2) is quite different from \( \omega^*_{\text{static}} \simeq -9.34 \) : this suggests, according to Ref. [8], that different resummation techniques, \( \text{viz} \) Padé approximants [10] and effective-one-body methods [11], which are designed to “accelerate” the convergence of the post-Newtonian series, do not in fact converge toward the same “exact” solution.

Let us now compute the ICO of two point particles (modelling black holes) at the 3PN order thanks to the previous body of works [1–7]. The circular-orbit energy \( E \), and angular-momentum \( J \), are deduced either from the 3PN harmonic-coordinates Lagrangian [7] or, equivalently, from the 3PN ADM-coordinates Hamiltonian [3] (we neglect the 2.5PN radiation damping). These functions are expressed in invariant form (the same in different coordinate systems), i.e. with the help of the angular orbital frequency \( \Omega \) of the circular orbit. The 3PN energy, per unit of total mass \( M \), is

\[
\frac{E(\Omega)}{M} = -\frac{\nu}{2} (M \Omega)^{2/3} \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) (M \Omega)^{2/3} + \left( -\frac{27}{8} + \frac{19}{8} \nu - \frac{\nu^2}{24} \right) (M \Omega)^{4/3} \\
+ \left( -\frac{675}{64} + \left[ \frac{209323}{4032} - \frac{205}{96} \pi^2 - \frac{110}{9} \lambda \right] \nu - \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right) (M \Omega)^2 \right\} .
\]

All over this paper we pose \( G = c = 1 \). Mass parameters are \( M = m_1 + m_2 \), and the symmetric mass ratio \( \nu = m_1 m_2/M^2 \) such that \( 0 < \nu \leq \frac{1}{4} \), with \( \nu = \frac{1}{4} \) in the equal-mass case and \( \nu \to 0 \) in the test-mass limit for one of the bodies. The 3PN angular momentum, scaled by \( M^2 \), reads

\[
\frac{J(\Omega)}{M^2} = \nu (M \Omega)^{-1/3} \left\{ 1 + \left( \frac{3}{2} + \frac{\nu}{6} \right) (M \Omega)^{2/3} + \left( \frac{27}{8} - \frac{19}{8} \nu + \frac{\nu^2}{24} \right) (M \Omega)^{4/3} \\
+ \left( \frac{135}{16} + \left[ \frac{209323}{5040} + \frac{41}{24} \pi^2 + \frac{88}{9} \lambda \right] \nu + \frac{31}{24} \nu^2 + \frac{7}{1296} \nu^3 \right) (M \Omega)^2 \right\} .
\]

The variations of the energy and angular momentum of the binary on the circular orbit during the inspiral phase obey the evolutionary (or “thermodynamic”) law

\[
\frac{dE}{d\Omega} = \Omega \frac{dJ}{d\Omega} ,
\]

(5)
which is equivalent, via the energy and angular-momentum balance equations, to the same relation but between the corresponding gravitational-wave fluxes at infinity. From Eq. (5), we see that the points of extremum for $E$ and $J$ are the same. In the limit $\nu \to 0$, Eqs. (3)-(4) reduce to the 3PN approximations of the known energy and angular momentum of a test particle in the Schwarzschild background:

$$
\frac{E^{\text{Sch}}(\Omega)}{M} = \nu \left\{ (1 - 2(M\Omega)^{2/3}) \left(1 - 3(M\Omega)^{2/3}\right)^{-1/2} - 1 \right\}, \quad (6a)$$

$$
\frac{J^{\text{Sch}}(\Omega)}{M^2} = \nu (M\Omega)^{-1/3} \left(1 - 3(M\Omega)^{2/3}\right)^{-1/2}. \quad (6b)
$$

We recall that in this case the location of the ICO is given by $M\Omega_{\text{ICO}}^{\text{Sch}} = 6^{-3/2}$, with $E^{\text{Sch}}_{\text{ICO}} = \nu M \left(\sqrt{\frac{8}{9}} - 1\right)$ and $J^{\text{Sch}}_{\text{ICO}} = \nu M^2 \sqrt{12}$.

We look for the point at which both $E(\Omega)$ and $J(\Omega)$ take some minimal values $E_{\text{ICO}} = E(\Omega_{\text{ICO}})$ and $J_{\text{ICO}} = J(\Omega_{\text{ICO}})$. As we see from Eq. (3), at the 3PN order $E(\Omega)$ is a polynomial of the fourth degree in the frequency-parameter $x \equiv (M\Omega)^{2/3}$. Therefore, the value of the minimum, $x_{\text{ICO}} = (M\Omega_{\text{ICO}})^{2/3}$, must be a real positive solution of an algebraic equation of the third degree (in general):

$$
1 + \alpha x + \beta x^2 + \gamma x^3 = 0. \quad (7)
$$

The coefficients are straightforwardly obtained from Eq. (3) as

$$
\alpha(\nu) = \frac{3}{2} - \frac{\nu}{6}, \quad (8a)$$

$$
\beta(\nu) = -\frac{81}{8} + \frac{57}{8} \nu - \frac{\nu^2}{8}, \quad (8b)$$

$$
\gamma(\nu, \lambda) = -\frac{675}{16} + \left[\frac{209323}{1008} - \frac{205}{24} \pi^2 - \frac{440}{9} \lambda\right] \nu - \frac{155}{24} \nu^2 - \frac{35}{1296} \nu^3. \quad (8c)
$$

The regularization constant $\lambda$ enters only the third-degree monomial (3PN order). Let us describe, in a qualitative way, the existence of solutions of Eq. (7). We find that the equation does not always admit a unique real positive solution, nor even several of them. This depends, for a given choice of the mass ratio $\nu$, on the constant $\lambda$. When $\lambda$ happens
to be smaller that some “critical” value $\lambda_0(\nu)$, depending on $\nu$, there is no (real positive) solution, and therefore there is no ICO at the 3PN order. When $\lambda$ is between $\lambda_0(\nu)$ and another “critical” value $\lambda_1(\nu)$, also depending on $\nu$, we obtain two real positive solutions. In this case, the energy function admits two extrema, a minimum and a maximum. The maximum occurs at a higher frequency than the minimum of the ICO, and is to be discarded on physical grounds (the corresponding frequency is generally too high, e.g. higher than $M^{-1}$, for being of physical interest). Finally, when $\lambda$ is larger than $\lambda_1(\nu)$, there is one and only one real positive solution: $x_{\text{ICO}}$, and this is a minimum of the energy. The latter regime, where the circular-orbit energy admits a unique extremum, which is a minimum (like for the Schwarzschild metric), is the best on the physical point of view. By contrast, when $\lambda_0(\nu) \leq \lambda < \lambda_1(\nu)$, it seems possible that the minimum be “attracted” to some unphysical value by the spurious behaviour of the energy function at high frequencies. Fortunately, the interesting values of $\lambda$ are all located in the best regime where $\lambda \geq \lambda_1(\nu)$. We summarize our discussion in Fig. 1.

It is not difficult to determine analytically the functions $\lambda_0(\nu)$ and $\lambda_1(\nu)$. Indeed, $\lambda_0(\nu)$ represents simply the minimal value of the function $x_{\text{ICO}} \to \lambda(\nu, x_{\text{ICO}})$ (see Fig. 1). Using also Eq. (7), we readily find the mathematical relation defining $\lambda_0(\nu)$:

$$\lambda = \lambda_0(\nu) \iff \gamma(\nu, \lambda) = \frac{2}{27} \left[ \alpha^2(\nu) - 3\beta(\nu) \right]^{3/2} - \alpha^3(\nu) + \frac{9}{2} \alpha(\nu) \beta(\nu),$$

from which the explicit expression of $\lambda_0(\nu)$ can be found using Eqs. (8). On the other hand, the function $\lambda_1(\nu)$ is determined by the cancellation of the third-degree coefficient in the equation (7), i.e.

$$\lambda = \lambda_1(\nu) \iff \gamma(\nu, \lambda) = 0.$$ 

The expression of $\lambda_1(\nu)$ then follows from using Eq. (8c). For allowed values of $\nu \in ]0, \frac{1}{4}]$, we find that both $\lambda_0(\nu)$ and $\lambda_1(\nu)$ are increasing functions of $\nu$, with maximal values
\( \lambda_0 \left( \frac{1}{4} \right) \simeq -2.2 \) and \( \lambda_1 \left( \frac{1}{4} \right) \simeq -0.96 \), and satisfy \( \lambda_0 (\nu) \to -\infty \) and \( \lambda_1 (\nu) \to -\infty \) when \( \nu \to 0 \). Furthermore we always have \( \lambda_0 (\nu) < \lambda_1 (\nu) \). This analysis shows that in the case of our preferred value \( \lambda = -\frac{1987}{3080} \simeq -0.64 \), as well as in the cases where \( \omega_{\text{static}} = -9.34 \) and \( \lambda = 0 \), the energy function \( E(\Omega) \), for any mass ratio \( \nu \), admits a unique extremum, which is a minimum, at some \( \Omega_{\text{ICO}} \). We show in Fig. 2 the graph of \( E(\Omega) \) for equal masses and \( \omega_{\text{static}} = 0 \). Anticipating on our discussion below, it is interesting to compare Fig. 2 with the result of the numerical simulation provided by the figure 16 in Ref. [13].

In Table I we present the values of the calculated frequency \( \Omega_{\text{ICO}} \), the corresponding energy \( E_{\text{ICO}} \) and angular momentum \( J_{\text{ICO}} \), at the 1PN and 2PN orders, and at the 3PN order in the three cases where \( \omega_{\text{static}} = 0 \), \( \lambda = 0 \), and \( \omega_{\text{static}} = -9.34 \). The 1PN and 2PN approximations are defined by the obvious truncation of Eqs. (3)-(4). Notice how close together already are the 2PN and 3PN approximations (however, the 1PN order seems to be quite inadequate).

We emphasize that the present computation is based on the post-Newtonian approximation, expanded in the usual way as a Taylor series in the frequency-related parameter \( x = (M \Omega)^{2/3} \) [see Eqs. (3)-(4)]. We did not use resummation techniques such as Padé approximants [10] and/or effective-one-body methods [11]. Nevertheless, as we now discuss, the 3PN approximation, in standard form (Taylor approximants), appears to be very good to locate the turning point of the ICO, in the sense that the prediction for that point is close to the recent result of numerical relativity.

A novel approach to the problem of the numerical computation of binary black holes in the pre-coalescence stage, has been proposed and implemented by Gourgoulhon, Grandclément and Bonazzola [12,13]. This approach uses multi-domain spectral methods [14], and is based on two approximations, the first one is essentially “technical”, the other one is “physical”. The technical assumption (which could be relaxed in future work) is the conformal flatness of the spatial metric: \( \gamma_{ij} = \Psi^4 \delta_{ij} \). On the other hand, an imposed “helicoidal” symmetry constitutes an important physical restriction to binary systems moving on exactly circular orbits. By helicoidal symmetry we mean that the space-time is endowed
with a Killing vector field of the type $\ell^\mu = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}$, where $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ denote respectively the time-like and space-like vectors that coincide asymptotically with the coordinate vectors of an asymptotically inertial observer. A crucial advantage of the helicoidal symmetry, especially in view of the comparison we want to make with the post-Newtonian calculation, is that the orbital frequency $\Omega$ is unambiguously defined as the rotation rate of the Killing vector. Thanks to these approximations, Gourgoulhon, Grandclément and Bonazzola \cite{12,13} were able to obtain numerically the energy $E(\Omega)$ and angular-momentum $J(\Omega)$ along the binary’s evolutionary sequence, i.e. maintaining Eq. (5) along the sequence, and to determine the minimum of these functions or ICO.

We display in Figs. 3 our Taylor-series-based values for $E_{ICO}$ and $J_{ICO}$ (they are indicated by the marks 2PN and 3PN), and contrast them with the numerical result of Ref. \cite{13} (marked by an asterisk), as well as with some results obtained by means of resummation techniques at 3PN order: Padé approximants \cite{10,8} and effective-one-body (EOB) methods \cite{11,8}. All these results agree rather well with each other. Even the 2PN (Taylor) approximation does well, as it predicts an ICO which is not too far from the 3PN and numerical points. However, the 1PN order lies far outside the range of the figures (see Table I). Therefore, we find that the location of the ICO computed by numerical relativity, under the helicoidal-symmetry approximation, is in good agreement with post-Newtonian predictions. This was already pointed out in Ref. \cite{13} from the comparison with Padé and EOB methods. This is a satisfying situation, because we recall that the earlier estimates of the ICO in post-Newtonian theory: $M \Omega_{ICO} \simeq 0.06$ and $E_{ICO}/M \simeq -0.009$ \cite{15}, and numerical relativity: $M \Omega_{ICO} \simeq 0.17$ and $E_{ICO}/M \simeq -0.024$ \cite{16,17}, strongly disagree with each other, and do not match with the present 3PN results (see Ref. \cite{13} for further discussion.

\footnote{The numerical calculation reported in Refs. \cite{12,13} has been performed in the case of corotating black holes, which are spinning with the orbital angular velocity $\Omega$. We leave for future work the consideration of spin effects in our computation.}
A point we make is that the sophisticated Padé approximants give about the same result as the standard post-Newtonian expansion, based on the much simpler Taylor approximants: indeed see in Figs. 3 the points referred to as the $e$ and $j$-methods, which are 3PN Padé resummations built respectively on the energy and angular-momentum [8]. For the case at hands – equal-mass binaries –, there is apparently no improvement from using Padé approximants. However, it is true that in the test-mass limit $\nu \to 0$ the Padé series converges rapidly toward the exact result [10]. For instance, the Padé constructed in this case from the 2PN approximation of the energy already coincides with the exact expression for the Schwarzschild metric [see Eq. (6a)]. But, the results of Figs. 3 suggest that this nice feature of the Padé approximants is lost when we turn on $\nu$ and consider the equal-mass case $\nu = \frac{1}{4}$. Notice also that the 2PN versions of these Padé (given in Ref [8]) differ much more significantly from the 3PN ones than in the case of Taylor. For instance, the 2PN $e$-method yields the values $M\Omega_{ICO} \simeq 0.09$ and $E_{ICO}/M \simeq -0.016$, which respectively differ by about 36% and 22% with the frequency and energy given by the $e$-method at 3PN. In the case of Taylor, the same figures are only 6% and 3%. Thus, on the point of view of the “Cauchy criterium”\(^3\), the Taylor series seems to converge better that the Padé approximants (for equal masses).

It is a pleasure to thank Eric Gourgoulhon for informative discussions, Alessandra Buonanno and Gilles Esposito-Farèse for useful remarks.

\(^3\)The Cauchy criterium for the series $\sum a_n$ is the fact that $|a_n - a_m| \to 0$ for any $n$ and $m$. 

9
REFERENCES


### TABLE I. Parameters for the ICO of equal-mass ($\nu = \frac{1}{4}$) binary systems.

<table>
<thead>
<tr>
<th></th>
<th>$M\Omega_{ICO}$</th>
<th>$\frac{E_{ICO}}{M}$</th>
<th>$\frac{J_{ICO}}{M^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1PN</td>
<td>0.522</td>
<td>-0.0405</td>
<td>0.621</td>
</tr>
<tr>
<td>2PN</td>
<td>0.137</td>
<td>-0.0199</td>
<td>0.779</td>
</tr>
<tr>
<td>3PN $\omega_{static} = 0$</td>
<td>0.129</td>
<td>-0.0193</td>
<td>0.786</td>
</tr>
<tr>
<td>3PN $\lambda = 0$</td>
<td>0.116</td>
<td>-0.0184</td>
<td>0.798</td>
</tr>
<tr>
<td>3PN $\omega_{static} = -9.34$</td>
<td>0.095</td>
<td>-0.0166</td>
<td>0.824</td>
</tr>
</tbody>
</table>
FIG. 1. The possible solutions as a function of the regularization constant $\lambda$. There is no solution when $\lambda < \lambda_0(\nu)$, two possible solutions when $\lambda_0(\nu) \leq \lambda < \lambda_1(\nu)$ [which become degenerate at $\lambda = \lambda_0(\nu)$], and a unique solution when $\lambda_1(\nu) \leq \lambda$. The upper branch, existing between $\lambda_0(\nu)$ and the vertical asymptote at $\lambda = \lambda_1(\nu)$, is actually a maximum of the energy.
FIG. 2. The 3PN energy function $E(\Omega)$ for equal-mass binaries and $\omega_{\text{static}} = 0$.

FIG. 3. Results for $E_{\text{ICO}}$ and $J_{\text{ICO}}$ in terms of $\Omega_{\text{ICO}}$ in the equal-mass case. The asterisk marks the result calculated by numerical relativity under the assumption of helicoidal symmetry. The $e$ and $j$-methods are Padé approximants at the 3PN order. EOB refers to the effective-one-body approach at the 3PN order. The points marked by 2PN and 3PN are obtained by means of the standard Taylor post-Newtonian series (this work).