An Exactly Conservative Integrator for the \( n \)-Body Problem

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Abstract. The two-dimensional \( n \)-body problem of classical mechanics is a non-integrable Hamiltonian system for \( n > 2 \). Traditional numerical integration algorithms, which are polynomials in the time step, typically lead to systematic drifts in the computed value of the total energy and angular momentum. Even symplectic integration schemes exactly conserve only an approximate Hamiltonian. We present an algorithm that conserves the true Hamiltonian and the total angular momentum to machine precision. It is derived by applying conventional discretizations in a new space obtained by transformation of the dependent variables. We develop the method first for the restricted circular three-body problem, then for the general two-dimensional three-body problem, and finally for the planar \( n \)-body problem. Jacobi coordinates are used to reduce the two-dimensional \( n \)-body problem to an \((n - 1)\)-body problem that incorporates the constant linear momentum and center of mass constraints. For the \( n \)-body problem, we find that a larger time step can be used with our conservative algorithm than with symplectic and conventional integrators.


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1. Introduction

The $n$-body problem is the study of motion of $n$ arbitrary particles in space according to the Newtonian law of gravitation. When $n = 2$ (the Kepler problem), the problem has a well-known analytic solution, but Poincaré has shown that the system is in general non-integrable for $n > 2$. To approximately solve these cases, one often attempts to discretize the equations of motion and study the evolution of the system numerically. However, discretization of a system of any differential equations typically leads to a loss of accuracy; first integrals of the motion may no longer be preserved and the phase portrait may become inaccurate. This often necessitates the use of small time steps, so that many iterations will be required. In this article, we demonstrate that conservative integration can be used to obtain an accurate picture of the dynamics even with a relatively large time step.

Conservative integration was introduced by Shadwick, Bowman, and Morrison [1, 2, 3]. These authors argued that a more robust and faithful evolution of the dynamics can be obtained by explicitly building in knowledge of the analytical structure of the equations; in this case, by preserving the known first integrals of the motion. They illustrated the method applied to a three-wave truncation of the Euler equations, the Lotka–Volterra problem, and the Kepler Problem. In this work, we extend the method to the equations of motion of $n$ bodies in space, first to the circular restricted three-body problem, then to the general three-body problem, and finally to the full $n$-body case. For simplicity we only consider two-dimensional motion (a reasonable assumption for all of the planets in the solar system except for Pluto); extending this work to three dimensions should be straightforward.

2. Conservative Integration

The equations describing the motion of the solar system form a conservative system: the friction that heavenly bodies sustain is so small that virtually no energy is lost. Both the total energy and total angular momentum are conserved. We argue that a robust integration algorithm should preserve both of these invariants.

One way to accomplish this is to transform the dependent variables to a new space where the energy and other conserved quantities are linear functions of the transformed variables, apply a traditional integration algorithm in this space, and then transform back to get new values for each variable [1, 2]. This approach is motivated by the following lemma.

**Lemma 1** Let $\mathbf{x}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^n$. If $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is orthogonal to $\mathbf{c}$, so that $I = \mathbf{c} \cdot \mathbf{x}$ is a linear invariant of the first-order differential equation $\frac{d}{dt} \mathbf{x} = f(t, \mathbf{x})$, then each stage of the explicit $m$-stage discretization

$$
\mathbf{x}_j = \mathbf{x}_0 + \tau \sum_{k=0}^{j-1} b_{jk} f(t + a_j \tau, \mathbf{x}_k), \quad j = 1 \ldots m, \quad (1)
$$
also conserves $I$, where $\tau$ is the time step and $b_{jk} \in R$. That is, $c \cdot x_j = c \cdot x_0$, for all $j = 1, \ldots, m$.

Proof. Since $c \cdot f = 0$, we have

$$c \cdot x_j = c \cdot x_0 + \tau \sum_{k=0}^{j-1} b_{jk} c \cdot f(t + a_j \tau, x_k) = c \cdot x_0, \quad j = 1 \ldots m. \quad \diamond$$

For example, given a system of ordinary differential equations $\frac{dx}{dt} = f(t, x)$, consider the second-order predictor–corrector (2-stage) scheme

$$\tilde{x} = x_0 + \tau f(t, x_0), \quad (2a)$$

$$x(t + \tau) = x_0 + \frac{\tau}{2}[f(t, x_0) + f(t + \tau, \tilde{x})], \quad (2b)$$

where we now write $\tilde{x}$ instead of $x_1$. In the conservative predictor–corrector algorithm, one seeks a transformation $\xi = T(x)$ of the dependent variable $x$ such that the quantities to be conserved can be expressed as linear functions of the new variables $\xi_i, i = 1, \ldots, n$. Then, keeping Eq. (2a) as the predictor, in the transformed space one applies the corrector

$$\xi(t + \tau) = \xi_0 + \frac{\tau}{2}[T'(x)f(t, x_0) + T'(\tilde{x})f(t + \tau, \tilde{x})], \quad (3)$$

where $\tilde{\xi}(t) = T(\tilde{x})$ and $T'$ is the derivative of $T$. The new value of $x$ is obtained by inverse transformation, $x(t + \tau) = T^{-1}(\xi(t + \tau))$. Often the inverse transformation involves radicals, and if the argument of the radical is negative, it is still possible to use a finite number of smaller time-step reductions to integrate the system [2]; this approach is particularly advantageous when the time step is chosen adaptively. Another way to deal with negative arguments is to switch to a conventional integrator (predictor–corrector) for that one time step. If the inverse transformation involves several branches (e.g. because of a square root), the correct branch can be distinguished to sufficient accuracy using the conventional predictor solution. Higher-order conservative integration algorithms are readily obtained in the same way, by coupling the first $m - 1$ “predictor” stages from Eq. (1) with the conservative corrector

$$\xi(t + \tau) = \xi_0 + \tau \sum_{k=0}^{j-1} b_{jk} T'(x)f(t, x_k), \quad j = 1 \ldots m. \quad (4)$$

According to Iserles [4], a major drawback of traditional non-conservative integration is that numbers are often “thrown into the computer.” Mathematical models are often discretized according to algorithms that have little to do with the original problem. Iserles argued that one should develop computational algorithms that reflect known structural features of the problem under consideration (e.g. see [5, 6]). The conservative predictor–corrector is an example of such an integrator. In the examples given by [1, 3], the transformation $T$ is tailored to the system at hand; there is obviously no generic transformation that can be used to integrate an arbitrary conservative system.
It is interesting to compare conservative integration (which conserves the value of the Hamiltonian) with symplectic integration (which conserves phase-space volume; e.g., see Refs. [7], [8], and [9]). According to Ge and Marsden (1988), if an integrator is both symplectic and conservative, it must be exact. Normally we do not have the luxury of having an exact discretization at our disposal. The drawback then with conservative integration is that the Hamiltonian phase-space structure will not be preserved, just as for symplectic integration the total energy will not conserved. Which method is preferable depends on the physical structure of the problem being investigated.

Another important advantage of conservative integration algorithms is that, unlike typical symplectic integration schemes, they are explicit. Although in some cases the inverse of the transformation $T$ may be defined by an implicit equation that requires iteration to solve (using the predicted value as an accurate initial guess), this is really nothing more than a special function evaluation; the time-stepping scheme itself, being causal, is explicit.

With conservative integration, one can get all of the known invariants of the $n$-body problem conserved exactly, even for large time steps. This can lead to a more accurate picture of the motion of the bodies [1, figure 9] for the same computational effort. In the next section, we motivate the extension of the method of conservative integration to the $n$-body problem by briefly revisiting the treatment of the Kepler problem in Ref. [1].

3. Kepler Problem

The Kepler problem describes the motion of two bodies $m_1$ and $m_2$ located at positions $r_1$ and $r_2$, respectively. The dynamics can be reduced to an equivalent one-body problem, the behaviour of a single particle of mass $m = m_1m_2/(m_1 + m_2)$ at the position $r = r_2 - r_1$ under the influence of a central gravitational force. This force may be expressed as the gradient of the potential function $V = -k/r$, where $k$ is the gravitational constant. The equations of motion can be written in terms of the radial velocity $v_r$ and the polar coordinate angle $\theta$ of the particle,

\[
\frac{dr}{dt} = v_r, \quad (5a)
\]

\[
\frac{dv_r}{dt} = -\frac{1}{m} \left( \frac{\partial V}{\partial r} \right) + r\dot{\theta}^2, \quad (5b)
\]

\[
\frac{d\theta}{dt} = \frac{\ell}{mr^2}, \quad (5c)
\]

where $\ell$ is the (constant) total angular momentum. It is convenient to rewrite the equations in terms of the linear momentum $p = mv_r$ and the angular momentum $\ell$:

\[
\frac{dr}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad (6a)
\]

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial r} = \frac{\ell^2}{mr^2} - \left( \frac{\partial V}{\partial r} \right), \quad (6b)
\]
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\[
\frac{d\theta}{dt} = \frac{\partial H}{\partial \ell} = \frac{\ell}{mr^2}, \quad (6c)
\]

\[
\frac{d\ell}{dt} = -\frac{\partial H}{\partial \theta} = 0, \quad (6d)
\]

where the Hamiltonian

\[
H = \frac{p^2}{2m} + \frac{\ell^2}{2mr^2} + V(r), \quad (7)
\]

is also conserved.

### 3.1. Integration

To set the framework for generalizing the two-body problem to the \(n\)-body problem, we slightly generalize the presentation in Ref. [1] to make the constant \(\ell\) a variable that is formally integrated, but which remains constant.

The predictor step of the conservative integrator is given by Eq. (2a), where \(x = (r, \theta, p, \ell)\). To derive the corrector, the vector \((r, p, \ell)\) is transformed to \((\xi_1, \xi_2, \xi_3)\), where

\[
\xi_1 = \frac{k}{r}, \quad (8a)
\]

\[
\xi_2 = \frac{p^2}{2m} + \frac{\ell^2}{2mr^2}, \quad (8b)
\]

\[
\xi_3 = \ell. \quad (8c)
\]

On differentiating these equations with respect to time and exploiting the fact that both \(H = \xi_1 + \xi_2\) and \(L = \xi_3\) are both conserved, one finds

\[
\dot{\xi}_1 = \frac{kp}{mr^2}, \quad (9a)
\]

\[
\dot{\xi}_2 = -\dot{\xi}_1, \quad (9b)
\]

\[
\dot{\xi}_3 = 0. \quad (9c)
\]

After applying Eq. (3), the inverse transformation

\[
r = \frac{k}{\xi_1}, \quad (10a)
\]

\[
\ell = \xi_3, \quad (10b)
\]

\[
p = \text{sgn}(\tilde{p})\sqrt{2m\xi_2 - \frac{\ell^2}{r^2}} \quad (10c)
\]

is used to update the values of the original variables at the new step. See Ref. [1] for details on how the invariance of the Runge–Lenz vector \(\textbf{A} = \textbf{v} \times \ell + V\textbf{r}\) is exploited to evolve \(\theta\).‡

Before generalizing the integrator of Shadwick et al. to the \(n\)-body problem, it is instructive to consider first the special case of the restricted three-body problem.

‡ We point out a typographical error in Eq. (54b) of Ref. [1], which should read

\[
v_r(t + \tau) = \text{sgn}(\tilde{v}_r) \sqrt{v_r^2 + \frac{\ell^2}{m^2} \left(\frac{1}{r^2} - \frac{1}{r^2(t + \tau)}\right) - 2\frac{\Delta}{m}}. \quad (11)
\]
4. Restricted Three-Body Problem

Suppose that two bodies of masses $m_1$ and $m_2$, called the primaries, revolve around their center of mass in circular orbits. The *circular restricted three-body problem* describes the motion of a third body, with a mass $m_3$ that is negligible compared to $m_1$ and $m_2$, at coordinates $(x, y)$ in the plane of motion of the other two bodies. The third body does not influence the motion of the other two. The derivation of the equations of motion for the restricted problem is described in [11]. The Hamiltonian is given by

$$ H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(y^2 + x^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}, $$

where

$$ r_1^2 = (x - \mu)^2 + y^2, \quad r_2^2 = (x + 1 - \mu)^2 + y^2. $$

In terms of the canonical variables

$$ q_1 = x, \quad q_2 = y, \quad p_1 = \dot{x} - y, \quad p_2 = \dot{y} + x, $$

the Hamiltonian appears as

$$ H = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}. $$

The equations of motion are then

$$ \dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 + q_2, $$

$$ \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2 - q_1, $$

$$ \dot{p}_1 = -\frac{\partial H}{\partial q_1} = p_2 - \frac{1 - \mu}{r_1^3} (q_1 - \mu) - \frac{\mu}{r_2^3} (q_1 + 1 - \mu), $$

$$ \dot{p}_2 = -\frac{\partial H}{\partial q_2} = -p_1 - \frac{1 - \mu}{r_1^3} q_2 - \frac{\mu}{r_2^3} q_2, $$

and the Hamiltonian can be rewritten as

$$ H = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_1^2 + q_2^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}. $$

4.1. Integration

The conventional predictor for this system is

$$ \tilde{q}_i = q_i + \dot{q}_i \tau, \quad \tilde{p}_i = p_i + \dot{p}_i \tau, $$

for $i = 1, 2$. Note that, unless specified otherwise, the variables are functions of $t$. Let

$$ \xi_1 = \frac{1}{2} q_1^2, $$

$$ \xi_2 = \frac{1}{2} q_2^2, $$
\[ \xi_3 = \frac{1}{2} \dot{q}_1^2 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}, \]  
(19c)

\[ \xi_4 = \frac{1}{2} \dot{q}_2^2. \]  
(19d)

Here

\[ H = -\xi_1 - \xi_2 + \xi_3 + \xi_4 \]  
(20)

and \( H \) is written as a linear function of the \( \xi \)s. Differentiating the \( \xi \)s with respect to time, we get

\[ \dot{\xi}_1 = q_1 \dot{q}_1, \]  
(21a)

\[ \dot{\xi}_2 = q_2 \dot{q}_2, \]  
(21b)

\[ \dot{\xi}_4 = \dot{q}_2 \ddot{q}_2 = \dot{q}_2 (\dot{p}_2 - \dot{q}_1), \]  
(21c)

\[ \dot{\xi}_3 = \dot{\xi}_1 + \dot{\xi}_2 - \dot{\xi}_4, \]  
(21d)

upon making use of Eq. (20) together with the conservation of \( H \). The corrector is given by

\[ \xi_i(t + \tau) = \xi_i + \frac{\tau}{2} (\dot{\xi}_i + \ddot{\xi}_i), \]  
(22)

for \( i = 1, \ldots, 4 \), where \( \ddot{\xi}_i \) is simply Eq. (4.1) evaluated at \( \ddot{q}_i, \ddot{p}_i \) and \( t + \tau \). Inverting, the new values of \( q_i \) and \( p_i \) can be expressed in terms of \( \xi_i \) as

\[ q_1 = \text{sgn}(\ddot{q}_1) \sqrt{2\xi_1}, \]  
(23a)

\[ q_2 = \text{sgn}(\ddot{q}_2) \sqrt{2\xi_2}, \]  
(23b)

and, on using Eqs. (16a) and (16b),

\[ p_1 = -q_2 + \text{sgn}(\ddot{p}_1 + \ddot{q}_2) \sqrt{2\xi_3 + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2}}, \]  
(24a)

\[ p_2 = q_1 + \text{sgn}(\ddot{p}_2 - \ddot{q}_1) \sqrt{2\xi_4}. \]  
(24b)

We used the same initial conditions as in Ref. [12]. In Fig. 1, the motion of the third body is plotted in the fixed frame, using a time step \( \tau = 0.0015 \) from time \( t = 0 \) to \( t = 17.1 \). The orbit for the predictor–corrector begins to converge to the large time-step orbit shown for the conservative predictor–corrector as the time step is reduced to \( \tau = 0.001 \).

This example assumes that the mass of one body is negligible to the other two masses, and that the other two masses are travelling in circular orbits. The rest of this paper discusses the general case of three or more bodies: no restrictions are placed on the masses of the bodies, and their orbits do not have to be circular, or even periodic.
5. General Three-Body Problem

The derivation of the equations of motion of the general three-body problem in a plane is described in Refs. [11], [13], and [14].

Given three bodies \( m_1, m_2, \) and \( m_3 \) with position vectors \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_3, \) where each \( \mathbf{r}_i \) is at location \((x_i, y_i)\), define \( \mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i, \) where \( i, j = 1, 2, 3 \). The potential is

\[
V = -\frac{km_1m_2}{r_{12}} - \frac{km_2m_3}{r_{23}} - \frac{km_1m_3}{r_{13}},
\]

where \( k \) is the gravitational constant and \( r_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \) is the distance between the \( i \)th and \( j \)th bodies.

The system consists of three second-order differential equations,

\[
m_1\ddot{\mathbf{r}}_1 = -\frac{\partial V}{\partial \mathbf{r}_1} = \frac{km_1m_2(r_2 - r_1)}{r_{12}^3} + \frac{km_1m_3(r_3 - r_1)}{r_{13}^3},
\]

\[
m_2\ddot{\mathbf{r}}_2 = -\frac{\partial V}{\partial \mathbf{r}_2} = \frac{km_1m_2(r_1 - r_2)}{r_{21}^3} + \frac{km_2m_3(r_3 - r_2)}{r_{23}^3},
\]

\[
m_3\ddot{\mathbf{r}}_3 = -\frac{\partial V}{\partial \mathbf{r}_3} = \frac{km_1m_3(r_1 - r_3)}{r_{31}^3} + \frac{km_2m_3(r_2 - r_3)}{r_{32}^3}.
\]

These equations conserve the total linear momentum \( \sum_{i=1}^{3} m_i\dot{\mathbf{r}}_i \) (which allows us to fix the center of mass at the origin) and total angular momentum \( \sum_{i=1}^{3} \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i \). The Hamiltonian

\[
H = \sum_{i=1}^{3} m_i\dot{\mathbf{r}}_i^2 + V.
\]
where $V$ is given by Eq. (25), is also conserved. We exploit the constancy of
the linear momentum and center of mass position to reduce the number of degrees
of freedom in the problem. It is convenient to implement this reduction by converting
to Jacobi coordinates (e.g., see Refs. [15], [16], and [17]). The remaining constraints of
constant total angular momentum and energy are built into the conservative integrator,
by transforming to a frame where these invariants are linear.

Letting $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = (r_x, r_y)$, $M = m_1 + m_2 + m_3$, and $\mu = m_1 + m_2$, the
location of the center of mass of $m_1$ and $m_2$ is seen to be at $\mu^{-1}(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)$, or, since
$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 = \mathbf{0}$, at $-\mu^{-1} m_3 \mathbf{r}_3$. Let $\mathbf{\rho} = (\rho_x, \rho_y)$ be the vector from the center
of mass of the first two bodies to the third body. Then $\mathbf{\rho} = \mathbf{r}_3 + \mu^{-1} m_3 \mathbf{r}_3 = M\mu^{-1} \mathbf{r}_3$
and we find

$$
\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r},
$$

$$
\mathbf{r}_3 - \mathbf{r}_1 = \mathbf{\rho} + m_2\mu^{-1} \mathbf{r},
$$

$$
\mathbf{r}_3 - \mathbf{r}_2 = \mathbf{\rho} - m_1\mu^{-1} \mathbf{r}.
$$

In these coordinates, following Eq. (27), the Hamiltonian can be written as

$$
H = \frac{1}{2} g_1 (\dot{r}_x^2 + \dot{r}_y^2) + \frac{1}{2} g_2 (\dot{\rho}_x^2 + \dot{\rho}_y^2) + V
$$

(29)
in terms of the reduced masses $g_1 = m_1 m_2 \mu^{-1}$ and $g_2 = m_3 M^{-1} \mu$, where $V$ is given by
Eq. (25).

Define $r_x = r \cos \theta$, $r_y = r \sin \theta$, $\rho_x = \rho \cos \Theta$, $\rho_y = \rho \sin \Theta$. In polar coordinates, the
Hamiltonian can be rewritten

$$
H = \frac{p_x^2}{2g_1} + \frac{p_y^2}{2g_2} + \frac{\ell^2}{2g_1 r^2} + \frac{L^2}{2g_2 \rho^2} + V(r, \rho, \theta, \Theta),
$$

(30)

where $p$ is the linear momentum of the first reduced mass, $\ell$ is the angular momentum of
the first reduced mass, $P$ is the linear momentum of the second reduced mass, $L$ is the
angular momentum of the second reduced mass, and $V = V(r, \rho, \theta, \Theta)$ is the potential
energy of the system. The Hamiltonian $H$ and the total angular momentum $\ell + L$ are
conserved, and the center of mass remains at the origin for all time.

The equations of motion in polar coordinates are

$$
\dot{r} = \frac{\partial H}{\partial p} = \frac{p}{g_1}, \quad \dot{\theta} = \frac{\partial H}{\partial \ell} = \frac{\ell}{g_1 r^2},
$$

$$
\dot{\rho} = -\frac{\partial H}{\partial r} = \frac{\ell^2}{g_1 r^3} - \frac{\partial V}{\partial r}, \quad \dot{\ell} = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta},
$$

$$
\dot{P} = \frac{\partial H}{\partial P} = \frac{P}{g_2}, \quad \dot{\Theta} = \frac{\partial H}{\partial L} = \frac{L}{g_2 \rho^2},
$$

$$
\dot{\rho} = -\frac{\partial H}{\partial \rho} = \frac{L^2}{g_2 \rho^3} - \frac{\partial V}{\partial \rho}, \quad \dot{L} = -\frac{\partial H}{\partial \Theta} = -\frac{\partial V}{\partial \Theta}.
$$
5.1. Integration

The variables are transformed as

\[ \xi_1 = \frac{p^2}{2g_1} + \frac{\ell^2}{2g_1r^2}, \quad \xi_2 = \frac{P^2}{2g_2} + \frac{L^2}{2g_2\rho^2}, \]

\[ \xi_3 = V, \quad \xi_4 = \rho, \quad \xi_5 = \ell, \quad \xi_6 = L, \quad \xi_7 = \theta, \quad \xi_8 = \Theta. \] (32a)

Note that the conserved quantity \( H \) becomes a linear function of the transformed variables:

\[ H = \xi_1 + \xi_2 + \xi_3. \] (33)

The time derivatives become

\[ \dot{\xi}_1 = \frac{p\dot{p}}{g_1} + \frac{\ell r^2 \dot{\ell} - r \ell^2 \dot{r}}{g_1r^4}, \] (34a)

\[ \dot{\xi}_2 = \frac{P\dot{P}}{g_2} + \frac{L\rho^2 \dot{L} - \rho L^2 \dot{\rho}}{g_2\rho^4}, \] (34b)

\[ \dot{\xi}_3 = \frac{\partial V}{\partial r} \dot{r} + \frac{\partial V}{\partial \theta} \dot{\theta} + \frac{\partial V}{\partial \rho} \dot{\rho} + \frac{\partial V}{\partial \Theta} \dot{\Theta}, \] (34c)

\[ \dot{\xi}_4 = \dot{\rho}, \quad \dot{\xi}_5 = \dot{\ell}, \quad \dot{\xi}_6 = \dot{L}, \quad \dot{\xi}_7 = \dot{\theta}, \quad \dot{\xi}_8 = \dot{\Theta}. \] (34d)

The integration procedure is an extension of the method used for the Kepler problem. We can invert to find the original variables as follows,

\[ \rho = \xi_4, \quad \ell = \xi_5, \quad L = \xi_6, \quad \theta = \xi_7, \quad \Theta = \xi_8, \] (35a)

\[ r = g(\xi_3, \rho, \theta, \Theta), \] (35b)

\[ p = \text{sgn}(\tilde{p}) \sqrt{2g_1 \left( \xi_1 - \frac{\ell^2}{2g_1r^2} \right)}, \] (35c)

\[ P = \text{sgn}(\tilde{P}) \sqrt{2g_2 \left( \xi_2 - \frac{L^2}{2g_2\rho^2} \right)}. \] (35d)

The value of the inverse function \( g \) defined by \( V(g(\xi_3, \rho, \theta, \Theta), \rho, \theta, \Theta) = \xi_3 \) is determined at fixed \( \rho, \theta, \Theta \) with by Newton–Raphson iteration, using the predicted value \( \tilde{r} \) as an initial guess.

In Fig. 2 we compare the predictor–corrector and conservative predictor–corrector solutions for the motion of one of the three unit masses, using the initial conditions determined by Simó [18] and cited in [19], with a fixed time step of \( \tau = 6.5 \times 10^{-5} \). Each mass travels once around the figure eight. As \( \tau \) is decreased, the predictor–corrector solution begins to look more like that of the (large time step) conservative predictor–corrector; when \( \tau = 5.1 \times 10^{-5} \), the two graphs become identical in appearance. This emphasizes that the conservative predictor–corrector can be viewed as a finite-time-step generalization of the conventional predictor–corrector, as argued in Ref. [1].

We now extend above results to the \( n \)-body case, where \( n \geq 2 \).
Figure 2. The predictor–corrector (dashed line) and conservative predictor–corrector (solid line) solutions for the general three-body problem.

6. General n-Body Problem

The Jacobi coordinates can be extended to n bodies in a plane, as discussed by [17] and [15], where \( n \geq 2 \).

Let n masses \( m_i \) have radius vectors \( r_i \), where \( i = 1, \ldots, n \). Define \( r_{ij} = r_j - r_i \) as the vector joining \( m_i \) to \( m_j \). Also define \( C_i \) to be the center of mass of the first \( i \) bodies, where \( i = 2, \ldots, n \), and choose the origin of the coordinate system so that \( C_n = 0 \). Let the vectors \( \rho_i \) be defined such that

\[
\begin{align*}
\rho_2 &= r_{12}, \\
\rho_3 &= r_3 - C_2, \\
&\quad \vdots \\
\rho_n &= r_n - C_{n-1}.
\end{align*}
\]

Also

\[
\begin{equation}
\rho_{k\ell} = \rho_\ell - \rho_k + \sum_{j=k}^{\ell-1} \frac{m_j \rho_j}{M_j},
\end{equation}
\]

where \( 1 \leq k < \ell \leq n \), and \( M_j = \sum_{k=1}^{j-1} m_k \).\(^\S\)

The reduced masses are

\[
g_2 = \frac{m_2 m_1}{M_2},
\]

\(^\S\) Here \( \rho_1 \) is a dummy variable that cancels out in the expression for \( r_{12} \).
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\[
g_3 = \frac{m_3(m_2 + m_1)}{M_3}, \tag{39}
\]

\[
\ldots
\]

\[
g_n = \frac{m_n M_{n-1}}{M_n}. \tag{41}
\]

The equations of motion in polar coordinates are just an extension of the three-body problem:

\[
\dot{\rho}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{g_i}, \tag{42a}
\]

\[
\dot{\theta}_i = \frac{\partial H}{\partial \ell_i} = \frac{\ell_i}{g_i \rho_i^2}, \tag{42b}
\]

\[
\dot{p}_i = -\frac{\partial H}{\partial \rho_i} = \frac{\ell_i^2}{g_i \rho_i^3} - \frac{\partial V}{\partial \rho_i}, \tag{42c}
\]

\[
\dot{\ell}_i = -\frac{\partial H}{\partial \theta_i} = -\frac{\partial V}{\partial \theta_i}, \tag{42d}
\]

where \( \rho_i, \theta_i, p_i \) and \( \ell_i \) are the radius, angle, linear momentum, and angular momentum, respectively, of the \( i \)th reduced mass, for \( i = 2, \ldots, n \). The potential is defined to be

\[
V = -\sum_{i,j=1}^{n} \sum_{i<j} m_i m_j \frac{1}{r_{ij}}, \tag{43}
\]

and the total kinetic energy is

\[
K = \frac{1}{2} \sum_{i=2}^{n} \left( \frac{p_i^2}{g_i} + \frac{\ell_i^2}{g_i \rho_i^2} \right). \tag{44}
\]

It is easy to verify that the Hamiltonian \( H = K + V \) is conserved by Eqs. (6). The total angular momentum, \( \sum_{i=2}^{n} \ell_i \), is also conserved and the center of mass remains at the origin for all time.

6.1. Integration

Transform \((\rho, \theta, p, \ell)\) to \((\zeta, \theta, \eta, \ell)\), where

\[
\zeta_2 = V, \tag{45a}
\]

\[
\zeta_i = \rho_i, \quad \text{for } i = 3, \ldots, n, \tag{45b}
\]

\[
\eta_i = \frac{p_i^2}{2 g_i} + \frac{\ell_i^2}{2 g_i \rho_i^2}, \quad \text{for } i = 2, \ldots, n. \tag{45c}
\]

Note that \( H \) is a linear function of the transformed variables:

\[
H = \sum_{i=2}^{n} \eta_i + \zeta_2, \tag{46}
\]
as is the total angular momentum $L = \sum_{i=2}^{n} \ell_i$. The time derivatives of $\zeta$ and $\eta$ are given by

$$\dot{\zeta}_{2} = \sum_{i=2}^{n} \left( \frac{\partial V}{\partial \rho_i} \dot{\rho}_i + \frac{\partial V}{\partial \theta_i} \dot{\theta}_i \right), \quad (47a)$$

$$\dot{\zeta}_i = \dot{\rho}_i, \quad \text{for } i = 3, \ldots, n, \quad (47b)$$

$$\dot{\eta}_i = \frac{p_i \dot{\rho}_i}{g_i} + \frac{\ell_i \rho_i^2 \dot{\ell}_i - \rho_i \ell_i^2 \dot{\rho}_i}{g_i \rho_i^4}, \quad \text{for } i = 2, \ldots, n. \quad (47c)$$

The predictor equations are

$$\tilde{\rho}_i = \rho_i + \dot{\rho}_i \tau, \quad \tilde{\theta}_i = \theta_i + \dot{\theta}_i \tau, \quad (48a)$$

$$\tilde{p}_i = p_i + \dot{p}_i \tau, \quad \tilde{\ell}_i = \ell_i + \dot{\ell}_i \tau \quad (48b)$$

and the corrector is given by

$$\zeta_i(t + \tau) = \zeta_i + \frac{\tau}{2} (\dot{\zeta}_i + \dot{\tilde{\zeta}}_i), \quad \theta_i(t + \tau) = \theta_i + \frac{\tau}{2} (\dot{\theta}_i + \dot{\tilde{\theta}}_i), \quad (49a)$$

$$\eta_i(t + \tau) = \eta_i + \frac{\tau}{2} (\dot{\eta}_i + \dot{\tilde{\eta}}_i), \quad \ell_i(t + \tau) = \ell_i + \frac{\tau}{2} (\dot{\ell}_i + \dot{\tilde{\ell}}_i), \quad (49b)$$

for $i = 2, \ldots, n$.

One then inverts to get the original variables as functions of the temporary transformed variables:

$$\rho_i = \zeta_i \quad \text{for } i = 3, \ldots, n, \quad (50a)$$

$$\rho_2 = g(\zeta_2, \rho_3, \ldots, \rho_n, \theta), \quad (50b)$$

$$p_i = \text{sgn}(\tilde{p}_i) \sqrt{2g_i \left( \eta_i - \frac{\ell_i^2}{2g_i \rho_i^2} \right)}, \quad \text{for } i = 2, \ldots, n. \quad (50c)$$

The value of the inverse function $g$ defined by

$$V(\zeta_2, \rho_3, \ldots, \rho_n, \theta), \rho_3, \ldots, \rho_n, \theta) = \zeta_2 \quad (51)$$

is determined at fixed $\rho_3, \ldots, \rho_n, \theta$ with a Newton–Raphson method, using the predicted value $\tilde{\rho}_2$ as an initial guess.

In Fig. 3, we illustrate the four-body choreography described by Simó [18]. The motions of one of the four unit masses, as predicted by the predictor–corrector and conservative predictor–corrector, are compared, using the fixed time step $\tau = 10^{-3}$ to integrate the system from time $t = 0$ to $t = 10$. For the same time step, we also compare these solutions to a second-order symplectic mapping with kinetic–potential energy splitting using Varadi’s NBI code. Of the three solutions, we note that the conservative predictor–corrector trajectory is the most accurate. It was also found to be more accurate than the second-order Wisdom–Holman scheme described by Varadi [20, 21].
7. Conclusion

Conservative integration algorithms can reduce the computational effort required to integrate a system of equations accurately. When the total energy and angular momentum of the $n$-body problem is conserved, it is possible to obtain accurate trajectories with a much larger time step than with conventional integration methods.

The $n$-body problem for planar motion has six invariants, all of which need to be considered during the integration. Jacobi coordinates for this problem were used to reduce the system to an $(n-1)$-body problem in which the linear momentum and center of mass constraints are implicitly built in, leaving fewer conservation laws to be explicitly built into the algorithm. The kinetic energy term of the Hamiltonian remains in diagonal form (a sum of squares) in Jacobi coordinates; this makes it easy to express the Hamiltonian as a linear function of new variables.

Future work in this area should include modifying the numerical code for the three-dimensional case, regularizing the equations of motions to handle collisions and close approaches, and building in precession, mutation, and tidal effects into the equations of motion.

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REFERENCES


