Towards Vacuum Superstring Field Theory: The Supersliver

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Abstract: We extend some aspects of vacuum string field theory to superstring field theory in Berkovits’ formulation, and we study the star algebra in the fermionic matter sector. After clarifying the structure of the interaction vertex in the operator formalism of Gross and Jevicki, we provide an algebraic construction of the supersliver state in terms of infinite–dimensional matrices. This state is an idempotent string field and solves the equation of motion of superstring field theory with a pure ghost BRST operator. We determine the spectrum of eigenvalues and eigenvectors of the infinite–dimensional matrices of Neumann coefficients in the fermionic matter sector. We then analyze coherent states based on the supersliver and use them in order to construct higher–rank projector solutions, as well as to construct closed subalgebras of the star algebra in the fermionic matter sector. Finally, we show that the geometric supersliver is a solution to the superstring field theory equations of motion, including the (super)ghost sector, with the canonical choice of vacuum BRST operator recently proposed by Gaiotto, Rastelli, Sen and Zwiebach.

Keywords: String Field Theory, Supersymmetry, Sliver, Tachyon Condensation.
Contents
1. Introduction and Summary

In the last two years, the search for nonperturbative information in string field theory [1, 2] has experienced a renewed interest mainly due to a series of conjectures by Sen [3, 4, 5] (also see [6] for a review and a list of references). These conjectures have been tested numerically to a high degree of precision in level truncated cubic string field theory, and some of them have been proven in boundary string field theory (see, e.g., [7] for a review and a list of references). In the meantime, the elegant construction of Berkovits [8, 9, 10, 11] has emerged as a promising candidate for an open superstring field theory describing the NS sector: in here, Sen’s conjectures about the fate of the tachyon in the non–BPS $D9$–brane have been successfully tested by level truncation to a high level of accuracy [12, 13, 14, 15], and kink solutions have been found that describe lower–dimensional $D$–branes [16] (see, e.g., [17] for a review and a more complete list of references).

So far, most of our understanding about tachyon condensation in both cubic string field theory and Berkovits’ superstring field theory is based on level–truncated computations and it would be of course desirable to have an analytical control over the problem. For the bosonic string, Rastelli, Sen and Zwiebach have proposed in a series of papers [18, 19, 20, 21, 22] a new approach to this problem called vacuum string field theory (VSFT). In VSFT, the form of the cubic string field theory action around the tachyonic vacuum is postulated by exploiting some of the expected properties it should have (like the absence of open string states). Then one can show that this theory has solutions that describe the perturbative vacuum and the various $D$–branes. In particular, the matter sector of the maximal $D25$–brane is described by a special state called the sliver. This state was first constructed geometrically by Rastelli and Zwiebach [23] and then algebraically by Kostelecky and Potting [24], and it is an idempotent state of the string field star algebra, in the matter sector. The construction of VSFT has been recently completed in [25], where Gaiotto, Rastelli, Sen and Zwiebach have proposed a canonical choice of the ghost BRST operator around the vacuum, with which they identified closed string states; and also in [26], where Rastelli, Sen and Zwiebach have found the eigenvalue and eigenvector spectrum of the Neumann matrices, which could allow for a proper definition of the string field space. The study of VSFT has also unveiled beautiful algebraic structures in cubic string field theory (for example, projectors of arbitrary rank in the star algebra have been constructed in detail in [20, 27, 28]).

The main purpose of this paper is to give the first steps towards the construction of vacuum superstring field theory around the tachyonic vacua of the non–BPS maximal $D9$–brane in Type IIA superstring theory, and to explore the algebraic structure of the star algebra in the fermionic part of the matter sector. In section 2, we begin with a review of Berkovits’ open superstring field theory for the NS sector and discuss the general features of vacuum superstring field theory. We shall show in detail that, assuming a pure ghost BRST operator around the vacuum as in VSFT, Berkovits’ equation of motion for the superstring field admits factorized solutions whose matter part is an idempotent state of the star algebra. In a sense, idempotency is even more useful in Berkovits’ theory since it drastically reduces the nonlinearity of the equation of motion. Idempotent string field solutions can be constructed in the GSO($+$) sector or in both GSO($\pm$) sectors.

In order to construct idempotent states in superstring field theory, one first has to understand in detail the structure of the star algebra in the fermionic matter sector. To do that, we use the
operator construction of the interaction vertex for the superstring due to Gross and Jevicki [29],
which extends their previous work on the bosonic string [30, 31] to the NSR superstring. In section
3 we review some of the relevant results and we further clarify the structure of the vertex. This
allows us to write the Neumann coefficients in terms of two simple infinite–dimensional matrices
which shall play a key rôle in the constructions of this paper.

Given any boundary conformal field theory (BCFT) one can construct geometrically a special
state which is an idempotent of the star algebra [21]. When the BCFT is that of a $D_{25}$–brane, this
state is called the sliver\(^1\) [23, 18]. This geometric construction extends in a very natural way to
the BCFT given by the NS sector of the open superstring which describes the unstable $D_9$–brane.
This yields an idempotent state that we call the supersliver. The matter part of the supersliver
is a product of two squeezed states: one made of bosonic oscillators (the bosonic sliver previously
considered in [24, 19]) and the other made of fermionic oscillators, that we shall call the fermionic
sliver. Although the geometric construction gives a precise determination of the fermionic sliver,
it is important for many purposes to have an algebraic construction as well. In section 4, and
making use the results of section 3 for the interaction vertex, we find a simple expression for the
fermionic sliver in terms of infinite–dimensional matrices, as in [24, 19], and we compare the result
to the geometric construction. We also briefly address the supersliver conservation laws. In section
5 we use the techniques recently introduced in [26] to determine the eigenvalue spectrum and the
eigenvectors of the various infinite–dimensional matrices involved in the fermionic star algebra,
including the matrices of Neumann coefficients.

Once the fermionic sliver has been constructed algebraically, one can take it as a sort of “vacuum
state” in order to build fermionic coherent states. This we do in section 6, where after constructing
these fermionic coherent states on the fermionic sliver, we study their star algebras. As in [20], one
can use these coherent states to construct higher–rank projectors of the fermionic star algebra. We
shall show that one can also construct closed fermionic star subalgebras. These star subalgebras
provide new idempotent states which at first seem to yield new solutions to the vacuum superstring
field theory equation of motion. However, they turn out to be related to the fermionic sliver by
gauge transformations.

In section 7 we show that if one chooses the vacuum BRST operator to be the recent canonical
choice of Gaiotto, Rastelli, Sen and Zwiebach, [25], then the geometrical sliver is a solution to
Berkovits’ superstring field theory equations of motion, \textit{i.e.}, we solve the equations of motion in the
(super)ghost sector.

Finally, in section 8 we state some conclusions and open problems for the future. In the Appendix
we give some of the details needed in the proof that the structure of the vertex found in section
3 agrees with the explicit expressions found by Gross and Jevicki in [29] using conformal mapping

\(^1\)Strictly speaking, the construction of the sliver state is purely geometric and is thus valid for arbitrary BCFT’s.
However, in this paper, we shall use the denotation of “sliver” for the particular BCFT associated to the maximal
brane in flat space.
2. Berkovits’ Superstring Field Theory

2.1. A Short Review of Berkovits’ Superstring Field Theory

In this paper, we shall study the non–GSO projected open superstring in the NS sector. In the matter sector, there are two fermions $\psi^\pm(\sigma)$ with the mode expansion

$$\psi^\mu_\pm(\sigma) = \sum_{r \in \mathbb{Z}_{12}} e^{\pm ir\sigma} \psi^\mu_r,$$  \hspace{1cm} (2.1)

where the modes satisfy the anticommutation relations

$$\{ \psi^\mu_r, \psi^\nu_s \} = \eta^{\mu\nu} \delta_{r+s,0}. \hspace{1cm} (2.2)$$

We will therefore write $\psi^\dagger_r = \psi_{-r}$ for $r > 0$. The ghost/superghost sector includes the $b, c$, and the $\beta, \gamma$, system and we bosonize the last one in the standard way [32]:

$$\beta = \partial \xi e^{-\phi}, \quad \gamma = \eta \phi.$$

A superstring field theory describing the GSO–projected NS sector of the open superstring was proposed by Berkovits in [8] (recent reviews can be found in, e.g., [10, 17]). In this theory, the string field $\Phi$ is Grassmann even, has zero ghost number and zero picture number. The action has the structure of a WZW model:

$$S[\Phi] = 12 \int \left( (e^{-\Phi} Q_B e^\Phi)(e^{-\Phi} \eta_0 e^\Phi) - \int_0^1 dt (e^{-t\Phi} \partial_t e^{t\Phi})(\{ (e^{-t\Phi} Q_B e^{t\Phi}), (e^{-t\Phi} \eta_0 e^\Phi) \}) \right), \hspace{1cm} (2.4)$$

where $Q_B$ is the BRST operator of the superstring and $\eta_0$ the zero–mode of $\eta$ (the bosonized superconformal ghost) [32]. In a WZW interpretation of this model, these operators play the role of a holomorphic and an anti–holomorphic derivatives, respectively. In this action, the integral and the star products are evaluated with Witten’s string field theory interaction [1]. The exponentiation of the string field $\Phi$ is defined by a series expansion with star products: $e^\Phi = \mathcal{I} + \Phi + 12 \Phi \star \Phi + \cdots$, where $\mathcal{I}$ is the identity string field. As usual, we refer to the first term in (2.4) as the kinetic term and to the second one as the Wess–Zumino term. It can be shown that the equation of motion derived from this action is [8]:

$$\eta_0 (e^{-\Phi} Q_B e^\Phi) = 0. \hspace{1cm} (2.5)$$

The action (2.4) has a gauge symmetry given by

$$\delta e^\Phi = \Xi_L e^\Phi + e^\Phi \Xi_R, \hspace{1cm} (2.6)$$
where the gauge parameters $\Xi_{L,R}$ satisfy

$$Q_B \Xi_L = 0, \quad \eta_0 \Xi_R = 0. \quad (2.7)$$

One can include GSO($-)$ states by introducing Chan–Paton–like degrees of freedom $[12, 13]$. The string field then reads,

$$\Phi = \Phi_+ \otimes 1 + \Phi_- \otimes \sigma_1, \quad (2.8)$$

where $\Phi_\pm$ are respectively in the GSO($\pm$) sectors, and $\sigma_1$ is one of the Pauli matrices. The $Q_B$ and $\eta_0$ operators also have to be tensored with the appropriate matrices:

$$\hat{Q}_B = Q_B \otimes \sigma_3, \quad \hat{\eta}_0 = \eta_0 \otimes \sigma_3. \quad (2.9)$$

The action is again given by one–half times (2.4), where the bracket now includes a trace over the Chan–Paton–like matrices (the $1/2$ factor is included to compensate for the trace over the matrices). The gauge symmetry is given again by (2.6), where $\Xi_{L,R}$ take values in both sectors as in (2.8).

It has been shown that Berkovits superstring field theory correctly reproduces the four–point tree amplitude in $[11]$, and it can be used to compute the NS tachyon potential in level truncation (see $[17]$ for a review), giving results which are compatible with Sen’s conjectures.

### 2.2. Superstring Field Theory Around a Classical Solution

In the cubic theory of Witten, one can consider a particular solution of the classical equations of motion, $\Phi_0$, and study fluctuations around it: $\Phi = \Phi_0 + \tilde{\Phi}$. It is easy to see that the action governing the fluctuations $\tilde{\Phi}$ has the structure of the original action for $\Phi$, but with a different BRST operator, $Q$. Bosonic VSFT, as formulated in the series of papers $[18, 19, 20, 21, 22]$, is based on two assumptions:

1) First, it is assumed that, when one expands around the tachyonic vacuum, the new BRST operator $Q$ has vanishing cohomology and is made purely of ghost operators.

2) Second, it is assumed that all $Dp$–brane solutions of VSFT have the factorized form:

$$\Phi = \Phi_g \otimes \Phi_m, \quad (2.10)$$

where $\Phi_{g,m}$ denote states containing only ghost and only matter modes, respectively. Since the star product factorizes into the ghost and the matter sector, and since we have assumed that $Q$ is pure ghost, the equations of motion split into:

$$Q \Phi_g + \Phi_g \star \Phi_g = 0, \quad (2.11)$$
The second equation says that the matter part is an idempotent of the star algebra (where the star product is now restricted to the matter sector). If these assumptions hold, the string field action evaluated at a solution of the form (2.10) is simply proportional to the BPZ norm of $|\Phi_m\rangle$, and this allows one to compare in a simple way ratios of tensions of different $D$–branes [19, 21].

An interesting question is to which extent are these assumptions valid in Berkovits’ superstring field theory. In order to answer this question, the first step is to analyze the fluctuations around a solution to the equations of motion. This was first addressed by Kluson [33], where it was shown that with an appropriate parameterization of the fluctuations, the equation of motion is identical to (2.5), albeit with a deformed $Q$ operator. It was thus concluded (without proof) in [33] that the action for the fluctuation should have the form (2.4) with the deformed operator. We shall now derive the equation of motion in a slightly different way from the one presented in [33], and this will allow us to show that the action is indeed of the required form by direct computation.

Let us define $G = e^\Phi$, the exponential of the string field that appears in Berkovits’ action. Let $\Phi_0$ be a solution to the classical equations of motion (2.5) and let us consider a fluctuation around this solution parameterized as follows [33],

$$G = G_0 * h, \quad G_0 = e^{\Phi_0}, \quad h = e^\phi.$$  \hspace{1cm} (2.13)

Since Berkovits’ action has the structure of a WZW theory, one should expect an analog of the Polyakov–Wiegmann equation [34] to be valid. In fact, it is easy to show (by using for example the geometric formulation of [33, 35]) that the action (2.4) satisfies

$$S[G_0 * h] = S[G_0] + S[h] - \int (G_0^{-1} Q_B G_0)(h \eta \eta h^{-1}),$$ \hspace{1cm} (2.14)

for arbitrary $G_0$ and $h$. The effective action for the fluctuations is then

$$S_{\text{eff}}[h] = S[h] - \int (G_0^{-1} Q_B G_0)(h \eta \eta h^{-1}).$$ \hspace{1cm} (2.15)

Let us now obtain the equation of motion satisfied by $h$. Varying $S[h]$, one obtains

$$\int h^{-1} \delta h \eta \eta (h^{-1} Q_B h),$$

and from the extra term in $S_{\text{eff}}[h]$ one gets

$$\int h^{-1} \delta h \eta \eta (h^{-1} A h),$$
where we denoted $A = G_0^{-1}Q_BG_0$ and we have used the equation of motion $\eta_0(A) = 0$. Putting both pieces together, one finds that

$$\eta_0(h^{-1}Q_Bh + h^{-1}Ah - A) = 0. \quad (2.16)$$

Therefore, the equation of motion is identical to (2.5) but with the deformed $Q$ operator:

$$Q_A(X) = Q_B(X) + AX - (-1)^X XA. \quad (2.17)$$

One can moreover easily prove [33] that the new operator satisfies all the axioms of superstring field theory (it is a nilpotent derivation and it anticommutes with $\eta_0$).

We shall now show that $S_{\text{eff}}[h]$ has in fact the structure of (2.4) but with the operator $Q_A$. For that, we simply need to notice that

$$\int A(h\eta_0h^{-1}) = 12 \int \left((h^{-1}Ah - A)(h^{-1}\eta_0h) - \int_0^1 dt A\partial_t(h\eta_0h^{-1} - h^{-1}\eta_0h)\right), \quad (2.18)$$

where we have used integration by parts with respect to $\eta_0$, and the fact that $\Phi_0$ satisfies its equation of motion. We have also denoted $\hat{h} = e^{t\phi}$. The first term in the RHS of (2.18) when added to the kinetic term in $S[h]$ gives a kinetic term with the $Q_A$ operator, while the second term when added to the Wess–Zumino term in $S[h]$ gives a Wess–Zumino term with $Q_A$. The conclusion of this computation is that the action for the fluctuations is simply

$$S_{\text{eff}}[h] = 12 \int \left((e^{-\phi}Q_Ae^{\phi})(e^{-\phi}\eta_0e^{\phi}) - \int_0^1 dt (e^{-t\phi}\partial_t e^{t\phi})\{(e^{-t\phi}Q_Ae^{t\phi}), (e^{-t\phi}\eta_0e^{t\phi})\}\right), \quad (2.19)$$

as anticipated in [33].

Let us now consider the superstring field theory describing the non–BPS $D9$–brane, i.e., Berkovits’ superstring field theory including both the GSO(±) sectors. It has been shown in level truncation that this theory has two symmetric vacua where the tachyon condenses. According to Sen’s conjectures, at any of these two vacua there are no open superstring degrees of freedom. Let us then choose one of these vacua and study the action for fluctuations around it. As we have seen, the action for the fluctuations has the same form as the original one, but with a different BRST operator, that we shall now denote by $Q$. According to Sen’s conjectures, at this chosen vacuum there are no open string degrees of freedom and it is thus natural to assume, as in the VSFT for the bosonic string, that the new BRST operator has vanishing cohomology and is made purely of (super)ghost operators. In addition we will also assume that this operator annihilates the identity,

$$QI = 0. \quad (2.20)$$
This condition, although very natural, is strictly not necessary in order to preserve some of the basic features of bosonic VSFT. In the superstring case however, it is crucial. It was noticed in [18] that operators of the form $Q = c_0 + \sum_n u_n c_n$ also have vanishing cohomology in the superstring case. In particular, the $Q$ operator recently proposed in [25] for the bosonic VSFT is of this form and annihilates the identity after some proper regularization, so that in principle it is a possible candidate for the superstring as well (where the superconformal ghost sector would be handled separately). We shall come back to this question in section 7.

With these assumptions at hand, and given the fact that the action around the vacuum has the same form as the original one but with a pure (super)ghost operator $Q$, it is now easy to show that the ansatz (2.10) solves the superstring field theory equations of motion if $\Phi_m$ is idempotent and $\Phi_g$ satisfies

\[ \eta_0 \left( e^{-\Phi_g} Q e^{\Phi_g} \right) = 0. \]  

(2.21)

In order to see this, notice that idempotency of $\Phi_m$ and factorization of the star product in matter and ghost parts yields

\[ e^{\Phi} = e^{\Phi_g} \otimes \Phi_m + \mathcal{I} - \Phi_m, \]  

(2.22)

and, since $Q$ kills the identity and is pure ghost, one has

\[ Q e^{\Phi} = (Q e^{\Phi_g}) \otimes \Phi_m. \]  

(2.23)

Using again idempotency of $\Phi_m$, the equation of motion becomes:

\[ \left( \eta_0 \left( e^{-\Phi_g} Q e^{\Phi_g} \right) \right) \otimes \Phi_m = 0. \]  

(2.24)

Therefore, the above conditions are sufficient to solve the equations of motion. In the same way, one can show that in these circumstances the action factorizes as

\[ S = K \langle \Phi_m | \Phi_m \rangle, \]  

(2.25)

where

\[ K = S[\Phi_g]. \]  

(2.26)

Let us now look at the gauge symmetry of the new action around the tachyon vacuum. We are particularly interested in transformations that preserve the structure of (2.22). Since both $Q$ and $\eta_0$ annihilate the identity, it is easy to see that the gauge transformation (2.6) with
Ξ_L = Ξ_m \otimes \mathcal{I}_g, \quad Ξ_R = -Ξ_m \otimes \mathcal{I}_g, \quad (2.27)

preserves (2.22). This gauge transformation leaves Φ_g invariant and changes Φ_m as follows:

δΦ_m = [Ξ_m, Φ_m]_s, \quad (2.28)

where [A, B]_s = A∗B - B∗A is the commutator in the star algebra. Notice that this transformation preserves idempotency of Φ_m at linear order. The gauge symmetry (2.28) is precisely the one that appears in bosonic VSFT when Q annihilates the identity [19, 21, 25].

The condition of idempotency of Φ_m in the non–GSO projected theory involves in fact two different conditions. In general, a matter string field Φ_m has components in both GSO(±) sectors,

Φ_m = Φ_m^+ \otimes 1 + Φ_m^- \otimes σ_1. \quad (2.29)

In this equation, Φ_m^± is Grassmann even (odd), and idempotency of Φ_m is equivalent to the following equations

Φ_m^+ * Φ_m^+ + Φ_m^- * Φ_m^- = Φ_m^+, \quad \Phi_m^+ * Φ_m^- + Φ_m^- * Φ_m^+ = Φ_m^-.

(2.30)

One particular solution is of course to take Φ_m^+ as an idempotent state and Φ_m^- = 0. The matter supersliver state that we will discuss later is an example of such a solution. Another possibility is to take Φ_m an idempotent and Φ_m^- a nilpotent state satisfying the second equation in (2.30). In section 6 we will construct solutions with these characteristics, although we will also show that they are related to the supersliver solution by gauge transformations at the vacuum.

3. Neumann Coefficients and Overlap Equations

In this section we review some of the results of [29] and we explain in detail the structure of the overlap equations involving the matter part of the fermionic sector.

3.1. The Identity

As in bosonic string field theory, the simplest vertex in superstring field theory is the integration, which corresponds to folding the string and identifying the two halves [1] thus defining the identity string field |I⟩,

∫ Φ = ⟨I|Φ⟩. \quad (3.1)
In the bosonic case, the overlap condition defining the identity is simply $x(\pi - \sigma) = x(\sigma)$. In the fermionic case, and due to the conformal weight $h = 1/2$, the precise conditions are as follows:

$$
(\psi_+(\sigma) - i\psi_+ (\pi - \sigma)) |I\rangle = 0, \quad 0 \leq \sigma \leq \pi 2,
(\psi_-(\sigma) + i\psi_-(\pi - \sigma)) |I\rangle = 0, \quad 0 \leq \sigma \leq \pi 2.
$$

The different sign in the second equation is due to the NS boundary conditions $\psi_{-}(0) = \psi_{+}(0), \psi_{-}(\pi) = -\psi_{+}(\pi)$. As usual, we can define a single antiperiodic fermion field $\psi(\sigma)$ in the interval $[-\pi, \pi]$ by declaring that $\psi(\sigma) = \psi_+(\sigma)$ for $0 \leq \sigma \leq \pi$, and $\psi(\sigma) = \psi_-(-\sigma)$ for $-\pi \leq \sigma \leq 0$. In terms of this single field, the overlap conditions (3.2) read

$$
\psi(\sigma) = \begin{cases} 
  i\psi(\pi - \sigma), & 0 \leq |\sigma| \leq \pi 2, \\
  -i\psi(\pi - \sigma), & \pi 2 \leq |\sigma| \leq \pi.
\end{cases}
$$

This condition leads to the following relation for the modes

$$
\begin{pmatrix} \psi_r \\ \psi_{-r} \end{pmatrix} = \begin{pmatrix} M_{rs} & \tilde{M}_{rs} \\ -\tilde{M}_{rs} & -M_{rs} \end{pmatrix} \begin{pmatrix} \psi_r \\ \psi_{-r} \end{pmatrix},
$$

where the matrices $M, \tilde{M}$, are defined by

$$
M_{rs} = -2\pi i^{r-s}r + s, \quad r = s \pmod{2},
\tilde{M}_{rs} = 2\pi i^{r+s}r - s, \quad r = s + 1 \pmod{2}.
$$

These matrices will play an important role in this paper. They satisfy the following properties:

$$
M^2 - \tilde{M}^2 = 1, \quad [M, \tilde{M}] = 0,
\overline{M} = M^T = M, \quad \overline{\tilde{M}} = -\tilde{M}^T = \tilde{M}.
$$

From (3.4) one obtains the following relation between positive and negative modes for the fermion fields that annihilate the identity,

$$
\psi_r = (\tilde{M}1 - M)_{rs} \psi_{-s}.
$$

Using this relation, one can then show that the identity is a squeezed state,
\[ |I\rangle = \mathcal{N}_I \exp \left[ 12\eta_{\mu\nu} \sum_{r,s \geq 1/2} \psi_{-r}^{\mu} I_{rs} \psi_{-s}^{\nu} \right] |0\rangle, \quad (3.10) \]

where

\[ I = \tilde{M} 1 - M. \quad (3.11) \]

This equation can be obtained acting with \( \psi_\mu^{\mu} \) on \( |I\rangle \) and using (3.9). In (3.10), \( \mathcal{N}_I \) is a normalization constant that we shall determine later, when we discuss the supersliver. One can also determine the coefficients \( I_{rs} \) explicitly by using conformal mapping techniques. The result, derived in [29], is the following. Defining the coefficients

\[ \hat{u}_{2n} = \hat{u}_{2n+1} = \left( -\frac{1}{2} \right)^n = (-1)^n(2n-1)!2^n n!, \quad (3.12) \]

one has

\[ I_{rs} = \delta^{r+s} \left( I_{nm}^+ r + s - I_{nm}^- r - s \right), \quad r = n + 1/2, \quad s = m + 1/2, \quad (3.13) \]

where

\[ I_{nm}^\pm = \begin{cases} 
-\mu \hat{u}_n \hat{u}_m, & n = \text{even}, \ m = \text{odd}, \\
\pm \mu \hat{u}_n \hat{u}_m, & n = \text{odd}, \ m = \text{even}, \\
0, & \text{otherwise}. 
\end{cases} \quad (3.14) \]

One can check that this explicit expression satisfies the equation (3.11) (see the Appendix).

3.2. Interaction Vertex and Overlap Equations

The interaction vertex, \( |V_3\rangle \), involves the gluing of three strings and determines the star algebra multiplication rule,

\[ |\Phi \ast \Psi\rangle_{(3)} = \langle (1)|\Phi|_2\langle \Psi||V_3\rangle_{(123)}. \quad (3.15) \]

In the operator formulation, this vertex involves a set of infinite–dimensional matrices whose entries are called Neumann coefficients. Usually, in order to find an explicit expression for these coefficients, one uses conformal mapping techniques. On the other hand, in order to understand the structural properties of these matrices, it turns out to be very convenient to analyze the overlap equations as well. In this section we shall deduce an expression for the Neumann coefficients in terms of the
matrices $M, \tilde{M}$, which will be very useful in the following. The starting point is the overlap equation for the three string interaction vertex. This overlap equation simply states that the interaction is obtained by gluing the halves of the three strings in the usual way \[1\]. In the fermionic case the equation reads \[29\]:

$$
\left( \psi^a(\sigma) - i\psi^{a-1}(\pi - \sigma) \right) |V_3\rangle = 0, \quad 0 \leq \sigma \leq \pi, \quad a = 1, 2, 3.
$$

(3.16)

The index $a$ labels each of the three strings. As in \[30\], it is convenient to diagonalize this condition by introducing the following discrete Fourier transforms

$$
q = 1\sqrt{3} \left( \psi^1 + \omega \psi^2 + \bar{\omega} \psi^3 \right),
$$

(3.17)

$$
q_3 = 1\sqrt{3} \left( \psi^1 + \psi^2 + \psi^3 \right),
$$

(3.18)

together with their adjoints,

$$
\bar{q}^\dagger = 1\sqrt{3} \left( (\psi^1)^\dagger + \omega (\psi^2)^\dagger + \bar{\omega} (\psi^3)^\dagger \right),
$$

(3.19)

$$
\bar{q}_3^\dagger = 1\sqrt{3} \left( (\psi^1)^\dagger + (\psi^2)^\dagger + (\psi^3)^\dagger \right),
$$

(3.20)

where $\omega = e^{2\pi i/3}$ is a cubic root of unity. The overlap conditions give the following condition for $q_3$:

$$
q_3(\sigma) = i q_3(\pi - \sigma), \quad 0 \leq \sigma \leq \pi,
$$

(3.21)

which is identical in structure to (3.3). On the other hand, for $q(\sigma)$ we find

$$
q(\sigma) = \begin{cases} 
\begin{align*}
 i\omega q(\pi - \sigma), & 0 \leq \sigma \leq \pi, \\
 -i\bar{\omega} q(\pi - \sigma), & \pi \leq \sigma \leq \pi.
\end{align*}
\end{cases}
$$

(3.22)

The overlap conditions for $q$ then yield the following relation between the modes,

$$
\left( \begin{array}{c}
q_r \\
\bar{q}_r^\dagger
\end{array} \right) = \begin{cases} 
\begin{align*}
 -12 \left( \begin{array}{cc}
 M_{rs} & \tilde{M}_{rs} \\
 -\tilde{M}_{rs} & -M_{rs}
\end{array} \right) + \sqrt{3} \left( \begin{array}{cc}
 0 & iC_{rs} \\
 -iC_{rs} & 0
\end{array} \right),
\end{align*}
\end{cases}
\right)
\left( \begin{array}{c}
q_r \\
\bar{q}_r^\dagger
\end{array} \right),
$$

(3.23)

where the matrix $C$ is defined by

$$
C_{rs} = (-1)^{r+1/2} \delta_{rs}.
$$

(3.24)

This matrix implements BPZ conjugation and satisfies the following conditions:
\[ C^2 = 1, \quad C^T = \overline{C} = C, \quad (3.25) \]
\[ CMC = M, \quad \overline{CMC} = -\overline{M}, \quad (3.26) \]
\[ CIC = -I, \quad (3.27) \]

which guarantee the consistency of (3.23).

We now write the three string vertex as:

\[ |V_3⟩ = \exp \left[ 12q_3^\dagger \cdot I \cdot q_3 + q_3^\dagger \cdot U \cdot \bar{q}^\dagger \right] |0⟩_{(123)}, \quad (3.28) \]

where \( I \) is the matrix (3.11). This is of course a consequence of (3.21). Since \( q|V_3⟩ = U\bar{q}^\dagger|V_3⟩ \), by using (3.23) we obtain an explicit expression for \( U \) in terms of \( M, \overline{M} \) and \( C \):

\[ U = -12 + M \cdot (\overline{M} - i\sqrt{3}C). \quad (3.29) \]

Using the above properties of \( M, \overline{M} \) and \( C \), it is easy to show that \( U \) satisfies,

\[ U = -U^T = -CUC, \quad IU = \overline{U}I, \quad I\overline{U} = UI. \quad (3.30) \]

The following formulae will also be useful:

\[ I^2 = M + 1M - 1, \]
\[ U^2 = \overline{U}^2 = M - 2M + 2, \]
\[ U + \overline{U} = -2\overline{M}M + 2, \]
\[ U - \overline{U} = 2\sqrt{3}iCM + 2. \quad (3.31) \]

Let us now find the structure of the Neumann coefficients for the three string vertex. These coefficients are defined through:

\[ |V_3⟩ = \exp \left[ 12\eta_{\mu\nu} \sum \psi^{(a)\mu}_{-r} K^{'(b)\nu}_{rs} \psi^{(b)\nu}_{-s} \right] |0⟩_{(123)}, \quad (3.32) \]

and satisfy the condition

\[ K^{'ab}_{rs} = -K^{'ba}_{sr}. \quad (3.33) \]

Using the above results, one finds that
\[ K^{ab} = 13(I + \omega^{b-a}U + \omega^{a-b}U), \]  
(3.34)

which has the same structure as the Neumann coefficients in the bosonic sector. We also have the cyclicity property, \( K^{a+1,b+1} = K^{ab} \). We shall frequently use the matrices \( K^{11}, K^{12} \) and \( K^{21} \), which are given by

\[
\begin{align*}
K^{11} &= 13(I + U + \overline{U}), \\
K^{12} &= 16I - 16(U + \overline{U}) + i\sqrt{36}(U - \overline{U}), \\
K^{21} &= 16I - 16(U + \overline{U}) - i\sqrt{36}(U - \overline{U}).
\end{align*}
\]
(3.35)

Again, one can use conformal mapping techniques to write explicit expressions for the Neumann coefficients [29]. The result is the following. Define the coefficients \( g_n \) through the expansion

\[ (1 + x)(1 - x)^{1/6} = \sum_{n=0}^{\infty} g_n x^n. \]
(3.36)

Next, define the following quantities:

\[
\begin{align*}
M_{nm}^+ &= -[(-1)^n - (-1)^m][(n + 1)g_{n+1}(m + 1)g_{m+1} - ng_n mg_m], \\
M_{nm}^- &= -[(-1)^n - (-1)^m][ng_n(m + 1)g_{m+1} - (n + 1)g_{n+1}mg_m], \\
\overline{M}_{nm}^+ &= [(1)^n + (-1)^m][(n + 1)g_{n+1}(m + 1)g_{m+1} - ng_n mg_m], \\
\overline{M}_{nm}^- &= [(1)^n + (-1)^m][ng_n(m + 1)g_{m+1} - (n + 1)g_{n+1}mg_m].
\end{align*}
\]
(3.37)

The Neumann coefficients are then given by,

\[
\begin{align*}
K_{rs}^{\alpha+1} &= 13I_{rs} + i^{r+s+1} \left[ M_{r-1/2,s-1/2}^+ r + s + M_{r-1/2,s-1/2}^- r - s \right], \\
K_{rs}^{\alpha-1} &= 12I_{rs} - 12K_{rs}^{\alpha} - 12\sqrt{3}i^{r+s-1} \left[ \overline{M}_{r-1/2,s-1/2}^+ r + s + \overline{M}_{r-1/2,s-1/2}^- r - s \right].
\end{align*}
\]
(3.38)

Using (3.34), (3.38), one can find explicit expressions for the matrices \( U + \overline{U} \) and \( U - \overline{U} \):

\[
\begin{align*}
(U + \overline{U})_{rs} &= 3i^{r+s} \left[ M_{r-1/2,s-1/2}^+ r + s + M_{r-1/2,s-1/2}^- r - s \right], \\
(U - \overline{U})_{rs} &= 3i^{r+s} \left[ \overline{M}_{r-1/2,s-1/2}^+ r + s + \overline{M}_{r-1/2,s-1/2}^- r - s \right].
\end{align*}
\]
(3.39)
Notice that the matrix $U - \overline{U}$ has nonzero diagonal terms. Using the results of [29], one finds,

$$
(U - \overline{U})_{rr} = 6i \left[ (n + 1)^2 g_{n+1}^2 - n^2 g_n^2 2n + 1 + 13 \sum_{i=0}^{n} (-1)^i g_{n-i}^2 \right].
$$

(3.40)

In the Appendix we show that these explicit expressions indeed agree with (3.29).

For the calculations in the next section, it will be useful to define the following matrices

$$
M^{ab} = CK^{ab}.
$$

(3.41)

Using (3.34) and the relations (3.30), it is easy to see that these matrices satisfy the following properties

$$
[M^{ab}, M^{a'b'}] = 0, \quad [CI, M^{ab}] = 0.
$$

(3.42)

These properties are of course similar to the properties of the matrices $M^{ab}$ in the bosonic case [24, 19].

4. The Supersliver

As we discussed in section 2, a factorized string field satisfies the equations of motion of vacuum superstring field theory, with a pure ghost BRST operator, $Q$, if the ghost part satisfies (2.21) and the matter part is idempotent. We shall now consider idempotent matter states with the factorized form

$$
|\Psi\rangle = |\Psi_b\rangle \otimes |\Psi_f\rangle,
$$

(4.1)

where $|\Psi_{b,f}\rangle$ denote states which are obtained from the vacuum by acting with bosonic and fermionic oscillators, respectively, and which are idempotent with respect to the star product in their respective matter sectors. In this section we will look for idempotent states in the fermionic sector. First, we provide an algebraic construction, in the spirit of [24]. Then we compare the solution to the geometric construction of the sliver given in [23].

4.1. Algebraic Construction

Our purpose here is to find a state in the fermionic part of the matter sector that star squares to itself. Our ansatz is a squeezed state of the form

$$
|\Psi_F\rangle = \mathcal{N}_F \exp \left[ -12 \eta_{\mu\nu} \sum_{r,s \geq 12} \psi^\mu_r F_{rs} \psi^\nu_s \right] |0\rangle,
$$

(4.2)
where $F_{rs}$ is an antisymmetric matrix. Recall that the star product of two states, $|\Psi\rangle$, $|\Phi\rangle$, defined as

$$|\Psi \star \Phi\rangle_{(3)} = \langle 1 | \langle 2 | (\Phi|V_3\rangle_{(123)} ,$$

involves the BPZ conjugate of the string field states. To obtain the BPZ conjugate of $|\Psi_F\rangle$, one has to take into account that

$$\text{bpz}(\psi^\mu_r) = (-1)^{r+1/2} \psi^\mu_r .$$

Therefore, the matrix that implements BPZ conjugation is $C$. It will be useful in the following to define:

$$H = CF .$$

In order to evaluate the star product, one still needs the following formula. Let $b_i, b_i^\dagger$ be fermionic oscillators with anticommutation relations $\{b_i, b_j^\dagger\} = \delta_{ij}$, let $\lambda_i, \mu_i$ be a set of Grassmann variables, and let $S_{ij}, T_{ij}$ be antisymmetric matrices. One then has

$$\langle 0 | \exp \left( \lambda^T \cdot b + 12b \cdot S \cdot b \right) \exp \left( \mu^T \cdot b^\dagger + 12b^\dagger \cdot T \cdot b^\dagger \right) | 0 \rangle =$$

$$= \left[ \det(1 + ST) \right]^{1/2} \exp \left[ \mu^T (1 + ST)^{-1} \lambda + 12\lambda^T T(1 + ST)^{-1} \lambda + 12\mu^T (1 + ST)^{-1} S \mu \right] .$$

Similar expressions for bosonic oscillators and for the ghost $bc$ system were presented in [24, 19]. Using this formula, one obtains the following expression:

$$|\Psi_F \star \Psi_F\rangle_{(3)} = N_F^2 \left[ \det(1 + \Phi \mathcal{K}) \right]^5 \cdot$$

$$\cdot \exp \left[ 12\eta_{\mu\nu} \left\{ \chi^{\mu T} (1 + \Phi \mathcal{K})^{-1} \Phi \chi^\nu + 12\psi^3_{-\mu} K_{rs}^3 \psi^3_{-\nu} \right\} \right] | 0 \rangle_{(3)} ,$$

where

$$\Phi = \begin{pmatrix} -HC & 0 \\ 0 & -HC \end{pmatrix} , \quad \mathcal{K} = \begin{pmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{pmatrix} , \quad \chi^\mu = \begin{pmatrix} K^{13} \psi^3_{-\mu} \\ K^{23} \psi^3_{-\mu} \end{pmatrix} .$$

Using (3.33) and the cyclicity property, one further obtains the following equation for $H$,

$$H = -M^{11} - (M^{12} M^{21}) \left( \begin{pmatrix} 1 - HM^{11} & -HM^{12} \\ -HM^{21} & 1 - HM^{22} \end{pmatrix} \right)^{-1} \begin{pmatrix} HM^{21} \\ HM^{12} \end{pmatrix} ,$$

16
and the following value for the normalization constant,

\[ \mathcal{N}_F = \left[ \det(1 + \Phi K) \right]^{-5}. \]  

(4.10)

Since the matrices \( M^{ab} \) commute, one can assume that \([H, M^{ab}] = 0\) and proceed as if we were dealing with commuting variables. After some simple algebra, one finds the following cubic equation for \( H \):

\[ A_3 H^3 + A_2 H^2 + A_1 H + A_0 = 0, \]  

(4.11)

where

\[ A_3 = M^{12} M^{21} - (M^{11})^2, \]
\[ A_2 = 3M^{11} M^{12} M^{21} - (M^{11})^3 - (M^{12})^3 - (M^{21})^3, \]
\[ A_1 = -1 - 2A_3, \]
\[ A_0 = -M^{11}. \]  

(4.12)

In the bosonic case analyzed in [24] and [19], the coefficients of the cubic equation for the bosonic piece of the sliver could be simplified by using relations among the matrices of Neumann coefficients. Here, it is convenient to express (4.12) in terms of the matrices \( U, \bar{U} \) and \( I \), which in turn can be expressed in terms of \( C, M \) and \( \bar{M} \). After some simple algebra, one finds the following:

\[ A_3 = 13(U + \bar{U})I + 16(U^2 + \bar{U}^2) = \]
\[ = MM + 2, \]
\[ A_2 = 12C(U^2 + \bar{U}^2)I + 23C(I + U + \bar{U}) = \]
\[ = -\bar{C}\bar{M}M - 2(M - 1)(M + 2), \]
\[ A_1 = -3M + 2M + 2, \]
\[ A_0 = \bar{C}\bar{M}M(M - 1)(M + 2). \]  

(4.13)

Since \( |I \star I| = |I| \), an important check of the above is whether \( H = -CI \) is a solution of (4.11). In fact, one can further write (4.11) as

\[ (H + CI)(MH^2 - 2\bar{C}\bar{M}H - M) = 0. \]  

(4.14)

In order to solve the quadratic equation, one has to be careful when extracting the square root. Since \( F \) must be antisymmetric, and remembering that \( F = CH \), one finds the two solutions,
\[ F^\pm = \tilde{M}M \left( 1 \pm 1\sqrt{1 - M^2} \right). \] (4.15)

Notice that \( CF^\pm C = -F^\pm \). It is also easy to check that \( H \) commutes with \( M^{ab} \), as assumed in our initial ansatz. Using the above result for \( F^\pm \), one can compute

\[ (1 + \Phi K)^{-1} = -14(M - 1)(M + 2) \left( 1 \mp M + 1\sqrt{1 - M^2} \right), \] (4.16)

which determines the normalization constant \( N_{F^\pm} \) through (4.10). Using again (4.6), one can further compute the norm of \( |\Psi^{F\pm}\rangle \) and find, for both signs,

\[ \langle \Psi^{F\pm}|\Psi^{F\pm}\rangle = \left[ \operatorname{det}((1 - M)(1 + M/2)^2) \right]^5. \] (4.17)

Finally, notice that in order for the identity to star square to itself, one needs

\[ N_I = \left[ \operatorname{det}((1 - M)(1 + M/2)) \right]^5, \] (4.18)

and its BPZ norm turns out to be

\[ \langle I|I\rangle = \left[ \operatorname{det}(2(1 - M)(1 + M/2)^2) \right]^5. \] (4.19)

4.2. Numerical Results and Comparison to the Geometric Sliver

The above results involve infinite–dimensional matrices. They can however be analyzed numerically by restricting the matrix rank to \( L < \infty \) and then using suitable numerics in order to study the limit \( L \to \infty \), as in [19]. The first thing to notice is that the determinant of \( M \) converges to zero very rapidly. As a consequence, the solution \( F^+ \), which behaves like \( 2\tilde{M}M^{-1} \), has diverging eigenvalues. The other solution, which behaves like \( \tilde{M}M/2 \), has a better behavior. This is the solution that we will discuss in the rest of the paper, and we shall henceforth simply denote it by \( F = F^- \).

It turns out that \( F \) is the matrix that appears in the fermionic part of the geometric sliver constructed by Rastelli and Zwiebach in [23]. Since the sliver can be defined purely in geometric terms, one can construct a supersliver in the CFT given by the NS sector of the superstring. Recall that the (super)sliver is defined by

\[ \langle \Xi| = \langle 0|U_f, \] (4.20)

where \( U_f \) is the operator associated to the conformal transformation given by
\[ f(z) = \arctan(z). \] (4.21)

The structure of the operator \( U_f \) was found in \([23]\). It is given by:

\[ U_f = e^{\sum_{n=1}^{\infty} a_n L_{-2n}}, \] (4.22)

where the coefficients \( a_n \) can be computed explicitly. The Virasoro operators split as \( L = L_b + L_f + L_g \), where \( b, f, g \) refer respectively to the bosonic matter, fermionic matter and ghost/superghost sectors. As a consequence, the supersliver will factorize as:

\[ |\Xi\rangle = |\Xi_b\rangle \otimes |\Xi_f\rangle \otimes |\Xi_g\rangle. \] (4.23)

The bosonic matter part is the one constructed algebraically in \([24]\). In the following, we will present evidence that the fermionic matter part is the idempotent state constructed above and corresponding to \( F \), \( i.e. \),

\[ |\Xi_f\rangle = |\Psi_F\rangle. \] (4.24)

The first step is, as in \([19]\), to write \( |\Xi_f\rangle \) as a squeezed state:

\[ |\Xi_f\rangle = \mathcal{N} \exp \left[ -12\eta_{\mu\nu} \sum_{r,s} \psi_{-r}^\mu \hat{F}_{rs} \psi_{-s}^\nu \right] |0\rangle. \] (4.25)

Using the CFT techniques of \([36, 37, 38]\), one finds for the matrix \( \hat{F} \):

\[ \hat{F}_{rs} = -\oint dw 2\pi i \oint dz 2\pi i z^{-r-1/2} w^{-s-1/2} (1 + z^2)^{1/2} (1 + w^2)^{1/2} (\tan^{-1}(z) - \tan^{-1}(w)). \] (4.26)

One can see that \( \hat{F}_{rs} = 0 \) if \( r + s = \text{odd} \), \( i.e. \), \( C\hat{F}C = -\hat{F} \), as follows from the algebraic description. Evaluating the residues, one finds for the first nonzero entries:

\[ \hat{F}_{1/2,3/2} = -16 \simeq -0.1666, \quad \hat{F}_{1/2,7/2} = 4360 \simeq 0.1194, \quad \hat{F}_{1/2,11/2} = -14695120 \simeq -0.0964, \]

\[ \hat{F}_{3/2,5/2} = -140 \simeq -0.0250, \quad \hat{F}_{3/2,9/2} = 7115120 \simeq .0046, \quad \hat{F}_{3/2,7/2} = -2397560 \simeq -0.0316. \] (4.27)
On the other hand, we can evaluate numerically the first few coefficients $F_{rs}$. Since $M, \tilde{M}$ do not commute at finite rank, we can approximate the matrix $F$ in two ways: multiplying $\tilde{M}$ on the right, or on the left. The results are shown, respectively, in the following tables:

<table>
<thead>
<tr>
<th>$L$</th>
<th>$F_{1/2, 3/2}$</th>
<th>$F_{1/2, 7/2}$</th>
<th>$F_{1/2, 11/2}$</th>
<th>$F_{3/2, 5/2}$</th>
<th>$F_{3/2, 9/2}$</th>
<th>$F_{5/2, 7/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.1929</td>
<td>0.1427</td>
<td>-0.1186</td>
<td>0.0033</td>
<td>-0.0178</td>
<td>-0.0448</td>
</tr>
<tr>
<td>100</td>
<td>-0.1876</td>
<td>0.1347</td>
<td>-0.1102</td>
<td>-0.0058</td>
<td>-0.0099</td>
<td>-0.0398</td>
</tr>
<tr>
<td>150</td>
<td>-0.1847</td>
<td>0.1335</td>
<td>-0.1091</td>
<td>-0.0074</td>
<td>-0.0087</td>
<td>-0.0391</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-0.1676</td>
<td>0.1235</td>
<td>-0.1098</td>
<td>-0.0268</td>
<td>0.0036</td>
<td>-0.0347</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L$</th>
<th>$F_{1/2, 3/2}$</th>
<th>$F_{1/2, 7/2}$</th>
<th>$F_{1/2, 11/2}$</th>
<th>$F_{3/2, 5/2}$</th>
<th>$F_{3/2, 9/2}$</th>
<th>$F_{5/2, 7/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.1140</td>
<td>0.0752</td>
<td>-0.0570</td>
<td>-0.0397</td>
<td>0.0163</td>
<td>-0.0076</td>
</tr>
<tr>
<td>100</td>
<td>-0.1299</td>
<td>0.0886</td>
<td>-0.0683</td>
<td>-0.0346</td>
<td>0.0122</td>
<td>-0.0155</td>
</tr>
<tr>
<td>150</td>
<td>-0.1328</td>
<td>0.0910</td>
<td>-0.0710</td>
<td>-0.0338</td>
<td>0.0115</td>
<td>-0.0168</td>
</tr>
<tr>
<td>$\infty$</td>
<td>-0.1726</td>
<td>0.1251</td>
<td>-0.1020</td>
<td>-0.0250</td>
<td>0.0045</td>
<td>-0.0335</td>
</tr>
</tbody>
</table>

The last entry shows an extrapolation to $L = \infty$ by fitting fifteen points $L = 10, 20, \ldots, 150$ to $a_0 + a_1/(\log L) + a_2(\log L)^2$. We see that there is good agreement with the exact result (4.27), and this provides good numerical evidence that the matrix $F$ is indeed given by the double residue (4.26).

It is also interesting to consider the behavior of the BPZ norms of the fermionic identity and the fermionic part of the sliver. The fermionic identity turns out not to be normalizable: the determinant in (4.18) grows very quickly as we increase the rank. On the other hand, the norm of $|\Xi_f\rangle$, given in (4.17), behaves like the norm of $|\Xi_b\rangle$ analyzed in [19]: an extrapolation to infinite rank, by fitting one hundred points $L = 10, 20, \ldots, 1000$, gives $\langle \Xi_f|\Xi_f\rangle^{1/5} = -0.0075$. This seems to indicate that the norm of the fermionic part of the supersliver is zero.

### 4.3. Conservation Laws

In this subsection we wish to derive conservation laws satisfied by the supersliver, involving the superconformal generators, $G_r$, and following [23, 19]. We shall be schematic, as the procedure is by now well known. Observe that due to its purely geometrical construction (4.20) the supersliver will clearly satisfy all the Virasoro conservation laws outlined in the Appendix of [19], involving the $L_n$ generators of the conformal algebra which now will have a fermionic matter piece as well as a bosonic and ghost pieces. Let us then outline how can one derive the conservation laws associated to the rest of the superconformal algebra, i.e., the ones depending on the $G_r$ generators.

The sliver surface state is defined in the upper half plane by the conformal map,

$$f_H(z) = \arctan(z), \quad (4.28)$$
while in the unit disk (coordinates that we will use in the following), it is given by

\[ f_U(z) = \frac{1 + i \arctan(z)}{1 - i \arctan(z)}. \]  

(4.29)

The usual contour deformation argument yields the expected conservation law,

\[ \langle \Xi \mid \oint dz \varphi(z) \ G(z) = 0, \]  

(4.30)

where \( G(z) \) is the super stress tensor, \( G(z) = \sum G_r/z^{r+\frac{3}{2}} \), and the conformal densities \( \varphi(z) \) now have weight \( -1/2 \). Precisely because of this non–integer weight, one has to be careful when taking the conformal transformation,

\[ \varphi(z) = \tilde{\varphi}(f(z)) \left( \frac{df(z)}{dz} \right)^{-\frac{1}{2}}, \]  

(4.31)

so that we shall adopt the standard conventions [17].

With the choice of conformal density,

\[ \varphi(z) = -\frac{4}{3} \sqrt{\frac{2}{3}} (1 - i) \left( 1 + \frac{1}{z - 1} \right), \]  

(4.32)

one obtains the following conservation law,

\[ \langle \Xi \mid \left( G_{-3/2} + \frac{11}{6} G_{1/2} + \frac{43}{360} G_{5/2} - \frac{1039}{15120} G_{9/2} + \cdots \right) = 0. \]  

(4.33)

If instead one chooses the conformal density,

\[ \varphi(z) = -\frac{4}{3} \sqrt{\frac{2}{3}} (1 - i) \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \frac{\sqrt{3}}{z - e^{\frac{2\pi i}{3}}} \right), \]  

(4.34)

one obtains the conservation law,

\[ \langle \Xi \mid \left( G_{-1/2} + \frac{1}{\sqrt{3}} G_{1/2} + \frac{11}{6} G_{3/2} + \frac{3}{2\sqrt{3}} G_{5/2} + \frac{7}{72} G_{7/2} + \cdots \right) = 0. \]  

(4.35)

Other conservation laws can be obtained in similar fashions.
5. Fermionic Star Algebra Spectroscopy

In this section we follow the methods of [26], in order to find the eigenvalue spectrum of the various infinite–dimensional matrices involved in the star algebra for the matter fermionic sector, as well as the corresponding eigenvectors. We first find by inspection an eigenvector of \( M \) and \( \tilde{M} \), and we then adapt the methods of [26] to find the rest of the spectrum. The star algebra spectroscopy has also been studied in [39, 40].

5.1. An Eigenvector of \( M \) and \( \tilde{M} \)

In this subsection we want to show that the matrices \( M \) and \( \tilde{M} \) have a common eigenvector with eigenvalues \(-1\) and 0, respectively. First define

\[
\nu_{n-1/2} = \begin{cases} \left( -\frac{1}{2}, k \right), & n = 2k + 1, \\ 0, & n = 2k. \end{cases}
\] (5.1)

Using (A.1), and setting \( r = n + 1/2 \), one easily finds

\[
\sum_s M_{rs} \nu_s = -2\pi \sum_{m=0}^{\infty} (-1)^{n-m}2m + 2n + 1 \left( -1/2 \right) = - \left( -1/2 \right) = -\nu_r,
\]

\[
\sum_s \tilde{M}_{rs} \nu_s = 2\pi \sum_{m=0}^{\infty} (-1)^{n+m}2m - 2n - 1 \left( -1/2 \right) = (1)^n \pi \Gamma(12) \Gamma(-n - 12) \Gamma(-n) = 0.
\] (5.2)

Therefore, \( \nu_r \) is a common eigenvector to \( M \) and \( \tilde{M} \) with eigenvalues \(-1\) and 0, respectively. This vector can be understood geometrically as follows. Notice that its components are the negative modes in the Fourier expansion of the function

\[
f(\sigma) = e^{-i\sigma^2} \sqrt{1 + e^{-2i\sigma}}.
\] (5.3)

This function is antiperiodic in \([-\pi, \pi]\) and satisfies the overlap equation \( f(\sigma) = if(\pi - \sigma) \). Therefore, its modes satisfy the equation (3.4). Since the positive modes are set to zero, it follows from (3.4) that the coefficients of the Fourier expansion give an eigenvector of \( M \) and \( \tilde{M} \) with the required eigenvalues. Finally, notice that the vector \( \nu \) is BPZ odd, since \( C\nu = -\nu \). A related discussion of the geometric meaning of the eigenvectors in the bosonic case can be found in [40].

5.2. Diagonalizing \( K_1 \)

To find the rest of the spectrum, we generalize the considerations of [26] to the fermionic sector. The derivation of the star algebra,
\[ K_1 = L_1 + L_{-1}, \] (5.4)

has a fermionic part which is a sum of bilinears in the modes \( \psi_{\pm r} \). This allows for a definition of an infinite–dimensional matrix as follows. Let \( \{ v_r \}_{r \geq 1/2} \) be an infinite–dimensional vector. Define then the matrix \( K_1 \) through

\[ [K_1, v \cdot \psi] = (K_1 v) \cdot \psi, \] (5.5)

where \( v \cdot \psi = \sum_{r=1/2}^{\infty} v_r \psi_r \). In what follows, it will be quite useful to label the positive half–integer indices with integer numbers by setting \( r = n - 1/2, n = 1, 2, \ldots \). Using the explicit expression for the Virasoro generators, we then find:

\[ (K_1)_{nm} = -(n - 1) \delta_{n-1,m} - n \delta_{n+1,m}. \] (5.6)

This is a symmetric matrix that anticommutes with \( C \), \( \{ K_1, C \} = 0 \). To find its spectrum, one associates to every vector \( w \) a function \( f_w(z) \) as follows

\[ f_w(z) = \sum_{n=1}^{\infty} w_n z^n. \] (5.7)

The infinite–dimensional matrix \( K_1 \) is then represented in the space of functions by the differential operator

\[ \mathcal{K}_1 = -(1 + z^2) ddz + 1z, \] (5.8)

and the problem of finding eigenvectors of \( K_1 \) now becomes the problem of finding eigenfunctions for this differential operator. The solution is immediate: for any \( -\infty < \kappa < \infty \) there is an eigenfunction of \( \mathcal{K}_1 \) given by

\[ f_{w(\kappa)}(z) = z \sqrt{1 + z^2} \exp(-\kappa \ \arctan(z)), \] (5.9)

with eigenvalue \( \kappa \). The normalization of this function has been chosen so that \( w_{1(\kappa)} = 1 \). One then concludes that \( K_1 \) has a non–degenerate, continuous spectrum, similar to the bosonic case studied in [26]. Also notice that

\[ f_{Cw}(z) = -f_w(-z), \] (5.10)

so that the BPZ matrix acts as

\[ Cw^{(\kappa)} = -w^{(-\kappa)}. \] (5.11)
5.3. Diagonalizing $M$ and $\tilde{M}$

We can now use this information in order to find the spectrum of $M$ and $\tilde{M}$. First, observe the following properties,

$$[K_1, CI] = 0,$$
$$[K_1, M^{ab}] = 0.$$  \hfill (5.12)

The first equation follows from the fact that $K_1$ kills the identity, and the second one from the fact that $K_1$ is a derivation of the star algebra, and then $(K_1^{(1)} + K_1^{(2)} + K_1^{(3)})|V_3⟩ = 0$ \cite{41}. To derive (5.12), we have also used the fact that $K_1$ anticommutes with $C$. Making use of (3.34), it follows that

$$[K_1, M] = [K_1, C\tilde{M}] = 0.$$  \hfill (5.13)

Therefore, and since the spectrum of $K_1$ is nondegenerate, an eigenvector of $K_1$ has to be an eigenvector of $M$ and $CM$ as well. Notice that this makes sense since $M$ and $CM$ are symmetric, real matrices, and so they have real eigenvalues.

We have then shown that the eigenvectors $w^{(κ)}$ given implicitly in (5.9) are also eigenvectors of $M$ and $CM$. Now, we have to find out which are the corresponding eigenvalues. This can be done with a trick from section 5.2 of \cite{26}. The eigenvalue equations are

$$M_{n-1/2, m-1/2} w^{(κ)}_m = m(κ) w^{(κ)}_n,$$
$$\tilde{M}_{n-1/2, m-1/2} w^{(κ)}_m = \tilde{m}(κ) w^{(κ)}_n.$$  \hfill (5.14)

Since we chose the normalization $w^{(κ)}_1 = 1$, one can consider the above equations with $n = 1$ and obtain for the eigenvalues:

$$m(κ) = 2\pi \sum_{q=1}^{∞} (-1)^q 2q - 1 w^{(κ)}_{2q-1},$$
$$\tilde{m}(κ) = -2\pi \sum_{q=1}^{∞} (-1)^q 2q - 1 w^{(κ)}_{2q}.$$  \hfill (5.15)

Define now the functions

$$\mu(z) = \sum_{q=1}^{∞} (-1)^q 2q - 1 w^{(κ)}_{2q-1} z^{2q-1},$$
$$\tilde{μ}(z) = \sum_{q=1}^{∞} (-1)^q 2q - 1 w^{(κ)}_{2q} z^{2q-1}.$$  \hfill (5.16)
which can be found to satisfy

\[ d\mu(z)dz = i2z(f_{w(\kappa)}(iz) - f_{w(\kappa)}(-iz)), \]
\[ d\tilde{\mu}(z)dz = i2z^2(f_{w(\kappa)}(iz) + f_{w(\kappa)}(-iz)). \] (5.17)

Using the explicit expression for \( f_{w(\kappa)}(z) \), the fact that \( \mu(0) = \tilde{\mu}(0) = 0 \), and integrating, one finally finds

\[ \mu(1) = -\pi 2 \text{sech}(\kappa \pi / 2), \]
\[ \tilde{\mu}(1) = \pi 2 \tanh(\kappa \pi / 2). \] (5.18)

This determines the eigenvalues of \( M \) and \( \tilde{C}M \) for the eigenvectors \( w^{(\kappa)} \):

\[ m^{(\kappa)} = -\text{sech}(\kappa \pi / 2), \]
\[ \tilde{m}^{(\kappa)} = -\tanh(\kappa \pi / 2). \] (5.19)

The spectrum of \( M \) lies in the interval \([-1, 0)\), while that of \( \tilde{C}M \) lies in \((-1, 1)\). The above results are of course compatible with the relation \( M^2 - \tilde{M}^2 = 1 \). Notice finally that, for \( \kappa = 0 \), we recover the results of the previous subsection, since

\[ f_{w^{(0)}}(z) = z\sqrt{1 + z^2} = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \frac{1}{n} z^{2n+1}, \] (5.20)

so \( w^{(0)} = \nu \) and from (5.19) we read that the eigenvalues with respect to \( M \) and \( \tilde{M} \) are in fact \(-1\) and \( 0 \), respectively, in agreement with the explicit computations of the previous subsection.

We can now diagonalize the symmetric matrix \( H = CF \) that defines the fermionic sliver. Since the derivation \( K_1 \) kills the supersliver [19], one has that

\[ [K_1, H] = 0, \] (5.21)

and by the same argument one has that \( w^{(\kappa)} \) are eigenvectors of \( H \). The corresponding eigenvalues will be denoted by \( h^{(\kappa)} \). In order to determine them first notice that, since \( H \) anticommutes with \( C \), one has

\[ h^{(\kappa)} = -h^{(-\kappa)}. \] (5.22)

We can determine \( h^{(\kappa)} \) from the explicit expression given in (4.15). However, one has to be careful when doing this. The reason is that \( \tilde{M}/(1 - M^2)^{1/2} \) gives an indeterminacy of the type \( 0/0 \) when
acting on $w^{(0)}$. If one naively substitutes the eigenvalues in (4.15), one seems to find that $h(0) \neq 0$, which contradicts (5.22). Of course the appropriate way to regularize this indeterminacy is by expanding $(1 - M^2)^{-1/2}$ in powers of $M$, and if this is done then at every order in the expansion one indeed finds the right value of the eigenvalue, which is $h(0) = 0$ (and can also be checked by computing $H w^{(0)}$ in level truncation). Related issues associated to the appearance of inverses of singular matrices have been considered in the bosonic case in [39]. Another subtlety (also present in the bosonic case analyzed in [26]) is that the quadratic equation determining $H$ gives two branches for the eigenvalues, and in fact there is a jump from one branch to the other at $\kappa = 0$. Since the numerical analysis of the spectrum shows that the eigenvalues of $H$ are in the interval $[-1, 1]$, one finally finds that the spectrum of $H$ is given by

$$h(\kappa) = \begin{cases} -\kappa |\kappa| e^{-|\kappa|\pi/2}, & \kappa \neq 0, \\ 0, & \kappa = 0. \end{cases}$$  (5.23)

in agreement with (5.22).

Using all these results, one can also diagonalize the rest of the matrices that we have encountered so far. For example, the eigenvalues of the real symmetric matrices $M^{11}, M^{12}, M^{21}$ are, respectively,

$$m^{11}(\kappa) = -\sinh(\kappa\pi/2)(1 + \cosh(\kappa\pi/2))(1 - 2 \cosh(\kappa\pi/2)),
$$

$$m^{12}(\kappa) = \cosh(\kappa\pi/2)(1 + \cosh(\kappa\pi/2) + \sinh(\kappa\pi/2))(1 + \cosh(\kappa\pi/2))(1 - 2 \cosh(\kappa\pi/2)),
$$

$$m^{21}(\kappa) = -\cosh(\kappa\pi/2)(1 + \cosh(\kappa\pi/2) - \sinh(\kappa\pi/2))(1 + \cosh(\kappa\pi/2))(1 - 2 \cosh(\kappa\pi/2))$$  (5.24)

Of course there is still the possibility that all of these matrices have other eigenvectors which are not eigenvectors of $K_1$. We have not performed a systematic numerical search, but we are led believe that, just as in the bosonic case, the above results determine the complete spectrum of eigenvectors and eigenvalues of the various infinite–dimensional matrices involved in the fermionic matter sector.

It is also interesting to observe that, again just as in the bosonic case [40, 26], the eigenvectors that we have found are not normalizable. This can be seen in detail as follows. Given two infinite–dimensional vectors $v$ and $w$, their inner product is given by

$$v \cdot w \equiv \sum_{n=1}^{\infty} v_n w_n = \int_0^{2\pi} d\theta 2\pi f^*_v(e^{i\theta}) f_v(e^{i\theta}).$$  (5.25)

The norm of a vector $v$ is defined as usual by $\|v\|^2 \equiv v \cdot v$. Using (5.9) and (5.25), one can find an explicit expression for the norm of $w^{(\kappa)}$,

$$\|w^{(\kappa)}\|^2 = \cosh(\kappa\pi/2)\|\nu\|^2,$$  (5.26)

where
\[ \|\nu\|^2 = 4 \int_0^{\pi/2} d\theta 2\pi 1\sqrt{2 + 2\cos(2\theta)}. \] (5.27)

This integral is logarithmically divergent, so the norm of \(\nu\) (and therefore of all \(w^{(\kappa)}\)) is infinite. Another way to see this is to compute directly the sum:

\[ \|\nu\|^2 = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^2. \] (5.28)

By using zeta–function regularization, we find that this series diverges as

\[ \lim_{\epsilon \to 0} \frac{2}{\pi} K(e^{-\epsilon}), \] (5.29)

where \(K(x)\) is the elliptic \(K\)–function, which indeed diverges logarithmically as \(x \to 1\).

### 6. Coherent States and Higher–Rank Projectors

Once the fermionic sliver has been constructed, it is natural to consider fermionic coherent states based on it, in analogy to the bosonic case [24, 20]. In this section we shall construct coherent states and determine their star products. This will be useful in order to construct higher–rank projectors of the fermionic star algebra—idempotent states that should represent multiple \(D\)–brane configurations [20].

#### 6.1. Coherent States on the Supersliver

We define fermionic coherent states as follows. Let \(\beta = \{\beta\}_r, r \geq 1/2\), be a Grassmannian vector. Then, the coherent state on the fermionic sliver associated to \(\beta\), that we shall denote by \(|\Xi_\beta\rangle\), is given by

\[ |\Xi_\beta\rangle = \exp[(-C\beta^T \cdot \psi^\dagger)]|\Xi_f\rangle. \] (6.1)

This definition guarantees that the BPZ conjugate of (6.1) has a simple expression,

\[ \langle\Xi_\beta| = \langle\Xi_f|\exp[\beta^T \cdot \psi]. \] (6.2)

The star product of two coherent states can be computed very easily by using (4.6), and one finds

\[ |\Xi_{\beta_1}\rangle \star |\Xi_{\beta_2}\rangle = \exp\left(\chi^T (1 + \Phi\mathcal{K})^{-1}\beta + 12\beta^T \mathcal{K}(1 + \Phi\mathcal{K})^{-1}\beta\right)|\Xi_f\rangle, \] (6.3)
where $\Phi$, $K$ and $\chi$ are given in (4.8), and $\beta$ denotes here the vector

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$  
(6.4)

An explicit computation yields

$$|\Xi_{\beta_1} \ast |\Xi_{\beta_2} = \exp[N(\beta_1, \beta_2)]|\Xi_{\rho_1\beta_1 - \rho_2\beta_2},$$

(6.5)

where

$$\rho_1 = -11 + \Phi K \left[H(M^{21})^2 + M^{12}(1 - HM^{11})\right] = 12(1 + MH - C\bar{M}),$$
$$\rho_2 = 11 + \Phi K \left[H(M^{12})^2 + M^{21}(1 - HM^{11})\right] = 12(1 - MH + C\bar{M}),$$

(6.6)

and

$$N(\beta_1, \beta_2) = 12 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & A \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = -12 \beta_1 (1 - C\bar{M}) \begin{pmatrix} HM + (M + 2)M^{11} \\ (M + 2)M^{21} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$  
(6.7)

The matrices $\rho_1$ and $\rho_2$ are real symmetric, and they have the following properties:

$$\rho_1 + \rho_2 = 1, \quad \rho_1\rho_2 = 0,$$
$$\rho_1^2 = \rho_1, \quad \rho_2^2 = \rho_2,$$

(6.8)

just as in the bosonic case studied in [20]. This means that $\rho_1$, $\rho_2$ are orthogonal projectors on complementary subspaces. We also have

$$C\rho_1 C = \rho_2.$$

Notice that the vectors $w^{(\kappa)}$ that we described in the previous section are eigenvectors of $\rho_{1,2}$. Let us denote by $\sigma_1(\kappa)$ and $\sigma_2(\kappa)$ the corresponding eigenvalues. By using (5.19) and (5.23), we find that

$$\sigma_1(\kappa) = \begin{cases} 1, & \kappa > 0, \\ 0, & \kappa < 0, \end{cases}$$

(6.9)

with $\sigma_2(\kappa) = 1 - \sigma_1(\kappa)$. Notice that the eigenvalues associated to the vector $\nu$ are $\sigma_1(\kappa) = \sigma_2(\kappa) = 12$. This contradicts in principle the statement that $\rho_1\rho_2 = 0$, and it gives yet another example of a fact noticed in [39]: formal computations involving inverses of matrices like $1 - M^2$ become ambiguous when acting on special eigenvectors.
6.2. Higher–Rank Projectors

It is obvious from (6.5) that the star multiplication law for coherent states becomes particularly simple when $\beta_{1,2}$ are eigenvectors of the projectors $\rho_{1,2}$ or combinations thereof. In this subsection, we will show that with this choice one finds states that form closed subalgebras of the fermionic star algebra. These states can be used to obtain new idempotent states that lead to higher–rank projectors, as in the bosonic situation [20]. The construction is indeed a direct generalization of section 5.2 of [20]. Let $v$ be an eigenvector of $\rho_2$,

$$\rho_1 v = 0, \quad \rho_2 v = v, \tag{6.10}$$

and define $w = -Cv$. Therefore, $\|v\| = \|w\|$, and it follows from $C \rho_1 C = \rho_2$ that one will have

$$\rho_1 w = w, \quad \rho_2 w = 0. \tag{6.11}$$

In addition, one has that $v \cdot w = v^T (\rho_1 + \rho_2) w = 0$, as in [20]. Using the explicit expressions for $\rho_{1,2}$, one can also show that

$$v^T A w = v^T C w = 12v^T M v,$$
$$v^T B w = 12v^T (1 + C\tilde{M}) v, \tag{6.12}$$

where the matrices $A$, $B$ and $C$ are the ones appearing in (6.7).

Consider now the following states, obtained by acting with fermionic creation operators on the fermionic sliver $\Xi_f$ (we suppress the brackets for notational convenience):

$$\Sigma_v = (v\|v\| \cdot \psi^\dagger) \Xi_f,$$
$$\Sigma_w = (w\|w\| \cdot \psi^\dagger) \Xi_f,$$
$$\Xi_{v,w} = (v\|v\| \cdot \psi^\dagger) (w\|w\| \cdot \psi^\dagger) \Xi_f. \tag{6.13}$$

Observe that the state $\Xi_{v,w}$ is Grassmann even, since fermions only appear via bilinears, while the $\Sigma_{v,w}$ states are Grassmann odd. Consider now the coherent states $\Xi_{\beta_1}, \Xi_{\beta_2}$, where $\beta_1 = \theta_1 v + \theta_2 w$, $\beta_2 = \hat{\theta}_1 v + \hat{\theta}_2 w$ and $\theta_{1,2}, \hat{\theta}_{1,2}$ are arbitrary Grassmann variables. It is simple to show, by computing the star product $\Xi_{\beta_1} \ast \Xi_{\beta_2}$, that the states defined in (6.13) satisfy the following subalgebra of the star product, in the fermionic matter sector:
\[ \Xi_f \ast \Sigma_v = 0, \quad \Xi_f \ast \Sigma_w = -\Sigma_w, \]
\[ \Sigma_v \ast \Xi_f = \Sigma_v, \quad \Sigma_w \ast \Xi_f = 0, \]
\[ \Sigma_v \ast \Sigma_v = 0, \quad \Sigma_w \ast \Sigma_w = 0, \]
\[ \Sigma_v \ast \Sigma_w = A_v \Xi_f - \Xi_{v,w}, \quad \Sigma_w \ast \Sigma_v = -B_v \Xi_f, \]
\[ \Xi_f \ast \Xi_{v,w} = A_v \Xi_f, \quad \Xi_{v,w} \ast \Xi_f = A_v \Xi_f, \]
\[ \Sigma_v \ast \Xi_{v,w} = A_v \Sigma_v, \quad \Sigma_w \ast \Xi_{v,w} = B_v \Sigma_w, \]
\[ \Xi_{v,w} \ast \Sigma_v = -B_v \Sigma_w, \quad \Xi_{v,w} \ast \Sigma_w = -A_v \Sigma_w, \] (6.14)

and finally
\[ \Xi_{v,w} \ast \Xi_{v,w} = A_v (A_v - B_v) \Xi_f + B_v \Xi_{v,w}. \] (6.15)

In these equations we have introduced the notation
\[ A_v = v^T A w \|v\|^2, \quad B_v = v^T B w \|v\|^2. \] (6.16)

One can also find the BPZ norm of these states by computing \( \langle \Xi_\beta | \Xi_\beta \rangle \), with \( \beta = \theta_1 v + \theta_2 w \). In this computation one has to evaluate the inner products
\[ v^T F 1 - F^2 w = -\|v\|^2 A_v, \]
\[ v^T 11 - F^2 v = w^T 11 - F^2 w = \|v\|^2 B_v, \] (6.17)

as it can be checked by using the explicit expression for \( F \) and the fact that \( v, w \) are eigenvectors of \( \rho_{1,2} \). One obtains in the end:
\[ \langle \Sigma_v | \Sigma_v \rangle = B_v \langle \Xi_f | \Xi_f \rangle, \]
\[ \langle \Sigma_v | \Sigma_w \rangle = 0, \]
\[ \langle \Xi_f | \Xi_{v,w} \rangle = -A_v \langle \Xi_f | \Xi_f \rangle, \]
\[ \langle \Xi_{v,w} | \Xi_{v,w} \rangle = -(A_v^2 + B_v^2) \langle \Xi_f | \Xi_f \rangle, \] (6.18)

together with their BPZ conjugates (notice that that BPZ conjugation exchanges \( v \leftrightarrow w \)).

One can now use this subalgebra in order to generate new solutions to the idempotency condition, and thus new solutions to the vacuum superstring field theory equations of motion. This one can do by taking the most general combination of the four states, \( \Xi_f, \Xi_{v,w}, \Sigma_v \) and \( \Sigma_w \), (and with the appropriate Chan–Paton factors, since the \( \Sigma \)'s are Grassmann odd). One finds in this way two types of new solutions:

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1) There is one new solution, which is Grassmann even. It is given by

$$\chi_f = \alpha \Xi_f + \beta \Xi_{v,w},$$

(6.19)

where $\chi_f \star \chi_f = \chi_f$ provided one chooses

$$\alpha = -\mathcal{A}_v \mathcal{B}_v, \quad \beta = 1 \mathcal{B}_v.$$  

(6.20)

One has that

$$\chi_f \star \Xi_f = \Xi_f \star \chi_f = 0,$$

(6.21)

and also

$$\langle \chi_f | \chi_f \rangle = \langle \Xi_f | \Xi_f \rangle, \quad \langle \chi_f | \Xi_f \rangle = 0.$$  

(6.22)

Therefore, we see that if one interprets the fermionic sliver as a projector in the space of string fields, the string field $\chi_f$ is a projector on an orthogonal subspace and their sum is then a higher rank projector, as in [20]. Indeed, the fermionic sliver is a rank–one projector on the fermionic sector of the space of half–string functionals. The best way to see this would be of course to construct a half–string formalism for the fermion fields. Unfortunately we have not been able to do that, as we have not found good boundary conditions for the split fermions. However, one can still bosonize the fermions and reduce the problem to the case already analyzed in [20, 27, 28]. In fact, bosonization was used in [28] to show that the ghost part of the bosonic sliver is also a rank–one projector on the ghost sector of the space of half–functionals2.

2) There are two families of new solutions, which have both a Grassmann even and a Grassmann odd piece. The first one is

$$\Xi_\ell = \Xi_f \otimes 1 + \ell \Sigma_v \otimes \sigma_1, \quad \ell \in \mathbb{R},$$

(6.23)

while the second one is

$$\chi_\ell = \chi_f \otimes 1 + \ell \Sigma_w \otimes \sigma_1, \quad \ell \in \mathbb{R}.$$  

(6.24)

These string fields are idempotent for arbitrary real $\ell$, since the Grassmann odd piece is a nilpotent state, and they have the same norm for any $\ell$, which is the norm of the fermionic sliver. Moreover,

2For a discussion on the bosonization of the interaction vertex of the superstring in the operator formalism, see [42].
one can show that $\Xi_\ell$ and $\chi_\ell$ are related to $\Xi$ and $\chi$ by gauge transformations at the vacuum. Indeed, using nilpotency of $\Sigma_{v,w}$ and the fact that

$$[\Sigma_v, \Xi_f]_* = \Xi_f, \quad [\Sigma_w, \chi_f]_* = \chi_f,$$

one finds

$$\Xi_\ell = e^\ell \Sigma_v \otimes \sigma_1 \star \Xi_f \star e^{-\ell} \Sigma_v \otimes \sigma_1,$$
$$\chi_\ell = e^\ell \Sigma_w \otimes \sigma_1 \star \chi_f \star e^{-\ell} \Sigma_w \otimes \sigma_1,$$

(6.26)
i.e., in the terminology of [20] $\Xi_\ell$ and $\chi_\ell$ are star rotations of $\Xi_f$ and $\chi_f$. But on the other hand, star rotations are indeed gauge transformations at the vacuum as it follows from (2.28). In the case we are considering, the gauge parameter is simply given by $\Xi_m = \ell \Sigma_{v,w} \otimes \sigma_1$.

In order to construct higher–rank projectors, we have used simultaneous eigenvectors of the projectors $\rho_1$ and $\rho_2$. These are precisely the $w(\kappa)$ that we have found in section 5, if one assumes that all the eigenvectors of these matrices are the eigenvectors of $K_1$. In this case, one can take for $v$ any vector $w(-\kappa)$ with $\kappa > 0$, and then $w = w(\kappa)$. The states defined in (6.13) give a family of fermionic subalgebras parametrized by $\kappa > 0$, with coefficients

$$A_v = -e^{-\kappa \pi/2} 1 + e^{-\kappa \pi/2}, \quad B_v = 1 + e^{-\kappa \pi/2}.$$  

(6.27)

Notice that we have normalized these states by introducing a factor $1/\|v\|$. In this way, the norms of $v, w$ do not appear in the star subalgebra nor in the BPZ products. Since the vectors $w(\kappa)$ have infinite norm, this normalization factor actually vanishes. Observe, however, that the norm of the (super)ssilver is also strictly zero since it contains a positive power of $\det(1+X)$, and the matrix $X$ is known to have an eigenvalue $-1/3$ [39, 40, 26]. In that respect, the states we have constructed are not essentially different. We should add that the same thing happens to the higher rank projectors constructed in [20]: they are constructed from eigenvectors of the bosonic projectors, which have infinite norm [40, 26], and the construction involves dividing by this norm. This is yet another manifestation of the rather singular structure of the idempotents of the string field star algebra.

7. The Geometric Supersilver and the (Super)Ghost Sector

So far, we have restricted ourselves to the matter sector. In order to have a complete picture, we still have to make a proposal for the the vacuum BRST $Q$ operator, and one has to solve the equations of motion in the ghost sector (2.21). In this section we will show that the ghost/superghost part of the geometric supersilver satisfies (2.21) if we take $Q$ to be the canonical BRST operator recently proposed by Gaiotto, Rastelli, Sen and Zwiebach [25] for the bosonic string, and that we shall denote in the following by $Q_{\text{GRSZ}}$. Observe that this implies that the full geometric supersilver is a solution to the full superstring field theory equations of motion.
Therefore, a natural proposal for vacuum superstring field theory is to take $Q = Q_{GRSZ}$, and postulate that the maximal $D9$–brane is described by the full geometric supersliver. Notice that the string field in Berkovits’ theory has ghost and picture number zero, and therefore the geometric supersliver is a good string field. This is in contrast to bosonic string theory, where the string field has ghost number one and therefore the sliver is not an acceptable string field. Indeed, the $D25$–brane is conjecturally described by the twisted sliver, whose algebraic construction was presented in [43] and has later been constructed in BCFT in [25]. The twisted sliver has in fact ghost number one, as required by cubic bosonic string field theory. We have seen in section 2 that idempotency of the string field seems to be even more useful in superstring field theory, where it reduces drastically the nonlinearity of the equation of motion. In fact, it is easy to see that an idempotent ghost/superghost state satisfying $\Phi g \cdot \Phi g = \Phi g$ reduces the WZW equation of motion (2.21) to a simpler form. If $\Phi_g$ is idempotent, the exponential is linearized as

$$e^{\Phi_g} = \mathcal{I} + (e - 1) \Phi_g;$$

and so the equation of motion becomes

$$\eta_0 \left\{ \left( \mathcal{I} + \left( \frac{1}{e} - 1 \right) \Phi_g \right) Q \Phi_g \right\} = 0.$$  

It is clear that this equation of motion is solved if

$$Q \Phi_g = 0.$$  

Let us then assume that the vacuum BRST operator is the one chosen recently by Gaiotto, Rastelli, Sen and Zwiebach [25] for the bosonic string,

$$Q_{GRSZ} = \frac{1}{2t} \left( c(i) - c(-i) \right).$$

In terms of oscillators, this operator is given by

$$Q_{GRSZ} = \sum_{n=0}^{\infty} (-1)^n C_n,$$

where

$$C_n = c_n + (-1)^n c_n, \quad n \neq 0,$$

$$C_0 = c_0.$$
We can now show that the ghost part of the supersliver is annihilated by $Q_{GRSZ}$, and therefore solves the equations of motion. First of all, notice that the ghost part of the supersliver factorizes into $bc$ and $\beta\gamma$ pieces,

$$|\Xi_g\rangle = |\Xi_{bc}\rangle \otimes |\Xi_{\beta\gamma}\rangle. \quad (7.7)$$

Since the choice $Q_{GRSZ}$ does not involve the superghosts, it is enough to show that

$$Q_{GRSZ}|\Xi_{bc}\rangle = 0. \quad (7.8)$$

We shall do this in two distinct ways. First, we use a geometric argument akin to that in [25]. Secondly, we shall prove it by using oscillator methods.

The geometric argument goes as follows. The supersliver is defined by the following relation,

$$\langle \Xi | \phi \rangle = \langle f \circ \phi \rangle, \quad (7.9)$$

where $f(z) = \arctan(z)$, and $|\phi\rangle$ is any Fock state. If one now acts with the arbitrary Fock state $|\phi\rangle$ on (7.8), one finds

$$\langle \phi \; Q_{GRSZ} | \Xi_{bc} \rangle = 12i \langle f \circ \phi(0) \left( (f'(i))^{-1}c(i\infty) - (f'(-i))^{-1}c(-i\infty) \right) \rangle. \quad (7.10)$$

But $(f'(\pm i))^{-1} = 0$, and therefore the above correlator is zero.

Let us now give an oscillator proof. Using the methods of [36], it is not too hard to show that the $bc$ part of the (super)silver is given by a squeezed state of the form

$$|\Xi_{bc}\rangle = \exp\left( \sum_{s,i} c_{-s} S_{si} b_{-i} \right) |0\rangle, \quad (7.11)$$

where $s = -1, 0, 1, \ldots$, $i = 2, 3, \ldots$, and $S_{si}$ is given by the double residue:

$$S_{si} = \oint_{0} dz 2\pi i 1z^{s-1} \oint_{0} dw 2\pi i 1w^{i+2}(f'(z))^{2}(f'(w))^{-1}f(z) - f(w)\left( f(w)f(z) \right)^{3}. \quad (7.12)$$

A different expression for this matrix has been given in [44]. If we now define $U = \sum_{s,i} c_{-s} S_{si} b_{-i}$, one has that for $n \geq 2$,

$$c_{n} U = U c_{n} - \sum_{s} c_{-s} S_{si}. \quad (7.13)$$

We thank L. Rastelli for pointing this out to us.
Using this result one can easily show that (7.8) holds if and only if the matrix $S$ satisfies

$$\sum_{n=1}^{\infty} S_{2k,2n}(-1)^n = (-1)^k,$$  

(7.14)

where we have also used that, due to twist invariance, $S_{si} = 0$ if $s + i = \text{odd}$. The above equation says essentially that $S$ has an eigenvector with eigenvalue 1. One can check that (7.14) is true by using the explicit representation of $S$ as a double residue and the techniques of [26]. Indeed, since

$$\sum_{n=1}^{\infty} (-1)^n w^{-2n-2} = w^{-2}1 + w^2,$$  

(7.15)

we have to deform the $w$ contour to pick the residue at $w = z$, and this yields

$$\sum_{n=1}^{\infty} S_{2k,2n}(-1)^n = \oint_0 dz \pi i z^{2k-1}11 + z^2 = (-1)^k,$$  

(7.16)

as we wanted to show. This gives yet another proof of (7.8), and also establishes a property of $S$ that may be relevant in future investigations.

Notice that in order to annihilate the identity the BRST operator of [25] has to be regularized in an appropriate way. It is also immediate to observe that this regularization does not affect the above computations.

8. Conclusions and Future Directions

In this paper we have taken the first steps towards the construction of vacuum superstring field theory. More concretely, we have shown that idempotent states play a distinctive role in Berkovits’ string field theory, and after clarifying the structure of the fermionic vertex in the NS sector we have given an explicit algebraic construction of the fermionic sliver. We have also explored some aspects of the star algebra. In particular, we have computed the spectrum of the fermionic Neumann matrices and we have constructed higher–rank projectors by using closed star subalgebras obtained from coherent states. Finally, we have shown that the geometric sliver is a solution to the superstring field theory equations of motion including both matter and ghost sectors.

Clearly, many things remain to be done. There are some obvious open problems that one should address to put this construction on a firmer ground, which we now list as directions for future research.

- The most pressing problem is to construct solutions describing the various BPS and non–BPS $D$–branes of Type IIA superstring theory. It is natural to conjecture that the supersliver describes the tachyonic vacuum of the $D9$–brane, but a necessary test is to verify that one can construct other
$D$–branes with the right ratio of tensions. In [19], lower dimensional $D$–branes were constructed in the bosonic case by first identifying the sliver state with the maximal $D25$–brane and then exploiting the spacetime dependence of the vertex. A more general construction was later described in [21] and implemented in detail in [45]. It should be possible to adapt this construction to the supersymmetric case, although there might be some subtle points that need to be addressed. For example, it is not obvious to us how one would reproduce the mod two behavior of the $D$–brane descent relations in the superstring case, i.e., the fact that in the IIA theory $Dp$–branes with odd $p$ are unstable while $Dp$–branes with even $p$ are stable, and in particular the fact that unstable and stable $D$–brane tensions differ with an extra $\sqrt{2}$ factor. One possibility is that this question of “BPS versus non–BPS” brane solutions could also be associated to the construction of solutions to vacuum superstring field theory only in the GSO$(+)$ sector or in both the GSO$(\pm)$ sectors. Another possibility may have to do with the introduction of the Grassmann odd state $G^{-1/2}_{-1/2}\Xi$ in the game. But surely the most straightforward way to proceed would be to follow the methods of [21, 45].

• One should also understand the structure of the ghost and superghost components of the sliver. Notice that in Berkovits’ theory the correlation functions that enter into the star product are defined in the large Hilbert space and, therefore, one should have a construction of the superghost vertex in terms of the bosonized superconformal ghosts. The full analysis of the ghost/superghost sector will be probably necessary in order to identify the closed superstrings at the nonperturbative vacuum, perhaps along the lines of [25].

• It would be interesting to develop a half–string formalism [46] in the fermionic sector. This would make clear some of the properties of the fermionic sliver, like the fact that it is a rank–one projector. As we pointed out in section 6, a way to see this is to bosonize the fermions, but it would be more convenient to have an explicit representation in terms of fermionic oscillators.

• Although in this paper we have focused on Berkovits’ superstring field theory, there exists another proposal for superstring field theory of the NS sector, which is cubic and has been also used to test Sen’s conjectures (see, e.g., the recent review [47]). In this cubic superstring field theory, where the string field has picture number zero and ghost number one, one can immediately extend all of the results of bosonic VSFT: assuming a pure ghost/superghost BRST operator, and factorization of the solution, the equation of motion in the matter part again reduces to idempotency of the string field. Since the star product is kept the same, all the results of this paper, concerning the fermionic matter sector, are as well valid for the cubic superstring field theory. The ghost sector, however, will probably required some sort of twisted ghost sliver as in [25].

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A. Appendix

In this appendix, we show that the explicit expressions given in (3.13) and (3.38) satisfy equations (3.11) and (3.29), respectively. Since this is very similar to the bosonic case analyzed in [30], we shall only give a few details. In the case of the identity, we are required to prove that \((1-M)I = \tilde{M}\).

The only thing we actually need is the following result:

\[
\sum_{\ell=0}^{\infty} (-1)^\ell \ell + a \left( -\frac{1}{2} \right) = \Gamma(12)\Gamma(a)\Gamma(a + 12). \tag{A.1}
\]

From this, one deduces that

\[
\sum_{\ell=0}^{\infty} \hat{u}_{2\ell} 2\ell + a + 1 = \begin{cases} 
\pi 2\hat{u}_a, & \text{a even,} \\
1a\hat{u}_a, & \text{a odd,}
\end{cases} \tag{A.2}
\]

and this is enough to prove (3.11). For the interaction vertex, one has to prove the following equations:

\[
(M + 2)(U + \overline{U}) = -2\tilde{M} - 2(U + \overline{U}),
\]

\[
(M + 2)(U - \overline{U}) = 2\sqrt{3}iC, \tag{A.3}
\]

where the matrices \(U + \overline{U}, U - \overline{U}\) are given in (3.39). The necessary ingredients to prove (A.3) are the following. First, one can show that the coefficients \(g_n\) defined in (3.36) satisfy the recursion relation:

\[
13g_n = (n + 1)g_{n+1} - (n - 1)g_{n-1}. \tag{A.4}
\]

Next, define as in [30] the following sums:

\[
O_n = \sum_{m=2\ell+1} g_m n + m,
\]

\[
E_n = \sum_{m=2\ell} g_m n + m. \tag{A.5}
\]

These sums can be written as integrals,

\[
O_n = 12 \int_{1}^{\infty} dt t^{n+1} \left( \left( t + 1t - 1 \right)^{1/6} - \left( t + 1t - 1 \right)^{1/6} \right),
\]

\[
E_n = 12 \int_{1}^{\infty} dt t^{n+1} \left[ \left( t + 1t - 1 \right)^{1/6} + \left( t + 1t - 1 \right)^{1/6} \right], \tag{A.6}
\]
and using this representation one can show that they satisfy the recursion relations:

\[(n + 1)E_{n+1} = 13O_n + (n - 1)E_n,\]
\[(n + 1)O_{n+1} = 13E_n + (n - 1)O_n.\]  
(A.7)

To evaluate these sums, we proceed as in [30]. On the one hand, we have

\[g_{2\ell} = 12\pi i \oint dz z^{2\ell+1} 12 \left[ \left( z + 1z - 1 \right)^{1/6} - \left( z + 1z - 1 \right)^{1/6} \right],\]  
(A.8)

where the contour is around the origin. On the other hand, when \(\ell\) is greater than zero, one can deform the contour in the above integral to the real axis and obtain

\[O_{2\ell} = \pi g_{2\ell}, \quad \ell \geq 1.\]  
(A.9)

Similarly, one proves that

\[E_{2\ell+1} = \pi g_{2\ell+1}, \quad \ell \geq 0.\]  
(A.10)

The value of \(O_0\) can be evaluated by direct integration. After performing the change of variables \(x = \tanh((\log t)/2)\), one finds

\[O_0 = \int_0^1 dx 1 - x^2(x^{-1/6} - x^{1/6}) = 12\left(\psi\left(\frac{712}{2}\right) - \psi\left(\frac{512}{2}\right)\right).\]  
(A.11)

We then find

\[O_0 = \pi - \sqrt{32}.\]  
(A.12)

Using this value and the recursion relations, one can obtain \(O_{-2\ell}, E_{-2\ell-1}\) as well. To evaluate the other sums, we follow the procedure in the Appendix of [30]. First, define the series

\[S_n = \begin{cases} E_n, & n = 2k, \\ O_n, & n = 2k + 1. \end{cases}\]  
(A.13)

Since the sums satisfy the recursion relation (A.7), \(S_n\) satisfies the recursion relation of the coefficients \(g_n,\) (A.4). There is another solution to this relation which is given by
\[ S_n = 3S_1g_n + 3 \sum_{m=0}^{n-1} (-1)^m g_m g_{n-m} + 1. \] (A.14)

To derive this, one first writes a differential equation for the function \( S(x) = \sum_{n=1}^{\infty} S_n x^n \) by using the recursion relation. The details are exactly like the ones in [30]. To have the complete solution to the problem, we then just have to evaluate \( S_1 \),

\[ S_1 = 1 + 1\sqrt{3} \log \left( \sqrt{3} - 1\sqrt{3} + 1 \right). \] (A.15)

Notice that \( S_m \sim 1/m \), and one has \( mS_m \to 1 \) as \( m \to 0 \). The recursion relation also implies that \( S_{-m} \) diverges for \( m = -1, -2, \cdots \), but \( mS_{-m-n} \) with \( n > 0 \) has a finite limit as \( m \) goes to zero that can be evaluated using the recursion relations.

With these ingredients, we can already prove very easily the first equation in (A.3). For example, in this proof one has to evaluate the quantity

\[ A_m = (-1)^m m(m+1) (g_{m+1}S_m - S_{m+1}g_m). \] (A.16)

Using the recursion relations, one can see that \( A_m \) does not depend on \( m \), therefore \( A_m = A_1 = 1/3 \). In order to prove the second equation in (A.3), one needs some extra ingredients to deal with the diagonal terms, since these involve the sums

\[
\tilde{O}_n = \sum_{m=2\ell+1} g_m(n+m)^2, \\
\tilde{E}_n = \sum_{m=2\ell} g_m(n+m)^2.
\] (A.17)

These sums have the integral representation

\[
\tilde{O}_n = 12 \int_{1}^{\infty} dt \log t t^{n+1} \left[ \left( x+1x - 1 \right)^{1/6} - \left( x+1x - 1 \right)^{1/6} \right], \\
\tilde{E}_n = 12 \int_{1}^{\infty} dt \log t t^{n+1} \left[ \left( t+1t - 1 \right)^{1/6} + \left( t+1t - 1 \right)^{1/6} \right],
\] (A.18)

and using them one can prove the recursion relations:

\[
(n+1)\tilde{E}_{n+1} = 13\tilde{O}_n + (n-1)\tilde{E}_n + E_{n+1} - E_{n-1}, \\
(n+1)\tilde{O}_{n+1} = 13\tilde{E}_n + (n-1)\tilde{O}_n + O_{n+1} - O_{n-1}.
\] (A.19)
These can be solved as in [30], but we shall not need their explicit expression, and in the evaluation of the relevant quantities it suffices to use the recursion relations they satisfy. For example, in the proof of the second equation in (A.3) one has to compute

\[ C_m = m(m + 1)(g_{m+1}\tilde{S}_m - \tilde{S}_{m+1}g_m), \]  

(A.20)

where \( \tilde{S}_n \) is defined as follows:

\[ \tilde{S}_n = \begin{cases} \tilde{O}_n, & n = 2k, \\ \tilde{E}_n, & n = 2k + 1. \end{cases} \]  

(A.21)

Using the recursion relations (A.4) and (A.19), as well as (A.12), one can show that

\[ C_m = \pi 3 \sum_{l=0}^{m} (-1)^l g_{m-l}^2 - \pi g_m g_{m+1} - \pi \sqrt{36}. \]  

(A.22)

Taking into account these results, the proof of the second equation in (A.3) is immediate.
References


