I. INTRODUCTION

During the past few years much research has been done in the field of analog laboratory models of general relativity. A particularly fruitful approach was the discussion of the propagation of signals in moving media. In an inhomogeneous moving medium, light [1] or sound [2] signals deviate from a straight path creating the impression of an attractive force drawing them towards some point. This effect has led to the notion of optical and acoustical black holes, velocity profiles where all signals coming sufficiently close to some point are so strongly attracted that they fall into it. From the theoretical point of view the most intriguing result of these investigations was the realization that these effects can actually be described in geometrical terms: an effective space-time metric creates a space-time curvature and thus leads to the deviation from the straight path. Although the idea of creating table-top experiments being models of astronomical objects triggered a particular interest in the physics of moving homogeneous media, they are by no means the only systems where differential geometry can be applied to describe the propagation of light in a dielectric.

The system described in the present paper is an inhomogeneous dielectric at rest, i.e., a dielectric with electric permittivity \( \varepsilon \) and magnetic permeability \( \mu \) depending on position. The best known everyday example of such a medium is the air above a hot road which reflects light coming from the sky. Here the effect is due to the different temperatures in adjacent layers of air which lead to a variable density. A more extreme example is the interface between air and glass where the index of refraction \( n = \sqrt{\varepsilon \mu} \) varies discontinuously, but this kind of system will not be considered here.

The propagation of a light ray in an inhomogeneous medium has usually been described in terms of Fermat’s principle: light moves along paths with the shortest optical length, i.e., the paths that give the shortest transit time. The fact that light rays extremize some measure of length suggests that they might be geodesics of the corresponding metric. That this actually is the case with a metric of the form \( ds^2 = n^2(dx^2 + dy^2 + dz^2) \) has been shown by Bortolotti in his 1926 paper [3]. The propagation of the field vectors along the ray has been addressed for the first time by Rytov [4] who discussed this topic in terms of classical differential geometry, calculating the rotation of the field vectors in relation to the principal normal of the trajectory. Kline and Kay show in their book [5] that the field vectors are actually parallel transported along the curve in the sense of Levi-Civita parallelism. Unfortunately, the authors stick to a mathematical notation as used in the differential geometry of curves which makes the calculations rather cumbersome and conceals the physical meaning of the results. Finally, Solimeno, Crosignani, and DiPorto [6] mention that it is possible to introduce a metric tensor and identify the rays as geodesics in this metric. Furthermore, they represent the ray equation in the form of a geodesic equation in terms of Christoffel symbols. They do not, however, describe the fields and the field vectors in this formalism.
The aim of the present paper is to give an account of how electrodynamics in an inhomogeneous medium can be represented in an alternative metric. We will extend the discussion beyond the ray equation discussed in [6]. With a few exceptions all calculations will be performed in the convenient notation of the Ricci calculus as used in the general theory of relativity. This keeps the calculations simple, gives them a clear meaning and makes them easily accessible to anyone acquainted with the methods of general relativity. We will confine our attention to metrics connected to the standard metric by a conformal transformation, i.e., they differ by a scalar factor. Some of the calculations are valid in the present form for a more general class of metrics, but as these metrics are of no physical significance for the systems considered here they will not be discussed. The more general system – the anisotropic medium – requires still more sophisticated mathematical methods.

The paper is built up as follows: In sec. II we introduce the general conformally transformed metric and show how the fields $\mu, \varepsilon$ and the wave equation are transformed under a conformal transformation of the metric. In sec. III the approximation of geometrical optics is introduced and the wave equation is rewritten as a series in powers of the inverse wave number. Imposing on the lowest order term the condition that the trajectory has to be a geodesic leads to the metric introduced by Bortolotti. In sec. IV the propagation of the field vectors is discussed together with the intensity flow. The wave equation in the new metric is given. Finally in sec. V the Riemann-Silberstein (RS) vector in an inhomogeneous dielectric is discussed. This concept was introduced by Riemann [7] and its main properties have been discussed by Silberstein [8]. Using the name Riemann-Silberstein vector, we follow Bia lynicki-Birula [9]. The RS-field allows for an alternative, complex, formulation of electrodynamics, and is considered a candidate for the photon wave function [9].

**II. FIELD EQUATIONS**

In the present section we consider the electromagnetic field equations in an inhomogeneous dielectric medium. As usual the electric and magnetic fields in this system can be described by Maxwell’s equations

\[
\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}.
\]

The properties of the medium are concealed in the connection between the fields, which in our case can be written in the form of the constitutive equations:

\[
\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}, \quad \mathbf{B} = \mu \mu_0 \mathbf{H}
\]

where we allow the electric permittivity and the magnetic permeability to depend on position. Maxwell’s equations (1) and the constitutive equations (2) allow us to derive the wave equation in the medium and, consequently, to discuss the propagation of light. Unfortunately, in general the equation obtained in this way includes many terms which do not have a clear physical interpretation. In this section we show that the introduction of an alternative space structure – a changed metric – can simplify the equations considerably and give them a clear physical meaning.

A metric is required, whenever one wishes to define lengths of vectors. The length of a vector $\mathbf{a}$ can then be written in the form

\[
|\mathbf{a}| = \sqrt{a^\mu a_\nu \gamma_{\mu\nu}}
\]

where $\gamma_{\mu\nu}$ is the metric tensor. For the reader familiar with general relativity we should emphasize that the calculations are performed in a three-dimensional space and that the metric tensor of flat space is the unit matrix. In (3) we assume Einstein’s summation convention, dropping the summation symbols and implicitly summing over repeated indices. In a space equipped with a nontrivial metric tensor, two different versions of the same vector, the co- and the contravariant one, have to be considered. They are mapped into each other by the metric tensor and its inverse. These tensors are said to raise and lower the vector’s indices because the co- and contravariant vectors are denoted by lower and upper indices, respectively:

\[
a_\mu = a^\nu \gamma_{\mu\nu}.
\]

In ordinary three-dimensional electrodynamics there is no need for an explicit introduction of the metric tensor which is simply the identity matrix, and the components of the co- and contravariant vectors are equal. In the present paper, however, we will allow for more general metric tensors which correspond to a more general form of Maxwell’s
When Maxwell’s equations are written in coordinates, the antisymmetry of the curl operator is represented by the antisymmetric tensor $e^{\alpha\beta\gamma}$. In the case of general metrics this has to be replaced by the generalized form:

$$\eta^{\alpha\beta\gamma} = \frac{1}{\sqrt{\gamma}} e^{\alpha\beta\gamma}$$

with

$$\gamma = \text{det}(\gamma_{\mu
u}).$$

The covariant metric tensor $\gamma_{\mu\nu}$ is the inverse of $\gamma^{\mu\nu}$:

$$\gamma_{\mu\nu} \gamma^{\nu\rho} = \delta^\rho_\mu.$$ 

Following the procedure used by Landau and Lifshitz [10] one can rewrite Maxwell’s equations in the form

$$\eta^{\alpha\beta\gamma} \nabla_\beta E_\gamma + \frac{\partial B^\alpha}{\partial t} = 0, \quad \eta^{\alpha\beta\gamma} \nabla_\beta H_\gamma - \frac{\partial D^\alpha}{\partial t} = 0$$

and

$$\nabla_\alpha D^\alpha = 0, \quad \nabla_\alpha B^\alpha = 0.$$

The symbol $\nabla_\beta$ denotes the covariant derivative, the generalization of the partial derivative $\partial/\partial x^\beta$ to curvilinear coordinates or curved spaces. In coordinates, the covariant derivative of a vector field $a^\mu$ acquires the form

$$\nabla_\alpha a^\mu = \frac{\partial a^\mu}{\partial x^\alpha} + \Gamma^\alpha_{\mu\beta} a^\beta.$$

The Christoffel symbol $\Gamma^\alpha_{\mu\beta}$ can be calculated when the spatial dependence of the metric tensor is known

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \gamma^{\lambda\rho} \left( \frac{\partial \gamma_{\rho\mu}}{\partial x^\nu} + \frac{\partial \gamma_{\rho\nu}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\nu}}{\partial x^\rho} \right).$$

It accounts for the fact that not only the vector components but also the coordinate vectors themselves depend on position. Consequently, the covariant derivative of a scalar field is equal to the partial derivative. Details can be found in any book on general relativity, e.g., [11,12]. In the present paper we will avoid the application of explicit coordinates, since our goal is to show the fundamental geometrical structure experienced by light propagating through an inhomogeneous medium. Obviously, as soon as one is interested in a concrete example, one has to choose a system of coordinates.

Maxwell’s equations in the form (8) and (9) have been introduced formally as an alternative to the standard form (1). They are obviously correct when the standard metric is used, but one has to check if they remain valid in the case of alternative metric tensors. The metric tensors we consider here are connected to the flat metric via a conformal transformation, i.e.

$$\gamma_{\mu\nu} = \Omega^2 \delta_{\mu\nu}$$

with $\Omega$ being some differentiable positive scalar-valued field.

When changing to a conformally equivalent metric it may become necessary to introduce also transformed fields of the form

$$\tilde{D}_\alpha = \Omega^s D_\alpha$$

where $s$ is an integer called the conformal weight. In order to calculate it one has to use the explicit form of the covariant derivative (10) with the Christoffel symbol in the form

$$\Gamma^\alpha_{\mu\nu} = \tilde{\nabla}_\nu (\ln \Omega) \delta^\alpha_\mu + \tilde{\nabla}_\mu (\ln \Omega) \delta^\alpha_\nu - \tilde{\nabla}_\beta (\ln \Omega) \delta^{\alpha\beta} \delta_{\mu\nu}.$$ 

Eq. (9) then acquires the form

$$\tilde{\gamma}^{\alpha\beta\gamma} \tilde{\nabla}_\alpha \tilde{D}_\beta = \Omega^{-2} \gamma^{\alpha\beta} \left( \partial_\alpha \Omega^s D_\beta - \Omega^s \Gamma^\gamma_{\alpha\beta} D_\gamma \right)$$

$$= \Omega^{-2} ((s - 2 + d) (\Omega^{s-1} D^\beta \partial_\beta \Omega) + \Omega^s \partial_\alpha D^\alpha).$$
where $d$ is the dimension of the space, which in our case is 3. For eq. (9) to be equivalent to the usual Maxwell equation the first term in (16) has to vanish, leading to the result

$$s = -1$$

which therefore implies

$$\tilde{D}_\alpha = \Omega^{-1} D_\alpha \quad \text{and} \quad \tilde{D}^\alpha = \Omega^{-3} D^\alpha$$

with an analogous relation for the $B$-field. Because the Christoffel symbol is symmetric in the two lower indices we can rewrite eq. (8) as

$$\frac{1}{\sqrt{\gamma}} \epsilon^{\alpha\beta\gamma} \partial_\beta \tilde{E}_\gamma + \frac{\partial \tilde{B}^\alpha}{\partial t} = 0$$

and, taking into account that $\sqrt{\gamma} = \Omega^3$, one gets

$$\tilde{E}_\alpha = E_\alpha \quad \text{and} \quad \tilde{E}^\alpha = \Omega^{-2} E^\alpha$$

with the equivalent result for the $H$-field. These results lead to the alternative constitutive equations

$$\tilde{B}_\alpha = \Omega^{-1} B_\alpha = \Omega^{-1} \mu_0 H_\alpha = \Omega^{-1} \mu_0 \tilde{H}_\alpha$$

$$\tilde{D}_\alpha = \Omega^{-1} D_\alpha = \Omega^{-1} \varepsilon_0 E_\alpha = \Omega^{-1} \varepsilon_0 \tilde{E}_\alpha,$$

valid in the transformed metric. This allows us to define the permeabilities of the system in the new metric as

$$\tilde{\mu} = \frac{\mu}{\Omega} \quad \text{and} \quad \tilde{\varepsilon} = \frac{\varepsilon}{\Omega}$$

In order to arrive at the general form of the wave equation we calculate as usual the curl of the first equation in (9) applying the covariant curl operator with $\eta_{\alpha\beta\gamma} = \sqrt{\gamma} \epsilon_{\alpha\beta\gamma}$, and get

$$\eta_{\alpha\rho\sigma} \tilde{\nabla}^\rho \left( \epsilon^{\rho\beta\gamma} \tilde{\nabla}_\beta \tilde{E}_\gamma + \frac{\partial \tilde{B}^\alpha}{\partial t} \right) = 0.$$  

This leads to the equation

$$\left( \delta^\beta_\rho \delta^\gamma_\sigma - \delta^\gamma_\rho \delta^\beta_\sigma \right) \tilde{\nabla}^\rho \tilde{\nabla}_\beta \tilde{E}_\gamma + \frac{\partial}{\partial t} \eta_{\alpha\rho\sigma} \tilde{\nabla}^\rho \tilde{B}^\alpha = 0.$$  

Here we applied the fact that the covariant derivative of the modified antisymmetric tensor $\eta^{\alpha\beta\gamma}$ vanishes. This can be easily seen, taking into account that the covariant derivative of the antisymmetric tensor $\epsilon^{\alpha\beta\gamma}$ does not vanish, but rather leads to the result

$$\tilde{\nabla}_\rho \epsilon^{\alpha\beta\gamma} = \Gamma^\sigma_\rho \epsilon^{\alpha\beta\gamma} = \frac{\partial \ln \sqrt{\gamma}}{\partial x^\rho} \epsilon^{\alpha\beta\gamma}$$

where in the last step the well-known result for the contracted Christoffel symbol has been applied [11]. Note the order of indices in the antisymmetric tensor:

$$\eta_{\alpha\rho\sigma} \tilde{\nabla}^\rho \tilde{B}^\alpha = - \eta_{\sigma\rho\alpha} \tilde{\nabla}^\rho \tilde{B}^\alpha$$

with the last term corresponding to the curl of $B^\alpha$. Thus we get for (25)

$$\tilde{\nabla}^\beta \tilde{\nabla}_\beta \tilde{E}_\sigma - \tilde{\nabla}^\gamma \tilde{\nabla}_\sigma \tilde{E}_\gamma - \frac{\partial}{\partial t} \eta_{\sigma\rho\alpha} \tilde{\nabla}^\rho \tilde{B} = 0.$$  

In order to be able to apply the divergence equation (9), one has to interchange the covariant derivatives in the second term. But, as one easily sees from the explicit form in eq. (10), covariant derivatives do not usually commute. The commutator of covariant derivatives is connected to the curvature of space, e.g., when applied to a vector the commutator gives
\[ [\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] E^\mu = R^\mu_{\alpha\beta\gamma} E^\gamma \]

(29)

with \( R^\mu_{\nu\alpha\beta} \) being the Riemann curvature tensor [11,12]. Note that the conformal change of the metric tensor may create a curvature, and the fact that the considered space is flat in the standard metric does not imply that this remains so after the transformation of the metric. In our case, the commutator gives

\[ [\tilde{\nabla}_\alpha, \tilde{\nabla}_\mu] E^\mu = R_{\alpha\mu} E^\mu \]

(30)

where

\[ R_{\alpha\mu} = R^\beta_{\alpha\beta\mu} \]

(31)

is the Ricci tensor. Finally, we get the wave equation in the form:

\[
\tilde{\nabla}^\beta \tilde{\nabla}_\beta \tilde{E}_\sigma + \tilde{\nabla}_\sigma (\tilde{E}_r \tilde{\nabla}^r \ln \tilde{\varepsilon}) - \frac{\tilde{\mu} \tilde{\varepsilon}}{c^2} \frac{\partial^2}{\partial t^2} \tilde{E}_\sigma + R_{\alpha\mu} \tilde{E}^\mu + (\tilde{\nabla}_\sigma \tilde{E}_\rho - \tilde{\nabla}_\rho \tilde{E}_\sigma) \tilde{\nabla}^\rho \ln \tilde{\mu} = 0.
\]

(32)

One easily notes that when the flat metric is applied, this is the usual wave equation for light in an inhomogeneous medium, e.g. [14]:

\[
\nabla^2 E + \frac{\mu \varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} + (\text{grad} \ln \mu) \times (\nabla \times E) + \text{grad}(E \cdot \text{grad} \ln \varepsilon) = 0.
\]

(33)

### III. GEOMETRICAL OPTICS

The physical content of the metric transformations becomes clear in the framework of geometrical optics, i.e., in the case of monochromatic light moving in a medium with the medium parameters not changing significantly within one wavelength. This approximation allows us to introduce the notion of light rays propagating through the medium. The field can then be written in the form

\[ \tilde{E}_\sigma = \tilde{\xi}_\sigma \exp(i(k_0 S - \omega t)). \]

(34)

\( \tilde{\xi}_\sigma \) is the envelope of the wave; other fields like \( \tilde{\mathcal{H}}^\sigma \) or \( \tilde{D}^\sigma \) are defined likewise. \( \omega \) is the constant frequency and \( k_0 \) is defined as \( k_0 = \omega/c = 2\pi/\lambda_0 \) with \( \lambda_0 \) being the vacuum wave length. The product of \( k_0 \) and \( S \) – the “optical path” – is the spatial phase of the wave. Inserting (34) into the wave equation (32) and expanding in orders of \( 1/k_0 \) gives:

\[
0 = - \tilde{\xi}_\sigma \tilde{\nabla}_\beta \tilde{\xi}^\beta S + \tilde{\mu} \tilde{\varepsilon} \tilde{\xi}_\sigma + \frac{i}{k_0} \left( 2\nabla^\beta \tilde{\xi}_\sigma \nabla_\beta S + \tilde{\xi}_\sigma \nabla^\beta \tilde{\nabla}_\beta S + \tilde{\xi}_r \nabla^r S \nabla^\gamma \ln (\tilde{\mu} \tilde{\varepsilon}) - \tilde{\xi}_\sigma \nabla_\rho S \nabla^\rho \ln \tilde{\mu} \right) \\
+ \frac{1}{k_0^2} \left( \tilde{\nabla}^\beta \tilde{\nabla}_\beta \tilde{\xi}_\sigma + \tilde{\nabla}^\gamma \ln (\tilde{\mu} \tilde{\varepsilon}) \nabla_\sigma \tilde{\xi}_\gamma + \tilde{\nabla}_\gamma \ln \tilde{\mu} \tilde{\varepsilon} \tilde{\nabla}_\beta \tilde{\xi}_\sigma - \tilde{\nabla}_\rho \tilde{\xi}_\sigma \nabla^\rho \ln \tilde{\mu} + R_{\alpha\mu} \tilde{E}^\mu \right).
\]

(35)

One should not be confused by the fact that the expansion coefficient has a dimension. The crucial point here is that two kinds of spatial variation are present here: the fast variation of the phase and the slow variation of the envelope. The clear separation of these scales is not a result of the calculations but rather a condition for the geometrical approximation to be valid. This validity is reflected in the size of \( k_0 \) which has to be large enough for the three terms in (35) to be well-separated.

Confining our considerations to the zeroth order contribution, we get the equation

\[
\nabla^\beta S \nabla_\beta S = \tilde{\mu} \tilde{\varepsilon}.
\]

(36)

In a homogeneous medium light rays follow straight lines, but in an inhomogeneous one this is usually not the case. In the case of a general metric the idea of the “straightest line” is represented by the concept of a geodesic – the line that “curves as little as possible” [11] and that extremizes the distance between two points. In mathematical terms the geodesic is most easily described as the line that “parallel transports its own tangent vector”, i.e., the covariant derivative of the tangent vector field in the direction of the curve vanishes. Calling a tangent vector \( t^a \) we thus get the condition
\[ t^\alpha \nabla_\alpha t^\beta = 0 \]  

(37)

for a curve to be a geodesic. In the case of propagating light, the tangent vector is the velocity vector \( \nabla_\alpha S \), the contravariant version of the wave vector \( \nabla_\alpha S \), which is the gradient of the phase \( S \). Consequently, for the light ray to be a geodesic the condition

\[ \nabla^\alpha S \nabla_\alpha \nabla^\beta S = 0 \]  

(38)

must be fulfilled. Note that \( S \) is a scalar and therefore its first derivative does not depend on the metric and two derivatives applied to it commute. Eq. (38) can be rewritten in the form

\[ \frac{1}{2} \nabla^\beta (\nabla^\alpha S \nabla_\alpha S) = 0. \]  

(39)

Since the expression in the brackets is a scalar, we end up with the condition that for the light trajectory to be a geodesic, \( \nabla^\alpha S \nabla_\alpha S \) has to be constant. This is equivalent to the condition that \( \varepsilon \mu \) has to be constant. This condition allows us to find the “most natural” metric for a given medium. Using (23) we conclude that the light ray is a geodesic in the metric

\[ \tilde{\gamma}_{\mu \nu} = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{pmatrix}, \]  

(40)

i.e., one has to apply the conformal transformation with

\[ \Omega^2 = n^2. \]  

(41)

In (40) we denoted the metric that allows us to represent the trajectories as geodesics with a bar over the symbol. All objects corresponding to this metric will be denoted by bars over the respective symbols, whereas symbols without bars correspond to arbitrary metrics of the form (12). Obviously, the metric tensor in (40) may still be multiplied by a constant but our choice makes \( \tilde{\mu} \) and \( \tilde{\varepsilon} \) inverse. Note in particular that the product of \( \tilde{\mu} \) and \( \tilde{\varepsilon} \) corresponds to an effective index of refraction. Consequently, the metric introduced here leads to a constant index of refraction and thus to a system with a constant effective velocity of light.

The quantity \( m \) introduced here is equal to

\[ m = \sqrt{\frac{\mu}{\varepsilon}}. \]  

(42)

Together with the index of refraction \( n \), \( m \) gives an alternative description of the medium properties. The interesting point is that \( m \) drops out in the calculations. Therefore it is possible to give a complete description of ray propagation in an inhomogeneous medium in terms of geodesics of a metric tensor. The quantity \( m \) is sometimes called the impedance [6] or the resistance [9] of the medium. This is due the the fact that it has the units of Ohms.

### IV. POLARIZATION TRANSPORT

Since the electromagnetic field is not a scalar field, the knowledge of the trajectory of a light ray – the set of the points it passes through – does not contain complete information about the propagation. One still needs to find out how the field strength vectors are propagated along the ray. In order to address this question we go back to eq. (35), assume that eq. (36) is fulfilled and demand that even the first order contribution vanishes. This leads to the equation

\[ 2 \tilde{\nabla}^\beta \tilde{\varepsilon}_{\sigma} \tilde{\nabla}_{\beta} S = -\tilde{\varepsilon}_{\sigma} \tilde{\nabla}^\beta \tilde{\nabla}_{\beta} S - \tilde{\varepsilon}_{\sigma} \tilde{\nabla}^\gamma \tilde{\nabla}^\gamma \ln (\tilde{\varepsilon} \tilde{\mu}) + \tilde{\varepsilon}_{\sigma} \tilde{\nabla}^\rho \tilde{\nabla}^\rho \ln \tilde{\mu}. \]  

(43)

In analogy to (37) the vector \( \tilde{E}_\sigma \) is said to be parallel transported, along the ray when the condition

\[ \nabla^\beta E_\sigma \nabla_\beta S = 0 \]  

(44)

is fulfilled, the left-hand side corresponding to the directional derivative of \( E_\sigma \) along the ray. In a more general sense the vector \( E_\sigma \) is still said to be parallel transported if

\[ \nabla^\beta E_\sigma \nabla_\beta S = \alpha E_\sigma \]  

(45)
and in equation by inserting the harmonic fields (34) into Maxwell’s equations (8) and (9) and keeping only the lowest order approximation all the angles are right angles. This can be seen from the geometrical approximation of Maxwell’s at one point on the ray in order to know them for the whole trajectory. One can easily convince oneself that in our geometric description is possible in the general case. As \( \bar{\mu} \) and \( \bar{\varepsilon} \) and \( \varepsilon \) differ only by a constant factor. This means that \( \bar{\mu} \) and \( \bar{\varepsilon} \) are constant. In this non-physical but interesting case, we get for the constitutive equations

\[
\bar{E}^\alpha = \bar{D}^\alpha \quad \bar{B}^\alpha = \bar{H}^\alpha. \tag{47}
\]

This means that the whole contribution of the medium is conserved in the metric structure of space, the light is propagating as if it were moving in empty but curved space. This result is not only valid in the geometrical approximation, but even allows us to write the wave equation in terms of a curved space as follows:

\[
\nabla_\rho \nabla_\sigma \bar{E}_\alpha^\gamma \rho^\beta + R_{\alpha\mu} \bar{E}^\mu - \frac{\partial^2}{\partial t^2} \bar{E}_\alpha = 0. \tag{48}
\]

This case corresponds to a constant value of the quantity \( m \) introduced in (42). In the general case of arbitrary \( \mu \) and \( \varepsilon \) we get for the transformed wave equation:

\[
\nabla^\beta \nabla_\beta \bar{E}_\sigma - \bar{\mu} \bar{\varepsilon} \frac{\partial^2}{\partial t^2} \bar{E}_\sigma + \bar{E}_\gamma \nabla_\sigma \nabla^\gamma \ln \bar{\varepsilon} + \nabla_\rho \bar{E}_\sigma \nabla^\rho \ln \bar{\varepsilon} = 0. \tag{49}
\]

As \( \bar{\varepsilon} = 1/m \), this equation shows that in the discussion of the propagation of the field vectors in the general case both \( n \) and \( m \) have to be used. Since the geometrical description is based on the index of refraction \( n \) only, no complete geometric description is possible in the general case.

With the field vectors being parallel transported along a geodesic, the angles between the field vectors and between the fields and the ray remain unchanged during the propagation along the ray. It is thus sufficient to find these angles at one point on the ray in order to know them for the whole trajectory. One can easily convince oneself that in our approximation all the angles are right angles. This can be seen from the geometrical approximation of Maxwell’s equation by inserting the harmonic fields (34) into Maxwell’s equations (8) and (9) and keeping only the lowest order in \( 1/k_0 \). We drop the tilde over the symbols when the general case is meant. We thus get

\[
\eta^{\alpha\beta}(\nabla_\beta S)\mathcal{H}_\gamma + \varepsilon_0 \bar{E}_\alpha^\alpha = 0 \quad \text{and} \quad \eta^{\alpha\beta}(\nabla_\beta S)\mathcal{E}_\gamma - \mu \mu_0 \mathcal{H}_\alpha = 0 \tag{50}
\]

and

\[
\mathcal{D}_\alpha \nabla_\alpha S = 0 \quad \text{and} \quad \mathcal{B}_\alpha \nabla_\alpha S = 0. \tag{51}
\]

The last two equations show that the field vectors are orthogonal to the ray. In order to show that they are orthogonal to each other one has to multiply the first equation of (50) by \( \mathcal{H}_\alpha \) which leads to

\[
\varepsilon_0 \bar{E}_\alpha^\alpha \mathcal{H}_\alpha = 0 \quad \tag{52}
\]

where the antisymmetry of the tensor \( \eta^{\alpha\beta\gamma} \) has to be taken into account. Finally, contracting the first equation of (50) with \( \mathcal{E}_\alpha \) and the second one with \( \mathcal{H}_\alpha \) we arrive at

\[
\varepsilon_0 \bar{E}_\alpha^\alpha \mathcal{E}_\alpha = \mu \mu_0 \mathcal{H}_\alpha \mathcal{H}_\alpha. \tag{53}
\]

Note that the terms in eq. (53) are – up to a constant – the electric and magnetic energy densities, respectively, showing that both densities are equal. The complete energy density of either field can be written in the form

\[
w_e = \frac{1}{4} \mathcal{E}_\alpha \mathcal{D}^\alpha = \frac{\varepsilon_0}{4} \mathcal{E}_\alpha \mathcal{E}_\alpha, \tag{54}
\]

\[
w_m = \frac{1}{4} \mathcal{H}_\alpha \mathcal{B}^\alpha = \frac{\mu \mu_0}{4} \mathcal{H}_\alpha \mathcal{H}_\alpha. \tag{55}
\]
The transformation of the fields to the new metric as described in (18) and (20) leaves the direction of the field vectors unchanged. Consequently, the result that they are orthogonal to each other and to the ray does not depend on the metric. But in the general metric the light ray is not a geodesic and consequently the vectors parallel transported along the ray will point in another direction than the physical ones. Thus what makes the metric (40) special is the fact that the parallel transported and the physical field vectors coincide along the whole ray.

Note that the energy densities, like the Poynting vector defined later, are time-averaged quantities with the fast temporal variation being disregarded. It should be mentioned that the numerical values of the energy densities (54) and (55) depend on the metric chosen. This is due to the fact that the physical quantity involved is the integral of the density over some volume, but as the line element is transformed according to

$$\tilde{\tilde{d}}x^\alpha = \frac{1}{\Omega} dx^\alpha$$

(56)

one has to change the density as well in order to keep the integral constant. Eq. (53) suggests that we introduce the auxiliary fields

$$K_\alpha = \sqrt{\varepsilon_0 \varepsilon} E_\alpha$$

and $$M_\alpha = \sqrt{\mu_0 \mu} H_\alpha.$$  

(57)

These vectors have equal absolute value and fulfill the simplified equation

$$2\nabla^\beta K_\sigma \nabla_\beta S = -K_\sigma \nabla^\beta \nabla_\beta S.$$  

(58)

Eqs. (46) and (58) do not only describe the propagation of field vectors along the ray but they are also the key to understanding the propagation of the field intensity. The intensity $$I$$ of a light ray is defined as the absolute value of the Poynting vector

$$I^\alpha = \frac{1}{2} \tilde{\varepsilon} \varepsilon^\gamma E_\beta H_\gamma$$

(59)

and it is easily shown that it can be written in the form

$$I = 2 \frac{c}{\sqrt{\varepsilon \mu}} w_e.$$  

(60)

Inserting (36) into (60) and taking into account that $$\tilde{\varepsilon} \tilde{\mu} = 1$$ we get for the intensity transport the result

$$\nabla^\beta S \nabla_\beta I = \frac{1}{2} \left( \frac{\varepsilon_0}{\mu_0} \nabla^\beta S \nabla_\beta (\tilde{\varepsilon} \tilde{\varepsilon} E^\sigma) \right)$$

(61)

$$= \frac{1}{2} \left( \frac{\varepsilon_0}{\mu_0} (2 \tilde{\varepsilon} \tilde{\varepsilon}^\sigma \nabla^\beta S \nabla_\beta \tilde{E}_\sigma + \tilde{E}^\sigma \tilde{E}_\sigma \nabla^\beta \nabla_\beta \tilde{E}^\sigma) \right)$$

(62)

$$= \frac{1}{2} \left( \frac{\varepsilon_0}{\mu_0} [\tilde{E}^\sigma \tilde{E}_\sigma (\nabla^\beta \nabla_\beta S + \tilde{E}^\sigma \nabla^\beta \nabla_\beta \ln \tilde{\varepsilon} + \nabla^\beta S \nabla_\beta \ln \tilde{\mu})] \right)$$

(63)

$$= -I \nabla^\beta \nabla_\beta S.$$  

(64)

The right-hand-side of (64) describes the change of the electromagnetic intensity along the ray. Due to the fact that $$\tilde{\mu}$$ and $$\tilde{\varepsilon}$$ are inverse, the terms containing these quantities cancel and the final result does not depend on their actual values. Consequently, light propagation in the geometrical approximation does not crucially depend on whether $$\tilde{\mu}$$ and $$\tilde{\varepsilon}$$ are constant or not, and neither the trajectory nor the energy flow are dependent on that. Clearly, the change of the fields $$E^\alpha$$ and $$H^\alpha$$ along the ray does explicitly depend on the variation of $$\tilde{\mu}$$ and $$\tilde{\varepsilon}$$, respectively. The purpose of the last term in eq. (46) is, in a sense, to adjust the field in a way that, despite the changing permeabilities, its intensity flow is half of (64). As has been pointed out before the product of $$\tilde{\mu}$$ and $$\tilde{\varepsilon}$$ can be seen as an effective index of refraction. Consequently, the mathematical framework of the changed metric leads to a constant velocity of light and therefore corresponds to an empty but possibly curved space, for all physical processes where only the index of refraction is involved.

The result eq. (64) describing the change in intensity along the ray sheds light on the wave structure of the discussed setup. This can be easily understood if one looks not just at one trajectory but rather considers a thin but finite bundle of trajectories. Clearly, the distance between the trajectories may depend on the curve parameter. In the homogeneous case, however, all light rays are straight lines, and consequently the dependence is linear making the second derivative of the distance vanish. One can easily show that in this case the Laplacian of the phase $$\partial^\alpha \partial_\alpha S$$
is proportional to the mean curvature, the arithmetic mean of the two principal curvatures, of the wave front. The
formula for the change of the element of surface area spanned by a bundle of straight lines is given in books on
elementary differential geometry (see e.g. [15]) where the term “parallel surfaces” is used to denote the system of
successive wave fronts. Consequently, the change in intensity is governed by the curvature of the wave fronts, whereas
the value of the intensity is inversely proportional to the area of the cross section of the bundle. In the inhomogeneous
case the right-hand side of eq. (58) cannot be interpreted in this simple way however because one has to add a term
containing a Christoffel symbol to the simple Laplace expression. The additional term can be traced to the fact that
in the inhomogeneous medium geodesics do not continue straight on, but rather curve through the dependence on
the structure of the medium – in general in a different way at different points. Usually, the distance between these
geodesics does not depend linearly on the curve parameter, i.e., the second derivative of the distance does not vanish.
This effect is known as the geodesic deviation and is a characteristic of a curved space. Consequently, the changed
law of intensity variation can be seen as a sign of the fact that the inhomogeneity of the medium creates an effectively
curved space.

More sophisticated mathematical methods should allow for a discussion of curved surfaces in a curved space. In
this way (64) might turn out to be the curved-space analogue of the mean curvature.

V. THE RIEMANN-SILBERSTEIN VECTOR IN INHOMOGENEOUS MEDIA

By analogy with the fields $M_\alpha$ and $K_\alpha$ in eq. (57) one can generally introduce the fields

$$K_\gamma = \frac{D_\gamma}{\sqrt{\varepsilon_0 \mu_0}} = E_\gamma \sqrt{\varepsilon_0}$$

and

$$M_\gamma = \frac{B_\gamma}{\sqrt{\mu_0}} = H_\gamma \sqrt{\mu_0}.$$  

(65)

(66)

These fields can be seen to be the real and imaginary parts of the Riemann-Silberstein vectors

$$(F_\pm)_\alpha = \frac{1}{\sqrt{2}} (K_\alpha \pm i M_\alpha).$$

(67)

They allow for an alternative description of electrodynamics with the complete Maxwell equations in vacuum taking
the form

$$i \frac{\partial F_+}{\partial t} = \pm c \nabla \times F_+ \quad \text{and} \quad \nabla \cdot F_+ = 0.$$  

(68)

Clearly, only one of the fields is needed for a complete description of electrodynamics. When used to describe
photons, the Riemann-Silberstein vector has to be a correctly defined wave function, i.e., it may only include positive
frequencies. In this case both vectors, $(F_+)_\alpha$ and $(F_-)_\alpha$, have to be used, corresponding to light which is left- and
right-circular polarized, respectively, i.e., they then correspond to positive and negative helicity states [9]. In an
inhomogeneous medium the field equations for the Riemann-Silberstein field become considerably more complicated
than eq. (68). Written in the general form with an arbitrary diagonal metric they attain the form

$$i \frac{\partial F_+^\alpha}{\partial t} + \frac{\epsilon_0 \alpha \beta \gamma}{\sqrt{\mu \varepsilon}} \left( \nabla_\beta (F_-) + \frac{1}{2} \left( (F_+)_\gamma \nabla_\beta \ln \sqrt{\frac{\mu}{\varepsilon}} + (F_-)_\gamma \nabla_\beta \ln \frac{1}{\sqrt{\mu \varepsilon}} \right) \right) = 0$$

(69)

$$i \frac{\partial F_-^\alpha}{\partial t} - \frac{\epsilon_0 \alpha \beta \gamma}{\sqrt{\mu \varepsilon}} \left( \nabla_\beta (F_+) + \frac{1}{2} \left( (F_-)_\gamma \nabla_\beta \ln \sqrt{\frac{\mu}{\varepsilon}} + (F_+)_\gamma \nabla_\beta \ln \frac{1}{\sqrt{\mu \varepsilon}} \right) \right) = 0$$

and

$$\nabla_\alpha F_+^\alpha = \frac{1}{2} \left( F_+^\alpha \nabla_\alpha \ln \frac{1}{\sqrt{\varepsilon \mu}} + F_-^\alpha \nabla_\alpha \ln \sqrt{\frac{\mu}{\varepsilon}} \right)$$

(70)

$$\nabla_\alpha F_-^\alpha = \frac{1}{2} \left( F_+^\alpha \nabla_\alpha \ln \frac{1}{\sqrt{\varepsilon \mu}} + F_-^\alpha \nabla_\alpha \ln \sqrt{\frac{\mu}{\varepsilon}} \right).$$
In the present paper a general geometrical description of electromagnetic phenomena in inhomogeneous dielectric media has been given with the emphasis being on light propagation in the approximation of geometrical optics. To find the conceptually simplest description we introduced a variation of the spatial metric allowing for all metric tensors connected to the unit matrix by a conformal transformation. The equations for the fields and for the propagation of light rays can easily be rewritten in this general geometry when care is taken to transform correctly the fields and the material parameters. It turns out that when the metric tensor is equal to the unit matrix multiplied by \( n^2 \) – the square of the index of refraction – the equations become particularly simple: light rays are geodesics of the metric and the field vectors are parallel transported along the ray. Even the wave equation simplifies significantly. As long as \( \varepsilon \) and \( \mu \) in the standard metric are independent quantities the system in the transformed metric corresponds to a curved space filled with a dielectric medium, but with the transformed quantities \( \bar{\mu} \) and \( \bar{\varepsilon} \) having a constant product. This property leads to significant simplifications in the description of light propagation. In the special case where \( \mu \) and \( \varepsilon \) differ only by a constant factor the situation becomes even simpler with the transformed quantities becoming constants. In this – non-physical – case the transformation fully eliminates the medium and all electromagnetic effects appear as if they were taking place in an empty but curved space. Whereas the wave equations contain significantly more terms when an effective medium is present, there are only slight differences between the two cases in the approximation of geometrical optics. The results there depend mostly on the behavior of the effective index of refraction which becomes constant.

The results presented here contribute to a better fundamental understanding of electromagnetism in dielectric media and show the power of geometrical concepts in classical fields of physics. In particular the mathematical techniques of the general theory of relativity turn out to be well-suited for calculations in inhomogeneous media. In a forthcoming paper the presented results will be extended to electromagnetic fields in moving inhomogeneous media, thus supplementing the calculations presented in our papers on light in moving media. A tempting problem might be a geometric theory of anisotropic media which will require more sophisticated mathematical methods than those presented here including the introduction of a more general space than the Riemannian space. Another mathematical challenge would be a discussion of the wave fronts in the spirit of differential geometry of curved surfaces in a curved three-dimensional space. Furthermore our results show that the correctly defined Riemann-Silberstein vector keeps a well-defined helicity during its propagation through the medium. This might trigger further investigations in the search for a photon wavefunction in a dielectric medium.
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