Tolman-Bondi collapse in scalar-tensor theories as a probe of gravitational memory

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In cosmological models with a varying gravitational constant, it is not clear whether primordial black holes preserve the value of $G$ at their formation epoch. We investigate this question by using the Tolman-Bondi model to study the evolution of a background scalar field when a black hole forms from the collapse of dust in a flat Friedmann Universe. Providing the back reaction of the scalar field on the metric can be neglected, we find that the value of the scalar field at the event horizon very quickly assumes the background cosmological value. This suggests that there is very little gravitational memory.

I. INTRODUCTION

Scalar-tensor (ST) theories of gravity provide a natural alternative to general relativity (GR). They describe gravity with not only a metric $g_{ab}$ but also a scalar field $\phi$. Derivatives of $\phi$ appear as source terms in the field equations and $\phi$ itself satisfies a wave equation. The strength of the gravitational coupling is determined by the function $\omega(\phi)$, where GR is recovered in the limits $\omega \rightarrow \infty$ and $\omega \omega^{-3} \rightarrow 0$.

ST theories can also be regarded as being equivalent to GR with a varying gravitational “constant” $G$. The most simple example of such a ST theory is Brans-Dicke theory \cite{1}, where $\omega(\phi)$ is constant and $G \propto \phi^{-1}$. However, weak field experiments have shown that $\omega \geq 500$ \cite{2} and so the deviation from GR is small. For more general ST theories, where $\omega$ is not constant, it is possible that $\omega$ was much smaller at earlier times. So observations allow such theories to greatly deviate from GR in the early universe.

There has been a renewed interest in ST theories in recent years due to the effective low energy actions of string theory involving one or more scalar fields. These scalar fields enter the field equations in much the same way as the scalar field in ST theories \cite{3}. Also, the increasing popularity of inflation and quintessence suggests that scalar fields might need to be incorporated into cosmological models.

The purpose of this paper is to study the effect of an evolving scalar field on the formation and evolution of a primordial black hole. In an asymptotically flat spacetime it is well known that a black hole radiates away any inhomogeneities in the scalar field until it becomes a stationary solution with constant $\phi$ \cite{4}. This is a consequence of the famous “no hair” theorem. However, in ST cosmological models the scalar field is evolving with time and this would modify how the black hole evolves during its lifetime.

Barrow \cite{5} was the first to examine this problem. He considered the two most extreme possibilities: scenario (A), where the scalar field evolves everywhere homogeneously in the same way as the cosmological background; and scenario (B), where the black hole forces the scalar field to remain constant in some local region around it. The second scenario Barrow called gravitational memory because the black hole would locally preserve the value of the scalar field from when it formed. Barrow & Carr \cite{6} studied the evolution of primordial black holes for these scenarios and found that either case results in a significant deviation from the usual GR analysis. They considered ST theories, where $G(\phi)$ varies as in the Brans-Dicke case. Since $\phi$ increases with cosmological time, this implies that $G(\phi)$ decreases, so black holes would take longer to radiate away their mass via Hawking evaporation than in the GR case. In both scenarios the black holes form when gravity is effectively stronger, so the rate of evaporation is less, but in scenario (B) the strength never decreases and so the lifetime is even longer.

As stated earlier the above scenarios are the two extremes and the reality is probably somewhere in between. Two more general scenarios have been proposed \cite{7}. In scenario (C) the scalar field evolves faster at the event horizon (EH) than at the particle horizon (PH), so $\dot{\phi}_{EH} \geq \dot{\phi}_{PH}$. Eventually the black hole must reach a stage where $\dot{\phi}_{EH} = \dot{\phi}_{PH}$, but this does not necessarily mean the scalar field is homogeneous since there could be some lag between the asymptotic and local increase. In Scenario (D), the scalar field is evolving locally but at a slower rate than asymptotically, so $\dot{\phi}_{EH} \leq \dot{\phi}_{PH}$. There is still some gravitational memory but not in the strict sense of scenario (B). In this scenario the gradient of the scalar field is increasing but one would expect there to be some limit, due to the influx of scalar gravitational waves.

Gravitational memory for black holes which are small compared to the cosmological scale (i.e. the particle horizon) has already been investigated by Jacobson \cite{8}. In this case, the scalar field evolution can be considered as an asymptotic perturbation to the Schwarzschild metric and the lag described above is found to be very small. This suggests that
II. BASIC EQUATIONS

For scalar-tensor theories of gravity with \( G(\phi) \propto \phi^{-1} \) the field equations are

\[
G_{ab} = \frac{8\pi}{\phi} T_{ab} + \frac{\omega(\phi)}{\phi^2} \left( \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} \partial_c \phi \partial^c \phi \right) + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \nabla^c \nabla_c \phi \right),
\]

\[\nabla^c \nabla_c \phi = \frac{8\pi T - (d\omega/d\phi) \phi' \phi \partial_c \phi}{3 + 2\omega(\phi)} \]  

where \( T_{ab} \) is the usual energy-momentum tensor, \( T \) is its trace and we have set \( c = 1 \). In Brans-Dicke theory Eq.(2.1) remains unchanged but \( d\omega/d\phi = 0 \) in Eq.(2.2). These equations are expressed in the Jordan frame but ST theories can also be expressed in a conformal frame known as the Einstein frame. The Einstein frame is related to the Jordan frame by the transformation

\[
\tilde{g}_{ab} = (G_0\phi) g_{ab} \quad \Rightarrow \quad \tilde{T}_{ab} = (G_0\phi)^{-1} T_{ab},
\]

where \( G_0 \) is the present value of the gravitational “constant” as measured in solar system experiments. It is called the Einstein frame because it can be expressed as GR with a scalar field. However, the Jordan frame will be used throughout this paper.
In GR the Tolman-Bondi solution is given by

\[ ds^2 = -dt^2 + A^2(t,r)dr^2 + R^2(t,r)[d\theta^2 + \sin^2 \theta d\psi^2], \tag{2.4} \]

where \( r \) is the comoving radial coordinate and \( R \) is given by

\[ t - t_s(r) = \sqrt{\frac{R'}{F}} G \left( -\frac{fR}{F} \right). \tag{2.5} \]

Here \( G(y) \) is a positive function given by

\[
G(y) = \begin{cases} 
\frac{\text{Arcsin} \sqrt{y} - \sqrt{1-y}}{y^{3/2}} & \text{or} \quad \frac{\pi - \text{Arcsin} \sqrt{y} + \sqrt{1-y}}{y^{3/2}} \quad (0 < y \leq 1) \\
\frac{2}{3} & (y = 0) \\
\frac{-\text{Arcsinh} \sqrt{-y} - \sqrt{1-y}}{(-y)^{3/2}} \quad (y < 0)
\end{cases}
\tag{2.6}
\]

\( t_s(r) \) is a constant of integration and \( F(r) = 2G0m(r) \) where \( m(r) \) is the mass within radius \( r \). \( A \) is given by

\[ A^2(t,r) = \frac{R^2(t,r)}{1 + f(r)}, \tag{2.7} \]

and \( R \) satisfies

\[ \dot{R}^2(t,r) = \frac{F(r)}{R} + f(r). \tag{2.8} \]

The density of the dust \( \rho \) is given by

\[ \rho = \frac{F'}{8\pi R^2 R'}. \tag{2.9} \]

In the above, a dot denotes \( \partial_t \) and a prime denotes \( \partial_r \).

There are two arbitrary functions in this solution: the mass function \( m(r) \) and the energy function \( f(r) \). Investigating collapse to a black hole with this solution just requires the appropriate choice for these functions. In this paper the energy function is chosen such that

\[ f(r) < 0 \quad \text{for} \quad r < r_0 \tag{2.10} \]

\[ f(r) = 0 \quad \text{for} \quad r > r_0 \tag{2.11} \]

for some \( r_0 \). This means that when a perturbation is applied to the background dust, all the matter interior to \( r_0 \) will eventually collapse to form a black hole, while the exterior matter will expand forever as in a flat Friedman universe [13]. The mass function \( m(r) \) is determined by putting \( R = r \) in Eq.(2.5).

The key to this approximation is that the back reaction of the scalar field is neglected. It is assumed that the effect of the scalar field on the spacetime is small compared to that of the matter. The initial configuration used is the general relativistic one with constant \( \phi \) and for simplicity Brans-Dicke theory is used. Then, to the lowest order, the evolution of \( \phi \) is determined by the wave equation:

\[ [\nabla_c \nabla_c]_{TB} \phi = \frac{8\pi}{3 + 2\omega} T_{TB} \tag{2.12} \]

where \([\nabla_c \nabla_c]_{TB}\) and \( T_{TB} \) are determined for the Tolman-Bondi metric and the general relativistic solution. Using the Tolman-Bondi metric the wave operator is given by

\[ [\nabla_c \nabla_c]_{TB} \phi = -\ddot{\phi} - \frac{(AR^2)'}{AR^2} \dot{\phi} + \frac{1}{A^2} \ddot{\phi} + \frac{1}{AR^2} \left( \frac{R^2}{A} \right)' \phi'. \tag{2.13} \]
The last equation needs to be rewritten in terms of a null coordinate suitable for the characteristic method. The retarded time coordinate $u$ is introduced such that $u = \text{constant}$ is an outgoing null geodesic. In the original coordinates the outgoing null geodesic is given by

$$\frac{dt}{dr} = A,$$  \hspace{1cm} (3.1)

so we can write

$$\frac{u'}{u} = -A.$$ \hspace{1cm} (3.2)

The coordinate system is now transformed from $(t, r)$ to $(u(t, r), \bar{r}(r))$ using the relations

$$du = \frac{1}{\alpha} (dt - Adr), \quad d\bar{r} = dr,$$ \hspace{1cm} (3.3)

where

$$\alpha \equiv \frac{1}{u}.$$ \hspace{1cm} (3.4)

The partial derivatives are then related by

$$\partial_t = \frac{1}{\alpha} \partial_u, \quad \partial_r = -\frac{A}{\alpha} \partial_u + \partial_{\bar{r}},$$ \hspace{1cm} (3.5)

In this coordinate system the metric becomes

$$ds^2 = -\alpha^2 du^2 - 2\alpha A(u, \bar{r})dud\bar{r} + R^2(u, \bar{r})[d\phi^2 + \sin^2 \theta d\psi^2].$$ \hspace{1cm} (3.6)

To use the characteristic method it is necessary to introduce the derivative along the ingoing radial null geodesic. In the original coordinates the ingoing null geodesic is given by

$$\frac{dt}{dr} = -A,$$ \hspace{1cm} (3.7)

which in the new coordinate system becomes

$$\frac{d\bar{r}}{du} = -\frac{\alpha}{2A}.$$ \hspace{1cm} (3.8)

Therefore the derivative along the ingoing null can be obtained:

$$\frac{d}{du} = \partial_u + \frac{d\bar{r}}{du} \partial_{\bar{r}} = \frac{\alpha}{2} \left( \partial_t - \frac{1}{A} \partial_r \right).$$ \hspace{1cm} (3.9)

The partial derivatives $\partial_t$ and $\partial_r$ can now be rewritten in terms of $d/du$ and $\partial_{\bar{r}}$:

$$\partial_t = \frac{1}{\alpha} \frac{d}{du} + \frac{1}{2A} \partial_{\bar{r}};$$ \hspace{1cm} (3.10)

$$\partial_r = \frac{A}{\alpha} \frac{d}{du} + \frac{1}{2} \partial_{\bar{r}}.$$ \hspace{1cm} (3.11)

The wave operator then becomes

$$[\nabla^c \nabla^c] \phi = -\frac{2}{\alpha AR} \frac{d\varphi}{du} - \frac{A'}{A^2 R} \varphi + \frac{1}{AR} \left[ (AR)' - \left( \frac{R'}{A} \right)^2 \right] \phi,$$ \hspace{1cm} (3.12)

where

$$\varphi \equiv \partial_{\bar{r}} (R\phi).$$ \hspace{1cm} (3.13)
Here \( \cdot \) and \( \cdot' \) refer to the operators given in Eq.(3.10) and Eq.(3.11) respectively. It is also necessary to obtain an equation for \( \alpha \). This is achieved by using \((\dot{u})' = (u')_\cdot \), which gives

\[ \partial_t \alpha = \dot{A} \alpha. \] (3.14)

Applying the full Tolman-Bondi solution, the basic equations that must be solved in Brans-Dicke theory are to first order

\[ \frac{d\varphi}{du} = \frac{\alpha \sqrt{1+f}}{2(1+f)} \left( \frac{f'}{R'} - \frac{R''}{R} \right) \varphi + \frac{\alpha}{2R} \frac{F'}{2F} - \frac{R'}{R} \right) \phi 
+ \frac{\alpha}{2} \frac{1}{3+2\omega} R \sqrt{1+f}, \] (3.15)

\[ \partial_t \alpha = \pm \frac{\alpha}{2} \frac{1}{\sqrt{(1+f)}} \left( \frac{F'}{R} - \frac{FR'}{R^2} + f' \right), \] (3.16)

where the upper and lower signs correspond to an expanding and collapsing phase respectively.

For numerical purposes it is also convenient to take the parametrized form of the Tolman-Bondi solution. For \( f = 0 \), \( R \) is given by

\[ R = \left( \frac{9F}{4} \right)^{1/3} (t - t_s(r))^{2/3} \] (3.17)

For \( f > 0 \), it is given by

\[ R = \frac{F}{2f} (\cosh \eta - 1), \] (3.18)

\[ t - t_s(r) = \frac{F}{2f^{3/2}} (\sinh \eta - \eta). \] (3.19)

For \( f < 0 \), it is given by

\[ R = \frac{F}{2(-f)} (1 - \cos \eta), \] (3.20)

\[ t - t_s(r) = \frac{F}{2(-f)^{3/2}} (\eta - \sin \eta). \] (3.21)

In the above, the signs are chosen so that they correspond to the big bang universe.

IV. MODELS

We choose the background primordial black hole model so that the following conditions are satisfied. (1) The big bang occurs at the same time everywhere. (2) The model is asymptotically flat Friedmann and compensated (in the sense described above). (3) The model is free of shell-focusing or shell-crossing naked singularities. (4) The central region is bounded, while the asymptotic region is marginally bound. (5) At the initial time \( t = t_0 \), the condition \( R' > 0 \) is satisfied everywhere.

In order to satisfy the above conditions, we set \( t_s(r) = 0 \) and choose the energy function \( f(r) \) to have the form

\[ f(r) = - \left( \frac{r}{r_c} \right)^2 \quad \text{for} \quad r < r_w \] (4.1)

\[ f(r) = - \left( \frac{r}{r_c} \right)^2 \exp \left( - \left( \frac{r - r_w}{r_w} \right)^4 \right) \quad \text{for} \quad r \geq r_w \] (4.2)

where \( r_c \) gives the curvature radius in the central closed Friedmann region and \( r_w \) gives the scale of the overdense region. We then determine \( F(r) \) so that \( r \) coincides with \( R \) at the \( t = t_0 \) spacelike hypersurface. In practice, the value of \( f \) is essentially zero for \( r \gtrsim 5r_w \) in the above choice of \( f \). We can then match to the Einstein-de Sitter universe.
at least for around ten initial Hubble times after the formation of the event horizon. Recall that $r$ imposes an even stronger condition since it implies $\phi$ event horizon runs through the underdense region and so the scalar field is reduced compared with by Eq. (4.3).

$\phi$ evolution of $\phi$ horizon runs through the overdense region and hence the scalar field is amplified compared with $t<0$. We then set the initial null hypersurface as the null cone whose vertex is at $(t, r) = (t_0, 0)$ and regard the cosmological solution as the initial data on this hypersurface. Although the value of the scalar field at the cosmological particle horizon must be given by this solution, the value in the perturbed region and the surrounding region may be different from this. To examine the sensitivity of the results to this alteration, we consider another form of the initial data which is different from the cosmological solution in the region $t \lesssim$ several $\times t_0$. We now choose

$$\phi \pm = \phi_c \left[ 1 \pm \exp \left[ - \left( \frac{t}{5t_0} \right)^2 \right] \right],$$

(4.4)

so that we have an ingoing wave in the perturbed region, and examine the evolution of the scalar field thereafter.

The numerical code has been checked by the following non-trivial test calculation. In the flat Friedmann universe, the code must reproduce the cosmological evolution Eq. (4.3) from the initial data. There is agreement to within 0.05 \% accuracy.

V. RESULTS

We denote the Hubble parameter in the Friedmann background (far from the perturbed region) as $H_0$ at $t = t_0$. Recall that $r_c$ gives the amplitude of the density perturbation, while $r_w$ gives the size of the perturbed region. For super-horizon scale perturbations, we cannot set the density perturbation to be very large else the overdense region closes up on itself and becomes disconnected from the rest of the universe [14]. Actually, the requirement $R' > 0$ imposes an even stronger condition since it implies $r_w \lesssim r_c$. We have set the Brans-Dicke parameter to be $\omega = 5$. If $r_c$ is much increased, then the amplitude of the overdensity is much decreased and the resulting black hole becomes very small compared with the horizon scale at the formation time. If $r_c$ is much decreased, the overdense region becomes a separate closed universe. Models and parameters are summarized in Table I. The difference between models A, B and C is in the choice of the background perturbation. The difference between models A, D and E is in the choice of the initial condition for the scalar field. A change of $\omega$ only scales the variation of the scalar field from $\phi_0$, as indicated by Eq. (4.3).

In Fig. 1, the energy function $f(r)$ and the initial density perturbation

$$\delta(t_0, r) = \frac{\rho(t_0, r) - \rho(t_0, \infty)}{\rho(t_0, \infty)}$$

(5.1)

are plotted. The trajectories of outgoing null geodesics are plotted in Fig. 2. It is seen that a nearly horizon-scale black hole is formed for model A, while the black hole is smaller than the horizon scale for models B and C. The initial data for the scalar field are set on the initial null Cauchy surface (see Fig. 3). We prepare three sets of initial data $\phi_0$, $\phi_0$ and $\phi_\infty$ and these are plotted in Fig. 4.

In Fig. 5, the profile of the scalar field is plotted for constant $t$. In Fig. 6, it is plotted for constant $R$. Generally, the configuration of the scalar field tends to be spatially homogeneous and the value of the scalar field around the black hole follows the cosmological evolution of the scalar field at each moment. To see this more clearly, we identify the value of the scalar field on the latest null ray within the calculation as the value on the black hole event horizon $\phi_{BH}$ because the null ray is very close to the event horizon for each model (at least for $H_0 t \lesssim 300$, see Fig. 2). Both $\phi_c$ and $\phi_{BH}$ are plotted in Fig. 7. We can see that $\phi_{BH}$ follows the evolution of $\phi_c$. There is a small deviation of $\phi_{BH}$ from $\phi_c$ but this can be explained by the central overdensity and the surrounding underdensity. For $H_0 t \lesssim 100$, the event horizon runs through the overdense region and hence the scalar field is amplified compared with $\phi_c$. Thereafter the event horizon runs through the underdense region and so the scalar field is reduced compared with $\phi_c$. The overall evolution of $\phi_{BH}$ is well described by $\phi_c$.

We have investigated several models for different parameter values and different initial data for the scalar field. The results are also seen in Figs. 5-7. Although there are minor differences among these models, we can conclude that the configuration of the scalar field is nearly spatially homogeneous and well described by the cosmological solution $\phi_c$, at least for around ten initial Hubble times after the formation of the event horizon.
VI. SUMMARY

We have calculated the evolution of the Brans-Dicke scalar field in the presence of a primordial black hole formed in a flat Friedmann background. We have found that the value of the scalar field at the event horizon almost always maintains the cosmological value. This suggests that primordial black holes “forget” the value of the gravitational constant at their formation epoch.

It should be stressed that this result has only been demonstrated for a dust universe in which the scalar field does not appreciably affect the background curvature and it remains to be seen whether the same conclusion applies when this assumption is dropped. As can be seen from Figs. 5 and 6, both the radial gradient and the time derivative of $\phi$ are large at early times and both of these act as source terms in the field equations. However, as long as $\omega$ is large, this is unlikely to stop the scalar field becoming homogeneous eventually, since it is clear that both the radial gradient and time derivative become small well after the initial collapse. Neglecting the back reaction should therefore be a reasonable assumption in this situation, so gravitational memory seems unlikely. However, this conclusion might not apply for $\omega \approx 1$.

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<table>
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FIG. 1. (a) The energy function $f$ and (b) the initial density perturbation $\delta$ are plotted for models A-E.

FIG. 2. Trajectories of outgoing null rays are plotted (a) for models A,D,E, (b) for model B, and (c) for model C.

FIG. 3. Penrose diagram of a primordial black hole in the Einstein de-Sitter universe. The null Cauchy surface is also depicted, on which the initial condition is set in our numerical calculations.

FIG. 4. Initial null Cauchy data sets for the scalar field, $\phi_c$, for models A,B,C, $\phi_+$ for model D, and $\phi_-$ for model E, are plotted. For clarity, the abscissa is in log-scale.

FIG. 5. The configuration of the scalar field at each moment $t = \text{const}$ is plotted for models (a) A, (b) B, (c) C, (d) D and (e) E. For comparison, the cosmological value $\phi_c$ at each moment is also plotted.

FIG. 6. The time variation of the scalar field along the world lines of constant $R$ is plotted for models (a) A, (b) B, (c) C, (d) D and (e) E. For comparison, the cosmological evolution $\phi_c$ is also plotted.

FIG. 7. The time evolution of the scalar field on the event horizon is plotted for models (a) A, (b) B, (c) C, (d) D and (e) E. For comparison, the cosmological evolution $\phi_c$ is also plotted.
FIG. 5(c)

FIG. 5(d)
FIG. 5(e)
FIG. 6(c)

FIG. 6(d)
FIG. 6(e)