Unitarity of Deconstructed Five-Dimensional Yang-Mills Theory

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Abstract

The low-energy properties of a compactified five-dimensional gauge theory can be reproduced in a four-dimensional theory with a replicated gauge group and an appropriate gauge symmetry breaking pattern. The lightest vector bosons in these “deconstructed” or “remodeled” theories have masses and couplings approximately equal to those of the Kaluza-Klein tower of massive vector states present in a compactified higher-dimensional gauge theory. We analyze the unitarity of low-energy scattering of the massive vector bosons in a deconstructed theory, and examine the relationship between the scale of unitarity violation and the scale of the underlying chiral symmetry breaking dynamics which breaks the replicated gauge groups. As in the case of compactified five-dimensional gauge theories, low-energy unitarity is ensured through an interlacing cancellation among contributions from the tower of massive vector bosons. We show that the behavior of these scattering amplitudes is manifest without such intricate cancellations in the scattering of the would-be Goldstone bosons of the deconstructed theory. Unlike compactified five-dimensional gauge theories, the amplitude for longitudinal vector boson scattering in deconstructed theories does grow with energy, though this effect is suppressed by $1/N$, with $N$ being the number of replicated gauge groups.

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The world may be consistently described by a compactified higher dimensional theory, manifested via additional towers of massive Kaluza-Klein (KK) states at low energies. Recently, it has been shown \[^{[1, 2]}\] that the low-energy properties of a compactified five-dimensional gauge theory may be reproduced from a four-dimensional theory with a replicated gauge group structure and an appropriate symmetry breaking pattern. A simple scheme is illustrated by the “aliphatic moose” shown in Figure 1. In this “moose” or “quiver” diagram \[^{[3, 4]}\], the circles represent \(N+1\) \(SU(m)\) gauge groups (labeled by \(j = 0, 1, 2, \cdots, N\)) and the directed lines represent the Goldstone bosons from the spontaneous symmetry breaking of the two adjacent \(SU(m)\) groups down to their diagonal subgroup. Thus, we have the gauge symmetry breaking pattern, \(SU(m)^{N+1} \rightarrow SU(m)_{\text{diag}}\), generating \(N(m^2 - 1)\) massive spin-1 vector states.

The Goldstone bosons may be collected into \(SU(m)\) matrix fields \(U_j (j = 1, 2, \cdots, N)\) which transform under the adjoining \(SU(m)_{j-1} \otimes SU(m)_j\) groups according to

\[
U_j(x) \rightarrow \Omega_{j-1}(x)U_j(x)\Omega_j^\dagger(x),
\]

where \(\Omega_{j(j-1)}\) is the \(SU(m)_{j(j-1)}\) gauge transformation. We thus write \(U_j\) as \[^{[2]}\],

\[
U_j(x) = \exp \left( \frac{i2\pi_j^a(x)T^a}{v} \right),
\]

where \(\{T^a\}\) are the \(SU(m)\) generators (normalized by \(\text{Tr}T^aT^b = \delta^{ab}/2\)), and \(v\) is the analog of the QCD pion decay constant \(f_\pi\) which characterizes chiral symmetry breaking.

Regardless of the underlying dynamics responsible for the gauge symmetry breaking, the low-energy properties of this model may be most economically described by an effective Lagrangian with only the gauge and Goldstone degrees of freedom. The leading terms in this description are

\[
\mathcal{L} = -\frac{1}{4} \sum_{j=0}^{N} F_{j}^{a\mu\nu} F_{j}^{a\mu\nu} + \frac{v^2}{4} \sum_{j=1}^{N} \text{Tr} \left( D^\mu U_j D^\mu U_j^\dagger \right),
\]

where the \(F_{j}^{a\mu\nu}\) is the field-strength of gauge group \(SU(m)_j\), and the covariant derivative is,

\[
D^\mu U_j = \partial^\mu U_j - igA_{j-1}^{a\mu} T^a U_j + igA_j^{b\mu} U_j T^b.
\]

Following Ref.\[^{[2]}\], we will refer the above model as a “deconstructed” theory. In the effective Lagrangian (3), there are \(N(m^2 - 1)\) massive gauge bosons which acquire their masses from absorbing
the corresponding would-be Goldstone bosons via the Higgs mechanism, and no scalars remain. The nonlinear sigma model(s) in the deconstructed theory are not renormalizable. Naive power counting \([?, ?, ?]\) implies such an effective theory is valid only for scales \(\lesssim 4\pi v\), and the underlying dynamics of chiral symmetry breaking must become manifest at or below this scale.

Note that we have chosen the gauge couplings \((g)\) and the vacuum expectation values \((v)\) of the deconstructed theory to be the same for all gauge groups and symmetry breakings. This pattern was chosen so as to reproduce the low-energy properties of a five-dimensional (5D) \(SU(m)\) Yang-Mills theory in which the fifth dimension is compactified to a line segment \(0 \leq x^5 \leq \pi R\). This compactification can be done consistently by an orbifold projection as follows: restrict the gauge fields of the five-dimensional theory \(\hat{A}^M(x^N) [M, N \in (\mu, 5) \text{ with } \mu \in (0, 1, 2, 3)]\) to those periodic in \(x^5\) with period \(2\pi R\) and further impose a \(Z_2\) symmetry,

\[
\hat{A}^\mu(x^\nu, x^5) = +\hat{A}^\mu(x^\nu, -x^5), \quad \hat{A}^5(x^\nu, x^5) = -\hat{A}^5(x^\nu, -x^5).
\]

These projections force the gauge-covariant boundary conditions,

\[
\hat{F}^{5N} = \hat{F}^{N5} = 0
\]

at \(x^5 = 0\) and \(\pi R\). Analyzing the KK modes of this compactified theory shows that, in unitary gauge, one has an infinite tower of massive \(SU(m)\) adjoint vector fields of mass \(n/R\) \((n = 0, 1, \cdots)\). In the compactified 5D theory, the self-interactions of the zero-mode fields are that of a four-dimensional (4D) Yang-Mills theory with gauge-coupling \(g = g_5/\sqrt{\pi R}\), where \(g_5\) is the 5D Yang-Mills coupling with dimension of \((\text{mass})^{-1/2}\). The interactions of the KK modes amongst themselves and with the zero mode gauge-bosons are given by Yang-Mills like couplings \([?, ?, ?]\). Compactified 5D Yang-Mills theory results in an effective 4D KK theory which has the remarkable property \([?, ?, ?]\) that low-energy unitarity is ensured through an interlacing cancellation among contributions from the relevant KK levels, and is delayed to energy scales higher than the customary limit of Dicus-Mathur and Lee-Quigg-Thacker \([?, ?, ?, ?]\) through the introduction of additional vector bosons rather than Higgs scalars. In this Letter, we analyze the unitarity of low-energy massive vector boson scattering in the deconstructed theory, and examine the relationship between the scale of unitarity violation and the scale of the underlying chiral symmetry breaking dynamics responsible for spontaneously breaking the replicated gauge groups. We show that the interactions of the massive vector bosons in the deconstructed theory are, for levels small compared with \(N\), precisely the same as those of the 4D KK theory. We explicitly show that, up to corrections suppressed by \(1/N\), the interactions among the would-be Goldstone bosons encoded in eqn. (3) match exactly with the interactions among the corresponding modes of \(A_n^{a5}\) absorbed through the geometrical Higgs mechanism in the compactified 5D gauge theory.

We begin by reviewing the correspondence between the 4D aliphatic moose model \([\text{cf. eqn. (3)}]\) and the orbifold compactification of 5D Yang-Mills theory \([?, ?, ?, ?]\). Diagonalizing the \(N + 1\) by \(N + 1\) dimensional mass-squared matrix in the aliphatic theory, we find the mass eigenvalues

\[
M_n = gv \sin \frac{n\pi}{2(N + 1)},
\]

with \(n \in (0, 1, 2, \cdots, N)\), which correspond to eigenstates

\[
\hat{A}_0^a = \frac{1}{\sqrt{N + 1}} (A_0^a + A_1^a + \cdots + A_N^a),
\]
for the remaining massless gauge field, and
\[ \tilde{A}_{n\mu}^a = \sqrt{\frac{2}{N+1}} \sum_{k=0}^{N} \cos \left( \left( k + \frac{1}{2} \right) \frac{n\pi}{N+1} \right) A_{k\mu}^a, \quad (n = 1, 2, \cdots, N), \] (9)

for the massive adjoint vector bosons. The massless fields \( \{ \tilde{A}_{0\mu}^a \} \) belong to the residual unbroken gauge group \( SU(m)_{\text{diag}} \) with coupling \( \tilde{g} = g/\sqrt{N+1} \).

Comparing the mass spectrum (7) of the deconstructed theory (for \( n \ll N + 1 \)) with the linear spectrum \( n/R \) in the KK theory, we see that the two coincide under the identification \( [?, ?] \),

\[ \frac{1}{R} = \frac{\pi g v}{2(N+1)} = \frac{\pi \tilde{g} v}{2\sqrt{N+1}}, \] (10)

and eqn. (7) can be expanded as,

\[ M_n = \frac{n}{R} \left[ 1 - \frac{\pi^2}{24} \left( \frac{n}{N+1} \right)^2 + O\left( \frac{n^4}{N^4} \right) \right] \equiv M_n [1 - \delta_n], \] (11)

where \( M_n = n/R \) and \( \delta_n = O(n^2/N^2) \). The identification (10) corresponds to interpreting the Moose diagram itself as a discretized fifth-dimension with a lattice spacing,

\[ a = \frac{\pi R}{N} = \frac{2(1 + N^{-1})}{\tilde{g} v} = \frac{2\sqrt{1 + N^{-1}}}{\tilde{g} v\sqrt{N}}. \] (12)

The spectrum of the deconstructed theory approximates the linear spectrum of the 4D KK theory so long as \( n \ll N + 1 \), i.e., \( M_n \ll 1/a \).

The correspondence between the deconstructed theory and 4D KK theory may be completed by identifying the couplings of the unbroken massless gauge group,

\[ \frac{1}{g^2} = \frac{\pi R}{g_5^2}, \] (13)

which yields

\[ g_5 = \sqrt{\frac{2g}{\tilde{g} v}}. \] (14)

This correspondence implies that

\[ \frac{a}{g_5^2} = \frac{1}{N g^2} = \frac{1 + N^{-1}}{g^2}, \] (15)

and the “bare” coupling of the deconstructed theory, \( g \), may be identified with the effective strength of the gauge coupling in the 5D Yang-Mills theory underlying the 4D KK theory.

As indicated in eqn. (11), the exact correspondence between the spectra of the deconstructed theory and the 4D KK theory is realized only for sufficiently large \( N \). We will explicitly show that the vector-boson scattering amplitudes agree in these theories as well, up to corrections suppressed by \( 1/N \). The “continuum limit” corresponds to \( a \to 0 \) for fixed \( R \) and \( \tilde{g} \), or, equivalently, \( N \to \infty \).
with\(^1\) \(g = \mathcal{O}(\sqrt{N})\). From eqn. (10), therefore, we deduce that \(v = \mathcal{O}(\sqrt{N})\) when approaching the continuum limit.

A 5D gauge theory is nonrenormalizable, and one manifestation of this is the bad high-energy behavior of massive vector-boson scattering. In Ref. [\(？\)], it is shown that tree-level gauge boson scattering in the 5D \(SU(m)\) Yang-Mills theory violates unitarity at an energy scale of the order

\[
\sqrt{s} = E_{\text{cm}} \leq \Lambda = \frac{96\pi}{23m} \frac{1}{g_5},
\]

(16)

and therefore this theory is, at best, a low-energy effective theory valid only up to a scale of order \(\Lambda\). In the deconstructed theory, from eqn. (13), \(\Lambda\) corresponds to an energy scale of order

\[
\Lambda \simeq \frac{96\pi}{23m} \frac{1}{\tilde{g}^2 N a},
\]

(17)

which is higher than \(1/a\) so long as

\[
\tilde{g} \lesssim \frac{3.6}{\sqrt{mN}}.
\]

(18)

When the deconstructed theory is embedded into a 4D renormalizable high-energy theory [\(？, ？\)], the 4D theory provides a “high-energy” completion of the compactified 5D Yang-Mills theory. From the considerations above, we see that for weak or moderate coupling and modest \(N\), a deconstructed theory provides a high-energy completion which respects the bound in eqn. (16).

As noted above, the deconstructed theory itself involves chiral symmetry breaking dynamics in order to provide the Goldstone-boson “link” fields that allow particles to “hop” in the fifth dimension. Power counting [\(？, ？, ？\)] shows that the non-linear sigma model low-energy description must break down at a scale \(\lesssim 4\pi v\). Given the effective lattice spacing eqn. (12), we see that, \(1/a < 4\pi v\), provided

\[
\tilde{g} \lesssim 8\pi \frac{\sqrt{N + 1}}{N}.
\]

(19)

In this case the non-linear sigma model description can remain valid up to the scale \(1/a\), at which the model no longer behaves like the compactified effective 4D KK theory. In what follows we will investigate the scattering of massive vector-bosons and their corresponding Goldstone bosons in the deconstructed theory for energy scales less than \(1/a\), therefore we need not be concerned about the underlying 4D chiral symmetry breaking dynamics.

To analyze the relevant scattering processes, we start by deriving the unitary gauge Lagrangian of the deconstructed theory in Fig. 1. In this gauge all link-fields \(\{U_j\}\) are set to the identity via appropriate \(SU(m)\) gauge transformations. Expressing all the vertices in terms of the mass-eigenstate gauge fields, we derive the interaction Lagrangian

\[
\mathcal{L}_{\text{gauge}} = -\tilde{g} G^{abc} \sum_{n=1}^{N} \left[ \partial_\mu \tilde{A}_a^{bn} \tilde{A}_b^{cn} + \partial_\mu \tilde{A}_a^{bn} (\tilde{A}_b^{cn} \tilde{A}_c^{bn} + \tilde{A}_b^{cn} \tilde{A}_c^{bn}) \right]
\]

\(^1\) As a practical matter, of course, the coupling of the replicated gauge groups \(g\) is bounded by \(\mathcal{O}(4\pi)\). Hence, there is a bound on how large \(N\) can be for a fixed size of the low-energy coupling \(\tilde{g}\). This is similar to the bound on the underlying scale of the 5D gauge theory relative to the compactification scale arising from the constraint that the 4D gauge coupling has a finite size.
so long as $\Delta$ unitarity at an energy scale, $n$ of longitudinally polarized vector bosons at level $\tilde{\Delta}$.

After a careful analysis of the gauge amplitudes with the mass expansion of (11) and the factor $\Delta_3$, the deviation of the vector-boson amplitude in the deconstructed theory is expected to enter the deconstructed theory. This provides a clue for how to proceed: the vector boson $\Delta_4$.

The deconstructed theory describes a set of massive self-interacting vector bosons with a characteristic coupling $\tilde{g}$. The traditional arguments $[?, ?, ?]$ suggest that the scattering amplitudes of longitudinally polarized vector bosons at level $n \ll N + 1$ would grow with energy and violate unitarity at an energy scale,

$$E^* \sim \frac{4\pi M_n}{\tilde{g}} \approx \frac{2\pi^2 n v}{\sqrt{N + 1}} = \frac{4n\pi^2}{\tilde{g} N a} = \frac{4n\pi^2}{g^2 N a}.$$  \hspace{1cm} (20)

This cannot be the case because, for given $v$, $E^*$ could be made much smaller than $4\pi v$ – the scale at which the chiral symmetry breaking dynamics is expected to enter the deconstructed theory.

Furthermore, $E^*$ could be made much smaller than the scale $1/a$ below which the theory should correspond to the 4D KK theory. This provides a clue for how to proceed: the vector boson masses in the 4D KK theory arise through a geometrical Higgs mechanism, and unitarity is ensured through an interlacing cancellation among the contributions of the relevant gauge KK modes $[?]$. We therefore expect that there will be similar cancellations in the deconstructed theory. Since the unitary-gauge Lagrangian (20) is identical to that derived in the compactified 5D theory $[?, ?, ?, ?]$ for low KK-levels, the deviations of the vector-boson amplitude in the deconstructed theory from that in the compactified 5D theory can only come from the modification term in the mass-spectrum (11). After a careful analysis of the gauge amplitudes with the mass expansion of (11) and the high energy expansion of $M_n/E$, we find that the individual $O(E^4)$ terms completely cancel and the non-canceled $O(E^2)$ terms appear only at the order $1/N$. For instance, we derive the following $O(E^2)$ amplitudes for the elastic scattering of the longitudinal gauge bosons, $\bar{A}_L^{\alpha n} A_L^{\beta n} \to \bar{A}_L^{\alpha n} A_L^{\beta n}$,

$$T \left[ \bar{A}_L^{\alpha n} A_L^{\beta n} - \bar{A}_L^{\alpha n} A_L^{\beta n} \right] = \frac{g^2 \delta_n}{M_n^2(N + 1)} \left[ -c C^{abc} C^{cde} + \left( \frac{9}{2} + \frac{11}{2} c \right) C^{ace} C^{bde} + \left( \frac{9}{2} - \frac{11}{2} c \right) C^{ade} C^{bce} \right], \hspace{1cm} (23)$$

$^2$If $(n, m, \ell, k)$ are such that $n + m + \ell$ or $n + m + \ell + k$ equals $2q(N + 1)$ for $q = 1, 2, \cdots$, the factors $\Delta_3, \Delta_4$ will have an additional contribution equal to $(-1)^q$. 

5
which, as expected, is proportional to the nonlinear modification $\delta_\theta$ in eqn. (11), and has the coefficient $(g^2\delta_\theta)/[M^2_n(N+1)] \simeq 1/[6u^2(N+1)]$ suppressed by $1/(N+1)$. Here, $c = \cos \theta$ and $\theta$ is the scattering angle. For the inelastic scattering $\tilde{A}_L^{an}\tilde{A}_L^{bn} \rightarrow \tilde{A}_L^{cn}\tilde{A}_L^{dn} (n \neq m)$, we arrive at

$$T \left[ \tilde{A}_L^{an}\tilde{A}_L^{bn} \rightarrow \tilde{A}_L^{cn}\tilde{A}_L^{dn} \right] = \frac{s}{6u^2(N+1)} \left[ -2c C^{abe} C^{cde} + (3+5c) C^{ace} C^{bde} + (3-5c) C^{ade} C^{bce} \right] ,$$

(24)

which, with the aid of Jacobi identity $C^{abe} C^{cde} + C^{ace} C^{bde} + C^{ade} C^{bce} = 0$, can be related to the $O(E^2)$ elastic amplitude (23) via

$$T \left[ \tilde{A}_L^{an}\tilde{A}_L^{bn} \rightarrow \tilde{A}_L^{cn}\tilde{A}_L^{dn} \right] \simeq \frac{2}{3} T \left[ \tilde{A}_L^{an}\tilde{A}_L^{bn} \rightarrow \tilde{A}_L^{cn}\tilde{A}_L^{dn} \right] .$$

(25)

The unitarity analysis above is performed in the vector boson sector of the deconstructed theory. Similar high energy behavior must also arise in the would-be Goldstone boson sector of the deconstructed theory. Since the deconstructed theory is based on the spontaneous symmetry breaking $SU(m)^{N+1} \rightarrow SU(m)_{\text{diag}}$, in the corresponding $R_\xi$ gauge gauge [cf. eqn. (29)] we can derive the equivalence theorem

$$T \left[ \tilde{A}_{nL}(p_n), \tilde{A}_{mL}(p_m), \cdots \right] = C_{\text{mod}} T \left[ \tilde{\pi}_n^a(p_n), \tilde{\pi}_m^b(p_m), \cdots \right] + O(M_{n,m,\cdots}/E) ,$$

(26)

where the levels $(n, m, \cdots) = 1, 2, \cdots, N$, and each external momentum is put on mass-shell, $p_n^2 = M_n^2$, etc. Here the fields $\{\tilde{\pi}_n^a\}$ are the would-be Goldstone bosons “eaten” by the corresponding mass eigenstate vector fields $\{\tilde{A}_n^a\}$. This is analogous to the traditional equivalence theorem in the Standard Model [?, ?, ?, ?, ?, ?, ?], and the modification factor $C_{\text{mod}} = 1 + O(\text{loop})$ appears at loop level [?, ?, ?, ?, ?].

To analyze the Goldstone boson scattering, we will derive the complete $R_\xi$ gauge Lagrangian for the deconstructed theory. From the nonlinear dimension-2 term of Goldstone boson kinetic energy in eqn. (3), we deduce the following bilinear gauge-Goldstone mixings,

$$L_{\text{mix}} = \sum_{j=1}^{N} -\frac{1}{2} \mu \left[ A_{j-1}^{a\mu} \partial_\mu \pi_j^a - A_j^{a\mu} \partial_\mu \pi_j^a \right] = \sum_{n=1}^{N} -M_n \tilde{A}_n^{a\mu} \partial_\mu \tilde{\pi}_n^a .$$

(27)

Here $M_n$ is given by eqn. (7) and the fields $\{\tilde{\pi}_n^a\}$ are the eigenstates of “eaten” Goldstone bosons defined by the orthogonal rotation,

$$\tilde{\pi}_n^a = \sum_{k=1}^{N} \frac{2}{N+1} \sin \left( \frac{nk\pi}{N+1} \right) \pi_k^a , \quad (n = 1, 2, \cdots, N) .$$

(28)

The bilinear mixing (27) can be eliminated by defining a general $R_\xi$ gauge fixing term,

$$L_{\text{GF}} = -\sum_{n=0}^{N} \frac{1}{2 \xi_n} \left( \partial^\mu \tilde{A}_n^{a\mu} - \xi_n M_n \tilde{\pi}_n^a \right)^2 ,$$

(29)

where $n = 0$ corresponds to the usual gauge-fixing of the unbroken group $SU(m)_{\text{diag}}$. The would-be Goldstone bosons $\tilde{\pi}_n^a$ acquire gauge-dependent masses $M_n^2 = \xi_n M_n^2$. The appropriate ghost Lagrangian can be derived as well, though it is not explicitly needed for the current analysis.
The interactions of the Goldstone bosons with the gauge bosons and among themselves arise from the nonlinear sigma model, à la Callan-Coleman-Wess-Zumino (CCWZ) [?]. However, the “geometric” Goldstone sector in compactified 5D Yang-Mills theory appears very different since its Goldstone bosons \(\{A_n^a\}\), the fifth components of the 5D gauge fields, interact at most bilinearly with other gauge modes, and have no self-interaction among themselves [?]. How does this highly nonlinear CCWZ Goldstone sector match with the geometric, linearized Goldstone sector in compactified 5D gauge theory? As we now show, the correspondence between the Goldstone sector of the deconstructed theory and that of the compactified 5D Yang-Mills theory works at \(\mathcal{O}(N^0)\), with the corrections suppressed by powers of \(1/N\).

Using the orthogonal rotations (9) and (28), and collecting the terms of order \(N^0\) and \(N^{-1}\), we arrive at, after a lengthy derivation, the following Goldstone interactions,

\[
L_{\text{GB}}^{\text{LO}} = +gC^{abc}\sum_{n=1}^{N} \bar{A}_0^n \bar{A}_n^a \left( \partial \pi^d_n - M_n \bar{A}_d^a \right) + \frac{g^2}{2} C^{abc} C^{ade} \sum_{n=1}^{N} \bar{A}_0^n \bar{A}_n^a \bar{A}_d^e \pi^c_{n} \pi^e_{d} \\
+ \frac{g}{\sqrt{2}} C^{abc} \sum_{n, \bar{m}, \ell = 1}^{N} \bar{\Delta}_3(n, \bar{m}, \ell) \bar{A}_n^b \bar{A}_\bar{m}^c \left( \partial \pi^d_n - M_\ell \bar{A}_d^a \right) \\
+ \frac{g^2}{\sqrt{2}} C^{abc} C^{ade} \sum_{n, \bar{m}, \ell = 1}^{N} \bar{\Delta}_3(n, \bar{m}, \ell) \bar{A}_n^b \bar{A}_\bar{m}^d \bar{A}_\ell^e \pi^c_{n} \pi^d_{\bar{m}} \pi^e_{\ell} \\
+ \frac{g^2}{4} C^{abc} C^{ade} \sum_{n, \bar{m}, \ell, k = 1}^{N} \bar{\Delta}_4(n, \bar{m}, \ell, k) \bar{A}_n^b \bar{A}_\bar{m}^d \bar{A}_\ell^c \bar{A}_k^e \pi^c_{n} \pi^d_{\bar{m}} \pi^c_{\ell} \pi^e_{k},
\]

(30)

\[
L_{\text{GB}}^{\text{NLO}} = \sum_{n, \bar{m}, \ell, k = 1}^{N} \frac{1}{12(N+1)N^2} \bar{\Delta}_4(n, \bar{m}, \ell, k) \left[ \frac{2}{m} \delta^{ab} \delta^{cd} + d^{abc} d^{cde} \right] \\
\times \left[ \pi^a_n \partial \pi^b_{\bar{m}} \pi^c_{\ell} \partial \pi^d_k - \partial \pi^a_n \partial \pi^b_{\bar{m}} \left( \pi^c_{\ell} \pi^d_k \right) \right] + \mathcal{O}(g), \quad (31)
\]

where

\[
\bar{\Delta}_3(n, \bar{m}, \ell) = \delta(n + \bar{m} - \ell) + \delta(n - \bar{m} + \ell) - \delta(n + \bar{m} - \ell), \\
\bar{\Delta}_4(n, \bar{m}, \ell, k) = \delta(n + \bar{m} + \ell - k) + \delta(n + \bar{m} - \ell + k) + \delta(n - \bar{m} + \ell + k) + \delta(n - \bar{m} - \ell + k) \\
- \delta(n + \bar{m} - \ell - k) - \delta(n - \bar{m} + \ell + k) - \delta(n - \bar{m} - \ell - k), \\
\bar{\Delta}_4(n, \bar{m}, \ell, k) = \delta(n + \bar{m} + \ell - k) + \delta(n + \bar{m} - \ell + k) + \delta(n + \bar{m} - \ell - k) \\
- \delta(n + \bar{m} - \ell + k) - \delta(n + \bar{m} + \ell - k) - \delta(n - \bar{m} + \ell + k) - \delta(n - \bar{m} - \ell - k).
\]

In (31), the factor \(\frac{2}{m}\) contains the \(m\) of \(SU(m)\); and the symmetric \(d\)-function is defined by \(\{T^a, T^b\} = \frac{1}{m} \delta^{ab} + d^{abc} T^c\). The unspecified terms of \(\mathcal{O}(g)\) or smaller in (31) contain at most one partial derivative and at least one gauge field, and are irrelevant to the \(\mathcal{O}(E^2)\) leading behavior of the Goldstone scattering amplitude to be derived shortly. Other contributions suppressed by \(1/N^2\) or higher will also not be needed below.

It is important to note that the leading order (LO) Goldstone Lagrangian \(L_{\text{GB}}^{\text{LO}}\) in (30) contains at
most two Goldstone fields. It *precisely* matches with that derived in the compactified 5D Yang-Mills theory [?], under the identification of $\pi_n^a \rightarrow A_n^{a5}$. To order $N^0$, the correspondence between the CCWZ Goldstone sector of the deconstructed theory and the geometric, linearized Goldstone sector in the compactified 5D gauge theory is exact. At this order all non-renormalizable interaction vertices containing of dimension > 4 disappear, and the leading order high-energy behavior of Goldstone boson scattering matches that of compactified 5D Yang-Mills theory [?].

The deviation of the deconstructed Goldstone Lagrangian from that of the compactified 5D gauge theory explicitly appears at the next-to-leading order (NLO) of the 1/N-expansion. For the scattering processes $\pi_n^a \pi_n^b \rightarrow \pi_n^a \pi_n^b$ and $\pi_n^a \pi_n^b \rightarrow \pi_m^a \pi_m^b$ ($n \neq m$), at $O(1/N)$ and $O(E^2/v^2)$, we derive

$$\mathcal{T}[\pi_n^a \pi_n^b \rightarrow \pi_m^a \pi_m^b] = \frac{3}{2} \mathcal{T}[\pi_n^a \pi_n^b \rightarrow \pi_m^a \pi_m^b] = \frac{3}{2(N+1)v^2} \left[ s X^{ab,cd} + t X^{ac,bd} + u X^{ad,bc} \right],$$

(33)

where $X^{ab,cd} \equiv \frac{2}{m} \delta^{ab} \delta^{cd} + d^{a b, c d e}$. For SU($m$) = SU(2), $X^{ab,cd} = \delta^{ab} \delta^{cd}$ and (33) reduces to the familiar form of the $\pi\pi$ scattering of low energy QCD [?] except an overall factor $\sim 1/(N+1)$. Making use of the SU($m$) identity

$$C^{a b c} C^{c d e} = \frac{2}{m} \left( \delta^{a c} \delta^{b d} - \delta^{a d} \delta^{b c} \right) + \left( d^{a c e} d^{b d e} - d^{a d e} d^{b c e} \right),$$

(34)

we find that the gauge amplitudes (23)-(24) fully agree to the corresponding Goldstone amplitudes (33) at the same order of 1/N, satisfying the equivalence theorem (26).

Projecting the elastic amplitude $\mathcal{T}[\pi_n^a \pi_n^b \rightarrow \pi_n^a \pi_n^b]$ to the isospin-singlet and spin-0 channel, i.e., $T_{00}[nn;nn] = [3/2(N+1)]s/16\pi^2 v^2$ for SU(2), we readily derive the unitarity bound,

$$T_{00}[nn;nn] \leq \frac{1}{2}, \quad \Rightarrow \quad \sqrt{s} \leq \sqrt{N+1} \frac{4\pi v}{\sqrt{3\pi}},$$

(35)

which is, apparently, delayed relative to the customary unitary limit for $\pi\pi$ scattering by a factor of $\sqrt{(N+1)/(2/3)}$.

However, the deconstructed theory has many "KK" levels of $\tilde{n}_n^a$, with $n = 1, 2, \cdots, N$, and we must consider coupled channels as well. Consider a normalized state, consisting of Goldstone boson pairs with "KK" levels up to $N_0$,

$$\left| \Psi^{ab} \right> = \frac{1}{\sqrt{N_0}} \sum_{\ell=1}^{N_0} \left| \tilde{n}_n^a \tilde{n}_n^b \right>,$$

(36)

\footnote{At the leading order of the 1/N-expansion the mass spectrum of gauge fields $\{ \tilde{A}_n^a \}$ also becomes identical to that of the the compactified 5D Yang-Mills theory, i.e., $M_n = (n/R) \left[ 1 + O(n^2/N^3) \right]$, as shown in eqn. (11).}

\footnote{Here we do not include the contribution of the leading order Lagrangian $\mathcal{L}_{GB}$ to the scattering amplitude since it behaves as constant and does not grow with the energy, as computed in Ref. [?].}
from which we deduce the $O(1/N)$ scattering amplitude, at high energies $\sqrt{s} \gg 2M_N$,

$$
T \left[ |\Psi^{ab}\rangle \rightarrow |\Psi^{cd}\rangle \right] = \sum_{\ell,k=1}^{N_0} \frac{1}{N_0} T \left[ \bar{\pi}^a_{\ell} \bar{\pi}^b_{\ell} \rightarrow \bar{\pi}^a_k \bar{\pi}^b_k \right]
$$

$$
= (N_0 - 1) T \left[ \bar{\pi}^a_n \bar{\pi}^b_n \rightarrow \bar{\pi}^a_m \bar{\pi}^b_m \right]_{n \neq m} + T \left[ \bar{\pi}^a_n \bar{\pi}^b_n \rightarrow \bar{\pi}^a_n \bar{\pi}^b_n \right]
$$

$$
= (N_0 + \frac{1}{2}) T \left[ \bar{\pi}^a_n \bar{\pi}^b_n \rightarrow \bar{\pi}^a_m \bar{\pi}^b_m \right] = \frac{(N_0 + \frac{1}{2})}{(N + 1)v^2} \left[ s X^{ab,cd} + t X^{ac,bd} + u X^{ad,bc} \right].
$$

(37)

Thus, we see that when the number of invoked “KK” levels reaches $N_0 \sim N$, we recover the customary unitarity limit, for $SU(m) = SU(2)$.\(^5\)

$$
\sqrt{s} \lesssim \sqrt{8\pi v} = \sqrt{\frac{2}{\pi} \frac{4g}{g^2_5}} = \frac{1}{\sqrt{\frac{1}{\pi} \frac{4(1 + N^{-1})}{g a}}},
$$

(38)

which is of the order $v \sim g/g^2_5 \sim 1/(ga)$, and is neither enhanced by extra $\sqrt{N + 1}$ [cf. the single channel analysis in eqn. (35)] nor further suppressed by $1/\sqrt{N + 1}$ [cf. the naive estimate in eqn. (22)].

In summary, we have systematically analyzed the gauge and Goldstone interaction Lagrangians in four-dimensional deconstructed Yang-Mills theory. For the low “KK” levels, the gauge sector differs from the compactified 5D theory only in the mass-spectrum, but the Goldstone sector explicitly differs in its interaction Lagrangian at order $O(1/N)$. We have analyzed the relationship between the scale of unitarity violation in longitudinal vector-boson scattering and the scale of the underlying chiral symmetry breaking dynamics responsible for spontaneously breaking the replicated gauge groups. As in compactified 5D gauge theory, the low-energy unitarity of longitudinal vector-boson scattering is ensured through an interlacing cancellation among contributions from various “KK” levels. We have shown that the behavior of these amplitudes can be also understood in the deconstructed theory by analyzing would-be Goldstone boson scattering via the equivalence theorem. Taking into account the non-cancelled $E^2$-contributions at the order $1/N$, we find that unitarity violation in the deconstructed theory is delayed to the intrinsic ultraviolet scale $1/g^2_5$ or $1/a$, and is above the customary Dicus-Mathur/Lee-Quigg-Thacker limit. We have also demonstrated explicitly the correspondence between the Higgs mechanism in the 4D deconstructed theory and the “geometric Higgs mechanism” in the compactified 5D gauge theory.

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\(^5\)The limit for the arbitrary $SU(m)$ varies only by a factor of $O(m/2) = O(1)$, with no conceptual difference.