The possibility of facing critical phenomena in nuclear fragmentation is a topic of great interest. Different observables have been proposed to identify such a behavior \([1,3,2,4,5]\), in particular, some related to the use of information entropy as a possible signal of critical behavior \([6]\). In this work we critically examine some of the most widespread used ones comparing its performance in bond percolation and in the analysis of fragmenting Lennard Jones Drops.

When nuclei are made to collide at energies of the order of hundreds of MeVs, the resulting highly excited systems breaks up into several medium-size fragments. This phenomena is usually referred as multifragmentation and attracts a lot of attention because it is conjectured that the system might be undergoing a phase transition of some order. Different observables have been proposed to properly characterize this phenomena. In particular the possibility of facing critical behavior (originally triggered by the seminal work of Purdue \([7]\)) has been accompanied by the development of different signatures to characterize such a behavior from the analysis of the observed mass spectra. Starting with the work of Bauer \([4]\) and Campi \([3]\) the percolation model has emerged as a quite useful test-bed for the analysis of the tools to be applied in the examination of nuclear multifragmentation.

In \([6]\) the information entropy has been proposed as a signature of critical behavior, and two systems has been analyzed: a lattice gas model and molecular dynamics model.

In this work, we test the information entropy as well as others signatures of critical behavior in systems that undergo a second order phase transition. One of these system is the percolation model, for which the behavior near the critical point (including the critical exponents) is well known. In this work we focus on a few of the different criticality signatures proposed in the literature besides the information entropy \((S_1)\): the normalized variance of the size of the maximum fragment \((NVM)\) \([2]\), the second moment of the distribution \((M_2)\) \([1]\), the best fitting power law \([5]\) and minimum \(\tau\) \([6,8]\).

Our test-bed are the mass distributions resulting from a bond percolation process on two lattices of sizes 30x30x30 and 6x6x6. It has been found that for the larger one, the finite size effects are negligible, meanwhile for the smaller one (with a nuclear system’s size) the finite size effects are important \([9]\).

Once we have tested the above mentioned tools in the percolation model, we can apply this signatures to the analysis of excited Lennard Jones drops that undergo fragmentation to detect if such a process can be understood as a second order phase transition.

In percolation, the number of clusters of size \(A\) per lattice site, \(n_A\), at a given lattice probability, \(p\), was proposed in \([1]\) to be given by:

\[
n_A(p) = q_0 A^{-\tau} f(z) \tag{1}
\]

with \(q_0\) a normalization factor, \(\tau\) a critical exponent and \(f(z)\) a scaling function. This equation shows that \(n_A(p)/q_0 A^{-\tau}\) depends on \(p\) and \(A\) via the combination \(z = ((p - p_c)/p_c) A^\epsilon = \epsilon A^\epsilon\). From this it is clear that at the critical point \(n_A(p_c) = q_0 A^{-\tau} f(0)\) with \(f(0) = 1\). All this is valid for not too small fragments and infinite systems removing the percolating cluster. In finite systems we won’t get a pure power law because finite size effects introduce distortions. Nevertheless, there exist a range of mass (depending of the size of the system) in which the mass spectra presents a power law like behavior, as can be seen in fig.(1f).

There are two main methods to find the critical point by tunning the power law mass spectra. The first one consists in fitting the mass spectra with a simple power law, with two independent parameters: \(q_0\) and \(\tau\), and then look for the probability (or the quantity taken as a control parameter) for which the exponent \(\tau\) is minimum \([6,8]\).

The second one was developed in \([5]\) and takes care of the constraint imposed by the normalization, by which \(q_0 = q_0(\tau)\) via a Riemann \(\zeta\) function. This dependence can be found if the first moment of the cluster distribution is considered at the critical point \([10]\):

\[
M_1(\epsilon = 0) = \sum_A n_A(\epsilon = 0) A = q_0 \sum_A A^{1-\tau} = 1 \tag{2}
\]

with the sum running over all clusters. Eq.(2) shows that the normalization constant, \(q_0\), is dependent on \(\tau\) via a Riemann \(\zeta\) function.

1

P. Balenzuela and C.O.Dorso

(The Date)
\[ q_0 = \frac{1}{\sum A^{{1-\tau}}} \] (3)

Therefore, if we take into account the dependence of \( q_0 \) with \( \tau \) via eq(3), we have to fit the cluster distribution not with a two parameter power-law but with a simple parameter, \( \tau \)-depending one. In this way, the critical point correspond to the best fitted spectra, i.e., minimum \( \chi^2 \).

The critical point could also be found by looking for the maximum (divergence in an infinite system) of the moments of the distribution [1] or of the normalize variance of the size of the maximum fragment (NVM).

The moments of the cluster distribution, \( M_k \), are defined as:

\[ M_k = \sum A^k n_A \] (4)

where \( n_A \) can is the cluster distribution.

The normalize variance of the size of the maximum fragment (NVM) [2] is defined as:

\[ NVM = \frac{\langle A_{max} - \langle A_{max} \rangle \rangle^2}{\langle A_{max} \rangle} \] (5)

where \( A_{max} \) is the biggest fragment of a given event and \( \langle ... \rangle \) is an average over an ensemble of events at a given probability.

Finally, the information entropy proposed in [6] is defined as:

\[ S_1 = -\sum_A p_A \ln(p_A) \] (6)

where \( p_A \) is the probability of detecting a fragment of mass \( A \) defined as \( p_A = N_A/N_t \), with \( N_A \) the number of fragments of size \( A \) detected in a set of experiments and \( N_t = \sum A N_A \) the total number of fragments detected in the same set of experiments. This magnitude is referred as \( S_1 \) because it belongs to the family of generalized Renyi entropies defined by [2]:

\[ S_q = \begin{cases} -\sum p \ln(p) & q = 1 \\ \frac{1}{q-1} \ln(\sum p^q) & q \neq 1 \end{cases} \] (7)

In figure (1) we show our calculations on three dimensional percolation lattices for the two different sizes specified above. The NVM in fig.(1a), \( M_2 \) in fig.(1b), \( \chi^2 \) in fig.(1c) (which minimum signals the best power law fitting with one-parameter fit, \( q_0 = q_0(\tau) \)), the \( \tau_{eff} \) in fig.(1d) (corresponding to a two-parameter fit of the cluster distribution. The minimum value of \( \tau \) would correspond to the critical point) and \( S_1 \) in fig.(1e).

We can see that \( NVM, M_2 \) (also \( \chi^2 \) not showed in this plot) and \( \chi^2 \) give the same critical probability (\( p_c = 0.25 \pm 0.01 \)) for 30x30x30. The signatures are very sharp and the critical probability is the one corresponding to the infinite limit, indicating that the finite size effects are negligible. On the other hand, the maximum of the information entropy is very smooth and also it is shifted (at \( p = 0.225 \pm 0.015 \)) respect the other magnitudes, showing that it is not a good signature of the critical point. Finally, we can see that when we fit the cluster distribution with two independent parameters power law, the exponent \( \tau \) is minimum at \( p = 0.24 \pm 0.01 \), slightly shifted from the other signals (but with overlapping error bars) and with a minimum smoother than when we fit the mass spectra with \( \tau \) and \( q_0(\tau) \) (minimum \( \chi^2 \)).

When we look at the results for the lattice of size 6x6x6, we can see that the magnitudes are slightly shifted from one another but with overlapping error bars, including the information entropy. The best fitted mass spectra according to the condition given by eq.(3) is at \( p = 0.31 \pm 0.01 \) (fig.(1e), the NVM is maximum at \( p = 0.30 \pm 0.01 \) (fig.(1a)), the \( M_2 \) at \( p = 0.29 \pm 0.01 \) (fig.(1b)) and \( S_1(p_i) \) at \( p_i = 0.27 \pm 0.03 \) (fig.(1c)).

As we can see, when the system is small, finite size effects can introduce distortion and smear the signature of the critical point. In particular, \( S_1 \) is shifted respect the critical point when finite size effects are negligible.

In order to understand the behavior of the information entropy in this kind of systems, we generate a family of clusters distributions according to eq.(1) with the critical exponents corresponding to an infinite percolation lattice, the normalization constant \( q_0 \) is taken as the inverse of the zeta Riemann function evaluated in \( \tau = 2.18 \) and the scaling function \( f(z) \) is the one fitted in [9]. We also choose \( p_c = 0.25 \), so the power law mass spectra correspond to this value of probability.

These clusters distribution have the same functional forms showed by the percolating system but without the percolating cluster and without finite size effects.
The information entropy can be calculated by replacing the eq.(1) for \( n_A(p) \) in the definition of \( S_1 \) given by eq.(6). This calculation is plotted in fig.(2c) and we can see that the maximum in the entropy does not correspond to the pure power law distribution \((p_c = 0.25 \text{ in this case})\) as was claimed in [6] but to the distribution corresponding to \( p = 0.22 \) showed at figure (2d). We can check this result if we calculate the derivative of \( S_1 \) against \( \epsilon = (p - p_c)/p_c \):

\[
\frac{dS_1}{d\epsilon} = q_0^2 \sum_{s_1,s_2} s_1^{(-\tau)} s_2^{(-\tau)} q_1 s_1 f(s_1,\epsilon) f(s_2,\epsilon) \log([s_1^{(-\tau)} f(s_1,\tau)]/[s_2^{(-\tau)} f(s_2,\tau)])
\]

with \( f(s,\epsilon) = f(z) \) the scaling function and \( f'(z) = \frac{df}{dz} \). If we evaluate the eq.(8) in the critical point, i.e. \( \epsilon = 0 \), we get:

\[
\left( \frac{dS_1}{d\epsilon} \right)_{\epsilon=0} = \frac{\tau}{\Sigma^2(\tau)} f'(0) \sum_{s_1,s_2} s_1^{\sigma s_1^{(-\tau)} s_2^{(-\tau)} [\log(s_1) - \log(s_2)] \approx -1.6}
\]

It is clear from this expression that the information entropy has not an extreme in the critical point for this family of distribution functions and therefore \( S_1 \) is not a signal of critical behavior.

We also plot the \( \chi^2 \) coefficient for the cluster distribution fitted with one parameter \((q_0 = q_0(\tau) \text{ according to eq.(3)})\) in fig.(2a) and the second moment of the distribution, \( M_2 \), in fig.(2b). It can be seen that these magnitudes signal the critical point very accurately.

Finally, we apply these tools to a molecular dynamics model. This model was described in many previous works [11–14] and consist in several particles (in the order of a few hundreds) interacting via a Lennard-Jones potential that undergoes multifragmentation.

Using this model and taking the excitation energy as a control parameter we perform the same analysis than in the percolation model.

In figure (3) we can see \( M_2 \) and \( NVM \) peak at \( E = 0.3 \pm 0.2\epsilon \), meanwhile the best fitted mass spectra according to eq.3 (minimum \( \chi^2 \)) is at \( E = 0.15 \pm 0.15\epsilon \). On the other hand the information entropy is monotonically increasing in the range analyzed and therefore it is not maximum when the system displays a power law mass spectra, i.e, in the energies around \( E = 0.2\epsilon \).

In conclusion, we have analyzed different quantities proposed in the literature as signatures of critical behavior. We found that the information entropy \( S_1 \) is not a signature of a second order phase transition as was proposed in [6]. For this purpose, we have analyzed systems that displays critical behavior: percolating lattice of two sizes. In the larger one, where the finite size effects can be neglected, it is clear that the signature of the information entropy is shifted respect the others quantities analyzed. But when the size is much smaller and the finite size effects can not be neglected, the signatures provided by the different quantities analyzed can be shifted between them and the maximum in the information entropy can be confused with a signature of critical behavior. This behavior is confirmed in the analysis of a Lennard-Jones hot drop that performs multifragmentation.

Finally, according to these results, we would choose the best fit mass spectra with a one-parameter power law (minimum \( \chi^2 \)), the normalize variance of the size of the maximum fragment \((NVM)\) or the second moment of the cluster distribution \((M_2)\) as signatures of critical behavior, and we would discard the information entropy \((S_1)\) to play this role.

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Figure Captions

Fig.1: The normalize variance of the size of the maximum fragment, $NVM$ (a), the ensemble average of the second moment of the cluster distribution $M_2$ (b), the best fitting of a one parameter power law coefficient ($\chi^2$) (c), the $\tau$ exponent from the fitting of two parameters power law (d) and the information entropy ($S_1$) (e) against bond probability for percolation lattices of sizes $30 \times 30 \times 30$ (full triangles) and $6 \times 6 \times 6$ (empty circles). In (f) we plot the cluster distribution for the critical probabilities at both sizes.

Fig.2: The $\chi^2$ coefficient (a) and $M_2$ (b) against probability for the cluster distribution generated from eq.(1) using the scaling function $f(z)$ obtained in [5]. In (c) we plot the information entropy calculated by replacing the cluster distribution given by eq.(1) in the definition of $S_1$ (eq.(6)). In (d) we can see both: the cluster distribution corresponding to the critical probability (chosen at $p_i = 0.25$, a pure power law) and the one that maximize the information entropy ($p_i = 0.22$). Here is clear that the distribution that maximize the entropy is not a power law.

Fig.3: The $\chi^2$ coefficient (a), the $M_2$ (b), the $NVM$ (c) and $S_1$ against the energy per particle for a molecular dynamics multifragmentation process for 147 particles interacting via a Lennard-Jones potential.