Orientifolds of $SU(2)/U(1)$ WZW Models

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Abstract

The orientifolds of $SU(2)/U(1)$ gauged WZW models are investigated. The D-branes in $SU(2)/U(1)$ WZW models are investigated by Maldacena, Moore and Seiberg and they determine the geometry of the branes. They also construct new types of D-branes other than so called Cardy type branes. In this paper we identify the geometry of the orientifolds and construct new types of orientifolds by the same method of the analysis of D-branes.
1 Introduction

In recent years the D-branes attract much attention to investigate several aspects of superstring theories. Much of the studies of D-branes are done in the flat background and it is natural to extend the analysis to the curved background. In the case of WZW models, the study of D-branes was started about ten years ago [1, 2]. Recently the geometrical aspects of D-branes in WZW models are investigated in [3, 4, 5, 6, 7, 8] and the stability of D-branes are investigated [9, 10, 11, 12].

The D-branes are described by so called boundary states in the conformal field theory and we can analyze perturbatively. It is known that the orientifolds also can be expressed by the crosscap states [13, 14] in the conformal field theory, however the orientifolds are not much investigated like D-branes. Nevertheless it is important to study the orientifolds because the orientifolds can be used to study the string duality and to construct the configuration with less supersymmetry. The orientifolds of WZW models were studied in [15, 16, 17, 18, 19, 20] by the algebraic method. Very recently, the geometrical aspects of the orientifolds of WZW models are studied in [21, 22, 23, 24] and it is interesting to apply to the other configurations.

In this paper we will consider the orientifolds in the parafermionic theories, which are obtained by the $SU(2)$ WZW models gauged by the $U(1)$ sector, e.g. $SU(2)/U(1)$ coset WZW models [25]. These theories appear in superstring theories on some background. For example, the near horizon geometry of NS5-branes can be described by the geometry [26, 27, 28]

$$R^{1,5} \times \mathbb{R}_\phi \times S^3,$$  \hspace{1cm} (1.1)

where $\mathbb{R}^{1,5}$ is the space parallel to the NS5-branes. The radial direction $\mathbb{R}_\phi$ is given by the Liouville theory and $S^3$ can be described by the $SU(2)$ WZW models. Moreover the parafermionic theories are used to describe the string theory on two dimensional black hole [29] or singular Calabi-Yau manifolds [30, 31], which can be described by the $SL(2,\mathbb{R})/U(1)$ WZW models.

The study of D-branes in $SU(2)_k/U(1)_k$ WZW models was started by [32] in the context of Gepner models and the geometrical picture of D-branes in the parafermionic theories was given in [33]. Some branes which preserve maximally symmetry can be obtained in the rational conformal field theory by Cardy construction [2], which will be called as A-type branes followed by [33]. The authors of [33] also construct new types of (B-type) boundary states which can not be obtained by Cardy construction. It can be shown that the $\mathbb{Z}_k$ orbifold of parafermionic theory is T-dual to the original theory. Therefore we can construct B-type boundary states by applying $\mathbb{Z}_k$ orbifold and T-duality. For example, in $U(1)$ WZW model, A-type boundary states represent the branes satisfying Dirichlet boundary condition and B-
type boundary states represent the branes satisfying Neumann boundary condition. In the parafermionic theory, the target space is given by the disk with radius one. In the paper [33], A-type branes are determined as D0-branes at the boundary of the disk and D1-branes connecting the points of the boundary. Moreover B-type branes are determined as D0-brane and D2-branes at the center of the disk. The recent developments about D-branes in coset WZW models are given in [34, 35, 36, 37, 38, 39, 40, 41].

In this paper we investigate the orientifolds in the parafermionic theories. In the general rational conformal field theory, several types of Cardy crosscap states are known [15, 16, 17, 18] and they are related by the simple currents [19, 20]. In the parafermionic theory we have level two simple currents and hence two types of Cardy crosscap states are constructed. We shall show that just as the D-brane case we can construct “B-type” orientifolds by using $\mathbb{Z}_k$ orbifold and T-duality, thus we get totally four types of orientifolds. Two A-type orientifolds are $O_0$-plane at the boundary of the disk and $O_1$-plane connecting the points of the boundary. Two B-type orientifolds are $O_0$-plane at the center of the disk and $O_2$-plane wrapping the whole disk.

The organization of this paper is as follows; in section 2, we review the known results of orientifolds which are constructed by the crosscap state technique. The amplitudes between the boundary states or the crosscap states can be seen in the direct channel by modular transformation. It is very important consistency condition that there appears only integer degeneracies in the direct channel. In section 3, we investigate the orientifolds of the simplest case, namely, $U(1)$ WZW models. We can construct B-type orientifolds from A-type ones even in this case. The orientifolds of $U(1)$ WZW models are the well-known ones in the one dimensional free bosonic theory; A-type ones can be identified as $O_0$-planes and B-type ones as $O_1$-plane. In section 4, we construct the orientifolds in the parafermionic theory. As we said above, we have four types of orientifolds and we examine the spectrum in the direct channel of the amplitudes. The geometry of orientifolds can be seen by the spectrum or the scattering with the closed string states as we explain in appendix B. In section 5, we extend the analysis to the supersymmetric case and obtain the similar results to the bosonic theory. We also discuss the stability of the orientifolds by reading the spectrum in the direct channel. Section 6 is devoted to the conclusions and discussions. In appendix A, the several functions and their modular transformations are denoted. In appendix B, we review the method to determine the geometry of branes by scattering the closed string states [33] and identify the shapes of orientifolds by applying this method.
2 General Analysis of Crosscap States

In the presence of orientifolds, the partition functions can be obtained by the Klein bottle worldsheet in addition to the torus one. If we add the open string sector, we have to consider the Möbius amplitudes and of course the annulus amplitudes. These amplitudes have to satisfy the constraints corresponding to the modular invariant. In this section, we use the rational conformal field theory and we later analyze the specific models.

The modular invariant condition of the torus amplitudes is well-known. The torus amplitudes can be written as

$$T(q) = \sum_{ij} \chi_i(q) Z_{ij} \chi_j(q) ,$$

(2.1)

where we use $\chi_i(q)$ as the character of representation $i$ and the moduli $q = \exp(2\pi i \tau)$ and $\bar{q} = \exp(2\pi i (-1/\tau))$. The torus amplitudes must be invariant under modular transformation, in other words, $S$ and $T$ transformation

$$\sum_{ij} S_{ij} Z_{ij} S_{jj'} = Z_{ij}, \quad \sum_{ij} T_{ij} Z_{ij} T_{jj'} = Z_{ij}. \quad (2.2)$$

The famous ones are given by the charge conjugation modular invariant $Z_{ij} = \delta_{i,j^*}$ and the diagonal modular invariant $Z_{ij} = \delta_{ij}$. The analysis of orientifolds is usually given in the charge conjugation modular invariant, however we will use the diagonal modular invariant in the subsequent sections.

When applying the orientifold projection, we have to add the Klein bottle amplitudes. The orientifold operation is given by the combination of the worldsheet orientation reversal ($\Omega : \sigma \rightarrow 2\pi - \sigma$) and the space $\mathbb{Z}_2$ isometry ($h$). Then the Klein bottle amplitudes are given by

$$\mathcal{K}(q) = \text{Tr}(\Omega h q^H_c) = \sum_i \mathcal{K}_i \chi_i(q),$$

(2.3)

where we define $H_c = \frac{1}{2} (L_0 + \tilde{L}_0 - \frac{c}{12})$ and $c$ is the central charge of the model. The coefficients $\mathcal{K}_i$ are the integers and they are related to how the fields behave under the orientifold operation, which must be consistent with OPEs of the fields[18]. The modulus of the Klein bottle is $\tilde{\tau} = 2it$ and $S$ transformation exchange the direct channel and the transverse channel. In the transverse channel, the Klein bottle amplitudes are given by the overlaps between the states called as crosscap states

$$\mathcal{K}(\bar{q}) = c \langle C | \bar{q}^H_c | C \rangle_C = \sum_i (\Gamma_i)^2 \chi_i(\bar{q}). \quad (2.4)$$
The coefficients $\Gamma^i$ are obtained by the one point amplitudes with closed string $i$ and the coefficients $\Gamma^i$ determine the crosscap states. The equations (2.3) and (2.4) are transformed by $S$ transformation each other. Therefore we obtain the constraint of the coefficients as

$$K^i = \sum_j S^i_j (\Gamma^j)^2,$$

which is the important condition to construct the crosscap states.

The open string sector is included in the presence of D-branes. The annulus amplitudes are given by

$$\mathcal{A}_{ab}(q) = \text{Tr}_{H_{ab}} q^{L_0 - c/24} = \sum_i A^i_{ab} \chi_i(q),$$

where $a,b$ are the labels of the boundary conditions, in other words, the labels of D-branes. The coefficients $A^i_{ab}$ are the non-negative integers. The transverse channel can be obtained by the amplitude between the boundary states $[1, 2]$ which describe D-branes

$$\mathcal{A}_{ab}(\tilde{q}) = C \langle B, a| \tilde{q} H_c |B, b \rangle = \sum_i B^i_a B^i_b \chi_i(\tilde{q}),$$

where $B^i_a$ are the one point disc amplitudes with the boundary condition $a$. The $S$ transformation connects the equations (2.6) and (2.7) and this constraint is known as the Cardy condition [2]

$$A^i_{ab} = \sum_j B^j_a B^j_b S^i_j.$$

The analysis of Möbius strip amplitudes is a little more complicated than the other cases. It can be seen that the following characters are convenient to study the Möbius strip amplitudes

$$\hat{\chi}_j(q) \equiv e^{-\pi i (h_j - \frac{c}{24})} \chi_j(-\sqrt{q}).$$

The transformation from the direct channel to the transverse channel can be obtained by the $P(\equiv \sqrt{TST^2 S\sqrt{T}})^1$ matrix [15]. In $SU(2)$ WZW model, this matrix can be written as

$$P_{jj'}^{SU(2)} = \frac{2}{\sqrt{k+2}} \sin \left( \frac{\pi (2j + 1)(2j' + 1)}{2(k+2)} \right) E_{k+2j+2j'},$$

where we use

$$E_n = \frac{1 + (-1)^n}{2}.$$
The modulus of Möbius strip can be assigned by \( \tilde{\tau} = 2it \) and \( P \) transforms \( t \to 1/(4t) \). The Möbius strip amplitudes in the direct channel are given by

\[
M_a(q) = \text{Tr}_{H_a}(\Omega q L^0 - \tilde{\tau}^i) = \sum_i M_a^i \hat{\chi}_i(q),
\]

(2.12)

where there is only one boundary condition \( a \) and the coefficients \( M_a^i \) are the integers. The amplitudes in the transverse channel are obtained by the amplitudes between the crosscap states and the boundary states

\[
\cal{M}(\tilde{q}) = C \langle C | \tilde{q}^H B, a \rangle_C = \sum_i \Gamma^i B^i_a \hat{\chi}_i(\tilde{q}).
\]

(2.13)

As we said above, the modular transformations are given by \( P \) transformation, thus the constraint becomes

\[
M_a^i = \sum_j \Gamma^j B^j_a P^i_j.
\]

(2.14)

The presence of \( P \) matrix in this equation makes things different from the case of the boundary states.

Moreover we have to care about so called completeness conditions [16]. The following set of crosscap states and boundary states which obey all consistency conditions was obtained [15, 16, 17, 18] by the coefficients

\[
B^i_a = \frac{S_{ai}}{\sqrt{S_{0i}}}, \quad \Gamma^i = \frac{P_{0i}}{\sqrt{S_{0i}}},
\]

(2.15)

If we use the Verlinde formula of the fusion coefficients

\[
N_{i,j}^l = \sum_m S_{mi} S_{mj} S_{ml},
\]

(2.16)

and define the integer valued tensor as [15]

\[
Y_{i,j}^l = \sum_m S_{mi} P_{mj} P_{ml},
\]

(2.17)

the amplitudes can be obtained as

\[
\cal{K}(q) = \sum_i Y_{0,0}^i \chi_i(q), \quad \cal{A}_{ab}(q) = \sum_i N_{a,b}^i \chi_i(q), \quad \cal{M}_a(q) = \sum_i Y_{a,0}^i \hat{\chi}_i(q).
\]

(2.18)

Let us see the example of \( SU(2)_k \) WZW models. The \( Y \) matrices are given as

\[
Y_{0,0}^i = (-1)^i, \quad Y_{a,0}^i = (-1)^{2a}(-1)^i N_{a,a}^i,
\]

(2.19)
Figure 1: The geometry of orientifolds of $SU(2)$ WZW models. (1) The boundary state of the type (2.15) describes two $O_0$-planes located at the opposite points of $S^3$. (2) The boundary state of the type (2.22) describes $O_2$-plane whose geometry is $S^2$ at the equator of $S^3$.

and fusion matrices are

$$N_{i,j} = \begin{cases} 1 & |i - j| \leq l \leq \min\{i + j, k - i - j\}, \ i + j + l \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$ (2.20)

The orientifolds in this model were investigated geometrically in [21, 23, 23] and this type of orientifold is identified as $O_0$-planes, see figure 1.

The other kind of orientifolds are elegantly expressed by using the simple current technique [19, 20]. The simple currents are the fields $J$ which satisfy

$$J \times i = i' \ (\forall i).$$ (2.21)

The trivial simple current is identity 0 and then $i' = i$. For $SU(2)_k$ WZW model, there is a non-trivial simple current $J = k/2$ and then $i' = k/2 - i$. By applying simple currents to the previous results, the set of crosscap states and boundary states which satisfy all consistency conditions are obtained by the coefficients

$$B_a^i = \frac{S_{ai}}{\sqrt{S_{j_i}}} , \ \Gamma^i = \frac{P_{ji}}{\sqrt{S_{j_i}}}.$$ (2.22)

We have changed also boundary states in the above equation for convenience, in fact these states are not new but the renamed ones. The amplitudes can be written as

$$\mathcal{K}(q) = \sum_i Y_{J,i}^i \chi_i(q) , \ \mathcal{A}_{ab}(q) = \sum_i N_{a,b}^J \chi_i(q) , \ \mathcal{M}_a(q) = \sum_i Y_{a,J}^i \tilde{\chi}_i(q).$$ (2.23)

$^2$We only analyze level 2 simple currents, namely, $J \times J = 1$. The level $N$ simple currents can be used to construct the crosscap states in $SU(N)$ WZW models [19].
In \(SU(2)_k\) WZW models, these \(Y\) matrices are given as
\[
Y^i_{\frac{k}{2}, \frac{k}{2}} = 1, \quad Y_a^i_{\frac{k}{2}} = N_{a, a}^{\frac{k}{2} - i},
\]
and this type of orientifold corresponds to \(O2\)-plane at the equator of the \(S^3\) \([21, 23, 23]\), see figure 1. We have not known yet that the above boundary states are all possible types of orientifolds. In fact, we shall construct other types of orientifolds in the following sections.

3 \(U(1)\) WZW Theories and Orientifolds

In the previous section, we have reviewed the crosscap states in the rational conformal field theory. In this section, we study the orientifolds of \(U(1)_k\) WZW model. It can be seen that the \(\mathbb{Z}_k\) orbifold of this theory is T-dual to the original theory, therefore we can apply the construction proposed by \([33]\) to the orientifolds. We shall show that the orientifolds which correspond to the solution (2.15) constructed in the previous section are \(O0\)-planes. Then we construct the B-type orientifold by applying \(\mathbb{Z}_k\) orbifold and T-duality and it can be identified as \(O1\)-plane wrapping the whole line. In the next subsection, we investigate the closed strings and D-branes in \(U(1)\) WZW models, and in subsection 3.2, we construct the orientifolds.

3.1 \(U(1)\) WZW Models and D-branes

The \(U(1)_k\) WZW model has the current \(J = i\sqrt{2k}\partial X\) and the primaries \(\Phi_n(X) = \exp(i\frac{n}{\sqrt{2k}} X)\) whose labels represent their \(U(1)\) charges and their conformal weights are \(h_n = n^2/(4k)\). The label \(n\) is defined modulo \(2k\) and we take \(-k + 1, -k + 2, \cdots, k\). The characters are defined as
\[
\psi_n(q, z) = \text{Tr}_{\mathcal{H}_n} q^{L_0 - 1/24} e^{2\pi i z J_0} \frac{\Theta_{n,k}(\tau, 2z)}{\eta(\tau)},
\]
and the characters with \(z = 0\) will be used
\[
\psi_n(q) = \frac{1}{\eta(q)} \sum_{r \in \mathbb{Z}} q^{k(r + \frac{n}{2k})^2}.
\]
Their modular transformations are given by
\[
\psi_n(\tau + 1) = e^{2\pi i (h_n - 1/24)} \psi_n(\tau), \quad \psi_n(-\frac{1}{\tau}) = \frac{1}{\sqrt{2k}} \sum_{n'} e^{-\pi i n'n/k} \psi_{n'}(\tau),
\]
and the diagonal modular invariant are given by
\[
T = \sum_n |\psi_n(q)|^2.
\]
This theory is same as the one free boson theory with radius $\sqrt{2k}$ where $\alpha' = 2$.

This theory has $\mathbb{Z}_k$ global symmetry under the transformation

$$g : \Phi_n \to e^{2\pi in/k}\Phi_n$$

and we can take the orbifold by this transformation. The most general modular invariants are given by $\mathbb{Z}_l$ orbifold procedure ($k = ll', l, l' \in \mathbb{Z}$) which are generated by $g'$

$$T = \sum_{n,\bar{n}} \psi_n(q)\psi_{\bar{n}}(\bar{q}) , \quad n + \bar{n} = 0 \mod 2l , \quad n - \bar{n} = 0 \mod 2l'. \quad (3.6)$$

From these modular invariant combinations, we can see that the original $U(1)_k$ WZW model is T-dual to its $\mathbb{Z}_k$ orbifold model since $\psi_n = \psi_{-n}$.

It is well-known that there are two types of D-branes in the $U(1)$ WZW model. The open strings on the one type of D-branes obey Neumann boundary condition and the ones on the other obey Dirichlet boundary condition. We will call the Dirichlet condition as A-type and the Neumann one as B-type. The D-branes are represented by the terms of the boundary states. The general boundary states which satisfy A-type boundary condition $J(z) + \bar{J}(\bar{z}) = 0$ are given by Ishibashi states [1]

$$|A, r, r\rangle_I = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n} \right) \sum_{l \in \mathbb{Z}} \left| r + \frac{2kl}{\sqrt{2k}}, r + \frac{2kl}{\sqrt{2k}} \right> . \quad (3.7)$$

The label $r$ runs $-k + 1, -k + 2, \cdots, k$. The boundary states corresponding to D-branes have to satisfy Cardy conditions (2.8) and they are obtained by (2.15)

$$|A, \hat{n}\rangle_C = \sum_n \frac{S_{\hat{n}n}}{\sqrt{S_{0n}}} |A, n, n\rangle_I = \frac{1}{(2k)^{1/4}} \sum_n e^{-\pi i n/k} |A, n, n\rangle_I . \quad (3.8)$$

These states describe D0-branes and their locations are represented by the label $\hat{n} = -k + 1, -k + 2, \cdots, k$. The $\mathbb{Z}_k$ symmetry corresponds to the rotation of the circle and $g$ transforms $|A, \hat{n}\rangle$ to $|A, \hat{n} - 2\rangle$. The annulus amplitudes between A-type branes are given by

$$\mathcal{A} = c\langle A, \hat{n}|\bar{q}^{H_{\alpha'}}|A, \hat{n}'\rangle_C = \psi_{\hat{n}' \hat{n}}(q) . \quad (3.9)$$

In non-Abelian WZW models, A-branes correspond to the maximally symmetric branes which are widely investigated.

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The boundary condition is assigned in the closed string channel and we use the notation $J = i\sqrt{2k}\partial X$ and $\bar{J} = -i\sqrt{2k}\bar{\partial} X$. 

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Dirichlet condition is described by the currents as $J(z) - \tilde{J}(\tilde{z}) = 0$ and B-type Ishibashi states are given by

$$|B, r, -r\rangle_I = \exp \left( -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \bar{\alpha}_{-n} \right) \sum_{l \in \mathbb{Z}} \left| r + 2kl \sqrt{2k}, -r + 2kl \sqrt{2k} \right> . \quad (3.10)$$

We should note here that there are only $r = 0, k$ states in the bulk spectrum. The Cardy states in this case are given by

$$|B, \eta = \pm 1\rangle_C = \left( \frac{k}{2} \right)^{1/4} \left( |B, 0, 0\rangle_I + \eta |B, k, -k\rangle_I \right) , \quad (3.11)$$

where the coefficients are chosen to have the consistent open spectrum. These boundary states describe the D1-branes and $\eta$ represents the Wilson line. These states can be also obtained by using $\mathbb{Z}_k$ orbifold and T-duality [33]. To get the boundary states in this orbifold theory, we sum over all image branes generated by $g^m (m = 0, 1, \cdots, k - 1)$. The amplitudes between B-type branes are obtained as

$$\mathcal{A} = C \langle B, \eta | \tilde{q}^{H_c} | B, \eta \rangle_C = \sum_{m} \frac{1 + \eta \eta' (-1)^m}{2} \psi_m (\tilde{q}) . \quad (3.12)$$

The amplitudes between A-type and B-type branes have the different open string spectrum. The amplitudes of Ishibashi states are given by

$$I \langle A, n, n | \tilde{q}^{H_c} | B, r, -r\rangle_I = \delta_{n,0} \delta_{r,0} \chi_{ND} (\tilde{q}) , \quad (3.13)$$

where we defined $\chi_{ND}$ as

$$\chi_{ND} (\tilde{q}) = \frac{1}{\tilde{q}^{1/24} \prod_{n=1}^{\infty} (1 + \tilde{q}^n)} , \quad (3.14)$$

and its modular transformation is given by

$$\chi_{ND} (\tilde{q}) = \sqrt{2} \chi'_{ND} (q) , \quad \chi'_{ND} (q) = \frac{q^{1/8}}{\prod_{n=1}^{\infty} (1 - q^{n + \frac{1}{2}})} . \quad (3.15)$$

Using these characters, the annulus amplitudes are obtained as

$$\mathcal{A} = C \langle A, \tilde{n} | \tilde{q}^{H_c} | B, \eta \rangle_C = \chi'_{ND} (q) . \quad (3.16)$$

It can be seen that the character $\chi'_{ND}$ correctly reproduces the spectrum of open strings which satisfy Neumann-Dirichlet boundary condition.
3.2 Crosscap States in $U(1)$ WZW Models

Next we shall construct the crosscap states which represent the orientifolds. The most famous example is the Type I string theory which can be regarded as the system with $O_{9}$-orientifolds of Type IIB string theory. Applying $(9-n)$ T-dualities to $O_{9}$-planes, we have $O_{n}$-planes. Therefore we have $O_{0}$-planes and $O_{1}$-planes in our $U(1)$ WZW models.

To construct the crosscap states, we have to obtain $P$ matrix in $U(1)$ WZW model as explained in the previous section. It is useful to calculate the next quantity by using the modular transformation of $U(1)$ characters as

$$ (ST^{2}S)_{mn} = \frac{1}{2k} \sum_{l=-k+1}^{k} e^{-\pi m l/k} e^{4\pi i (\frac{l^2}{4k} - \frac{1}{12})} e^{-\pi i n/k} = \frac{1}{2k} \sum_{l=-k+1}^{k} e^{\frac{\pi i}{k} (l - \frac{m+n}{2})^2} e^{-\pi i (m+n)^2 - \frac{\pi i}{k}}. \quad (3.17) $$

Using the formula of Gaussian sum

$$ \sum_{n=0}^{\kappa-1} e^{2\pi i n^2/\kappa} = \frac{1}{2} (1 + i)(1 + i^{-\kappa}) \sqrt{\kappa}, \quad \sum_{n=0}^{\kappa-1} e^{2\pi i (n+1/2)^2/\kappa} = \frac{1}{2} (1 + i)(1 - i^{-\kappa}) \sqrt{\kappa}, \quad (3.18) $$

we can obtain the $P$ matrix for $U(1)$ WZW models as

$$ P_{mn} = \frac{1}{\sqrt{k}} e^{-\pi i mn/2k} E_{k+m+n}, \quad (3.19) $$

where $E_{k+m+n}$ is the projection to the even elements (2.11).

In the presence of orientifolds, we have to add some conditions to the right and left moving currents just like the boundary conditions. In our case, these conditions are written as

$$ (J_{n} \pm (-1)^{n} \tilde{J}_{-n})|O\rangle = 0. \quad (3.20) $$

We call the crosscap states obeying the (+) condition as A-type and the ones obeying the (−) condition as B-type. The Ishibashi crosscap states can be easily constructed and A-type ones are

$$ |OA, r, r\rangle = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} \alpha_{-n} \tilde{\alpha}_{-n} \right) \sum_{l \in \mathbb{Z}} \frac{r + 2kl}{\sqrt{2k}}, \frac{r + 2kl}{\sqrt{2k}} \right), \quad (3.21) $$
where the label $r$ runs $-k + 1, -k + 2, \cdots, k$. This type of crosscap states are the ones discussed in the previous section and the Cardy crosscap states are given by (2.15)

$$|OA, \hat{n}\rangle_C = \sum_n \frac{P_{\hat{n}n}}{\sqrt{S_{0n}}} |OA, n, n\rangle_I .$$  \hspace{1cm} (3.22)

The Klein bottle amplitude can be obtained by using the crosscap states as

$$A = C \langle OA, \hat{n}|\tilde{q}^{H_c}|OA, \hat{n}\rangle_C = \psi_0(q) + (-1)^{\hat{n}+k} \psi_k(q) .$$  \hspace{1cm} (3.23)

We can see that this is the appropriate orientifold projection because there is the summation of the “winding number” $l$ in the character (3.2). From this reason, this crosscap state can be identified with $O_0$-orientifolds.

The Möbius strip amplitudes can be given by the amplitudes between the boundary states and crosscap states. If we use A-type boundary states we obtain

$$M = C \langle A, \hat{n}|\tilde{q}^{H_c}|OA, \hat{n}\rangle_C = \hat{\psi}_{2\hat{n} - \hat{n}'}(q) .$$  \hspace{1cm} (3.24)

This amplitude shows that the positions of orientifolds are the same points of D-branes with the label $\hat{n} = \hat{n}'/2, \hat{n}'/2 + k$ in the even $\hat{n}'$ case. The odd $\hat{n}'$ orientifolds are located at the middle point between $\hat{n} = (\hat{n}' - 1)/2$ and $\hat{n} = (\hat{n}' + 1)/2$ and the opposite points of the circle. These points correspond to the fixed points under the reflection. In fact, the amplitudes with mirror branes are

$$A = C \langle A, \hat{n}|\tilde{q}^{H_c}|A, \hat{n}' - \hat{n}\rangle_C = \hat{\psi}_{2\hat{n} - \hat{n}'}(q) ,$$  \hspace{1cm} (3.25)

whose spectrum is same as in the Möbius strip amplitudes.

The calculation of the amplitudes with B-type boundary states needs $P$ transformation of the character $\chi_{ND}$ (3.14). Using the fact that this character can be rewritten as

$$\chi_{ND}(q) = \frac{1}{q^{1/24} \prod_{n=1}^{\infty} (1 + q^n)} = \frac{\eta(\tau)}{\eta(2\tau)}$$  \hspace{1cm} (3.26)

and the modular transformation of $\eta$ functions (A.2), we get

$$\hat{\chi}_{ND}(\tilde{q}) = \hat{\chi}_{ND}(q) ,$$  \hspace{1cm} (3.27)

namely, $P = 1$. Hence we can get the amplitudes as

$$M = C \langle B, \eta|\tilde{q}^{H_c}|OA, \hat{n}\rangle_C = E_{\hat{n}+k} \hat{\chi}_{ND}(q) .$$  \hspace{1cm} (3.28)

This is the spectrum of the open strings between D1-brane in the system with $O_0$-planes.
Figure 2: Boundary states and crosscap states in $g$ twisted sector. (1) Boundary conditions are denoted by $\alpha$, which must be invariant under the operation $g$. (2) Opposite points of crosscap states must be related by $\Omega h$, where $(\Omega h)^2 = g$.

The B-type orientifolds can be constructed in the similar way as B-type branes. The Ishibashi crosscap states obeying the condition (3.20) with $(-)$ are given by

$$|OB, r, -r\rangle_I = \exp \left( -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \alpha_{-n} \bar{\alpha}_{-n} \right) \sum_{l \in \mathbb{Z}} \left| r + 2kl \sqrt{2}k, -r + 2kl \sqrt{2}k \right\rangle,$$

with $r = 0, k$ which is same as the B-brane case. The Cardy crosscap states in this case can be obtained by using $Z_k$ orbifold and T-duality. In $g$ twisted sector, the orientifold operation $\Omega h$ must be $(\Omega h)^2 = g$ [42], see figure 2. $g^{1/2}$ generates the half sift of $g$ action to the branes and the states $|OA, \hat{n}\rangle$ are transformed to the states $|OA, \hat{n} - 2\rangle$. Summing over all image A-type crosscap states in even $k$ case, we get

$$\frac{1}{\sqrt{k}} \sum_{n=0}^{k-1} |OA, 2\hat{n}\rangle_C = (2k)^{1/4} |OA, 0, 0\rangle_I.$$

In odd $k$ case, we can use the odd $\hat{m}$ crosscap states instead. Applying T-duality, we can get the B-type crosscap state as

$$|OB\rangle_C = (2k)^{1/4} |OB, 0, 0\rangle_I,$$

which is the same as the usual crosscap states for $O1$-plane. In fact, the Klein bottle amplitude is given by

$$A = c\langle OB|q^Hc|OB\rangle_C = \sum_n \psi_n(q),$$

which is the summation of the states with all momentum. The Möbius strip amplitudes with B-type boundary states can be obtained as

$$M = c\langle B, \eta|q^Hc|OB\rangle_C = \sum_n E_{n+k} \psi_n(q),$$

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and the amplitudes with A-type boundary states are given by

\[ \mathcal{M} = C\langle A, \hat{n} | \tilde{q}^H | OB \rangle_C = \hat{\chi}_{ND}(q) , \]  

(3.34)

where we use the \( P \) transformation of \( \hat{\chi}_{ND} \) (3.27).

In summary, we can construct B-type orientifolds as well as A-type orientifolds by using \( \mathbb{Z}_k \) orientifold and T-duality. A-type orientifolds correspond to \( \mathcal{O}_0 \)-planes and B-type orientifold corresponds to \( \mathcal{O}_1 \)-plane. It is the important consistency condition that the spectrum of the amplitudes in our system has integer degeneracies.

4 \( SU(2)/U(1) \) WZW Models and Orientifolds

In this section, we will construct the orientifolds of \( SU(2)_k/U(1)_k \) WZW models. In contrast to the orientifolds of \( U(1) \) WZW models, there are two types of A-orientifolds given by the solution (2.15) and (2.22). They can be determined as \( \mathcal{O}_0 \)-planes at the boundary of the disk and \( \mathcal{O}_1 \)-plane connecting the boundary points. Furthermore, applying the \( \mathbb{Z}_k \) orbifold and T-duality, we can get two types of B-orientifolds, which are \( \mathcal{O}_0 \)-plane and \( \mathcal{O}_2 \)-plane at the center of the disk. In the next subsection, we study the closed strings in the parafermionic theory and construct D-branes followed by [33]. In subsection 4.2, the crosscap states in parafermionic theory are constructed.

4.1 \( SU(2)/U(1) \) WZW Models and D-branes

The parafermionic theory can be described by the \( SU(2)_k/U(1)_k \) coset WZW model [25] which is defined by gauging \( U(1) \) sector in the \( SU(2) \) WZW models. We denote the Hilbert space of \( SU(2)_k \) WZW model as \( \mathcal{H}^{SU(2)}_{j,m} \), where \( j \) is the Casimir number and \( m \) is the \( J_{3}^0 \) eigenvalue of the highest weight representation. These values take \( j = 0, 1/2, \ldots, k/2 \) and \( m = -2j, -2j + 1, \ldots, 2j \). We also denote the parafermionic Hilbert space as \( \mathcal{H}^{PF}_{j,n} \), where \( j = 0, 1/2, \ldots, k/2 \) and \( n \in \mathbb{Z}_{2k} \) and \( U(1) \) Hilbert space of momentum \( n \) is denoted as \( \mathcal{H}^{U(1)}_{n} \). These Hilbert spaces are related as

\[ \mathcal{H}^{SU(2)}_{j,m} = \mathcal{H}^{PF}_{j,m} \otimes \mathcal{H}^{U(1)}_{m} . \]  

(4.1)

The right hand side does not include all Hilbert space, in fact, the parafermionic Hilbert space has the following identification

\[ \mathcal{H}^{PF}_{j,n} = \mathcal{H}^{PF}_{k/2 - j, n + k} , \]  

(4.2)
which will be often called as the “spectral flow” identification. There is also the relation between \( j \) and \( n \) as \( 2j + n \in 2\mathbb{Z} \). From now on, we will call the distinct irreducible representations as \( PF(k) \).

The characters of parafermionic theory can be given by the characters of the \( SU(2) \) and \( U(1) \) currents as

\[
\chi_j^{SU(2)}(\tau, z) = \sum_{n=-k+1}^k \chi_{j,n}(\tau) \psi_n(\tau, z),
\]

which have the following identities as

\[
\chi_{j,n}(q) = \chi_{j,-n}(q) = \chi_{k/2-j,k-n}(q).
\]

The lowest conformal weights of \( PF(k) \) are given by

\[
h_{j,n} = \begin{cases} 
  \frac{j(j+1)}{k+2} - \frac{n^2}{4k} & -2j \leq n \leq 2j \\
  \frac{j(j+1)}{k+2} - \frac{n^2}{4k} + \frac{n-2j}{2} & 2j \leq n \leq 2k - 2j,
\end{cases}
\]

where we use \( n = -2j, -2j + 2, \ldots, 2k - 2j - 2 \). The modular transformations of the parafermionic characters are given by

\[
\chi_{j,n}(\tau + 1) = e^{2\pi i \left( h_{j,n} - \frac{c}{24} \right)} \chi_{j,n}(\tau),
\]

\[
\chi_{j,n}(-1/\tau) = \sum_{(j',n') \in PF(k)} S_{j,n}^{PF} j',n' \chi_{j',n'}(\tau),
\]

where we use

\[
S_{j,n}^{PF} j',n' = \sqrt{2k} S_{SU(2)} j' \psi_{in'/k}.
\]

The fusion coefficients can be written as

\[
N_{PF}^{j,n''} (j',n') = N_{j,j'}^{j'',n''} \delta^{(2k)}_{n+n'-n''} + N_{j,j'}^{k/2-j'',n''} \delta^{(2k)}_{n+n'-n'-k},
\]

where we define \( \delta^{(2k)} \) modulo \( 2k \).

The diagonal modular invariant is given by

\[
T = \sum_{(j,n) \in PF(k)} |\chi_{j,n}(q)|^2.
\]

Moreover the orbifold procedure under the global \( \mathbb{Z}_k \) symmetry

\[
g : \Phi_{j,n} \rightarrow e^{2\pi i n/k} \Phi_{j,n}
\]
gives other modular invariants. The torus amplitudes of $\mathbb{Z}_l$ orbifolds ($k = ll'$) are given by
\[
T = \frac{1}{2} \sum_{j,n,\bar{n}} \chi_{j,n}(q) \bar{\chi}_{j,\bar{n}}(\bar{q}) , \quad n - \bar{n} = 0 \mod 2l , \quad n + \bar{n} = 0 \mod 2l'.
\] (4.11)
Just as the $U(1)$ case, the $\mathbb{Z}_k$ orbifold and T-duality gives the same spectrum because of the character identities (4.4).

Next we study D-branes in the parafermionic theory. It was found in [33] that there are two types of branes, which are called as A-type and B-type branes. A-type branes correspond to the solution (2.15). Using A-type Ishibashi states $|A,j,n\rangle_I$, they can be constructed as
\[
|A,\hat{j},\hat{n}\rangle_C = \sum_{(j,n) \in \text{PF}(k)} S^{PF}_{j,n} \sqrt{S^{PF}_{j,n}} |A,j,n\rangle_I.
\] (4.12)

The annulus amplitudes between these A-type boundary states can be calculated by using the Verlinde formula as
\[
C \langle A,\hat{j},\hat{n}|q^{H_{\text{c}}}|A,\hat{j}',\hat{n}'\rangle_C = \sum_{(j,n) \in \text{PF}(k)} N^{PF}_{j,-\hat{n},j',\hat{n}'} \chi_{j,n}(q).
\] (4.13)
In appendix B, we study the geometry of the target space of parafermionic theory and determine the geometry of the branes. The target space is given by the disk with radius one and the geometry of the branes can be determined by scattering with the closed string states [33]. The boundary states with $\hat{j} = 0$ describe D0-branes at $k$ special points of the boundary of the disk and the general $\hat{j}$ states connect $2\hat{j}$ separated points, see figure 3. The rotation of the disk gives the states with different $\hat{n}$.

The B-type branes can be constructed by using $\mathbb{Z}_k$ orbifold procedure and T-duality from A-type branes. The $\mathbb{Z}_k$ orbifold projects to the states with $n = 0, k$ and we will only use $n = 0$ states by making use of the spectral flow identification (4.2). The T-duality in parafermionic theory can be realized by the operator $\exp(\pi i J_0^i)$. We can see that this operator changes $\Phi_{j,n}$ into $\Phi_{j,-n}$ because of the property
\[
e^{\pi i J_0^i} J_3^3 e^{-\pi i J_0^i} = -J_0^3.
\] (4.14)
In this way, we can define the B-type Ishibashi crosscap states as
\[
(1 \otimes e^{\pi i J_0^1}) |j\rangle^{|SU(2)}_I = \sum_{r=-k+1}^k |B,j,r,-r\rangle^{PF}_I \otimes |B,r,-r\rangle^{U(1)}_I ,
\] (4.15)
where $|B,r,-r\rangle^{U(1)}$ are (3.10). By combining the $\mathbb{Z}_k$ orbifold procedure and T-duality, the Cardy states can be constructed from these Ishibashi states as
\[
|B,\hat{j}\rangle_C = (2k)^{1/4} \sum_{j \in \mathbb{Z}} S_{j}^{\hat{j}} \sqrt{S_{0}^{\hat{j}}} |B,j,0,0\rangle_I ,
\] (4.16)
and the annulus amplitudes between two B-branes are calculated as

\[
A = c \langle B, \hat{j}|q^H_c|B, \hat{j}' \rangle_C = \sum_{j=0}^{k/2} \sum_{n=-2j}^{2k-2j-2} N_{j, \hat{j}} \chi_{j, n}(q). \quad (4.17)
\]

The geometry of B-branes is given in figure 3. From A-branes we can construct B-branes by summing all image branes and taking T-duality. Therefore we can see that the state with \( \hat{j} = 0 \) corresponds to D0-brane at the center of the disk and the states with the general \( \hat{j} \) correspond to D2-branes. In even \( k \) case, we can construct \( \hat{j} = k/4 \) state and there are two identity states among the open string spectrum in the amplitudes. Therefore we need to redefine this state to be the irreducible one, however we do not deal with this state for simplicity and analyze only general \( \hat{j} \neq k/4 \) B-branes.

The annulus amplitudes between A-type and B-type branes are obtained by using the overlaps between Ishibashi states

\[
_1\langle A, j, 0, 0|q^H_c|B, j', 0, 0 \rangle_I = \delta_{j,j'} \chi'_{j}(q), \quad (4.18)
\]

where

\[
\chi'_{j}(q) = \frac{\chi^{SU(2)}_{j}(q, z = 1/2)}{\chi_{ND}(q)}. \quad (4.19)
\]

Their modular transformations are given by

\[
\chi'_{j}(\tilde{q}) = \sum_{j'=0}^{k/2} \frac{1}{\sqrt{2}} S_{j}^{j'} \tilde{\chi}_{j'}(q), \quad (4.20)
\]

Figure 3: The geometry of D-branes in \( SU(2)_k/U(1)_k \) WZW model with \( k = 8 \). The line in figure (1) describes A-type D1-brane with \( \hat{j} = 3/2 \). The summation of image branes is given in figure (2). Applying T-duality, we obtain B-brane with \( \hat{j} = 3/2 \) as figure (3).
where we define
\[ \tilde{\chi}_j(q) = e^{\pi i k/8} \chi_j^{SU(2)}(\tau, \tau/2) q^{-1/48} \prod_{n=1}^{\infty} (1 - q^n)^{-1/2} . \] (4.21)

Now we can get the amplitudes as
\[ A = c \langle B, \hat{j} \mid H^{H_c} \mid A, \hat{j}', \hat{n}' \rangle_c = \sum_{j=0}^{k/2} N_{j,j'} \tilde{\chi}_j(q) . \] (4.22)

4.2 Crosscap States in \( SU(2)/U(1) \) WZW Models

Now we turn to the orientifolds of parafermionic theory. We shall see that there are four types of orientifolds. Two types of them are the solutions (2.15) and (2.22), which correspond to A-type orientifolds. The other two types of orientifolds are obtained by \( \mathbb{Z}_k \) orbifolds and T-duality from the two types of A-orientifolds just as the construction of B-branes.

Before constructing the crosscap states, we have to know the explicit form of \( P \) matrix. Since there is the spectral flow identification (4.2), we have to be careful to make the \( P \) matrix. We find
\[ \hat{\chi}_{j,n}(q) = \frac{1}{2} \sum_{j'=0}^{k/2} \sum_{n'=-2j'}^{2k-2j'-2} \left( P_{j,j'} P_{n,n'} \hat{\chi}_{j',n'}(q) + P_{j,j'} P_{n,n'}^{k/2-j'} \hat{\chi}_{k/2-j',n'+k}(q) \right) \]
\[ = \frac{1}{2} \sum_{j'=0}^{k/2} \sum_{n'=-2j'}^{2k-2j'-2} \left( P_{j,j'} P_{n,n'} + (-1)^{j'-n'/2} P_{j,j'}^{k/2-j'} P_{n,n'}^{k/2-j'} \right) \hat{\chi}_{j',n'}(q) . \] (4.23)

where we use \( P_{j,j'} \) for \( SU(2) \) \( P \) matrix and \( P_{n,n'} \) for the inverse of \( U(1) \) \( P \) matrix
\[ P_{n,n'} = \frac{1}{\sqrt{k}} e^{\pi i n'/2k} E_{n+n'+k} . \] (4.24)

In the last line, we used that
\[ \hat{\chi}_{k/2-j,n+k}(q) = (-1)^{j-n/2} \hat{\chi}_{j,n}(q) , \] (4.25)
since we have defined the character as (2.9) and the conformal weights are given by (4.5). Therefore the \( P \) matrix which is consistent with the spectral flow identification (4.2) is given by
\[ P_{j,n}^{PF} j',n' = P_{j,j'} P_{n,n'} + (-1)^{j'-n'/2} P_{j,j'}^{k/2-j'} P_{n,n'}^{k/2-j} . \] (4.26)

The \( P \) matrix has the following property as
\[ P_{j,j'}^{k/2-j'} = (-1)^j P_{k/2-j} . \] (4.27)
which will be often used in the calculation.

Now we can construct the crosscap states of parafermionic theory. The crosscap states of (2.15) are obtained by using parafermionic $P$ matrix (4.26) as

$$|OA, 0, \hat{n}\rangle_C = \sum_{(j,n) \in PF(k)} \frac{P_{0,\hat{n}}^{PF}}{\sqrt{S_{0,0}^{PF}}} |OA, j, n\rangle_I ,$$

(4.28)

where $|OA, j, n\rangle_I$ are Ishibashi crosscap states. The $P$ matrices restrict to the label $\hat{n} \in 2\mathbb{Z}$.

The Klein bottle amplitudes can be calculated as

$$K = C \langle OA, 0, \hat{n} | \tilde{q}^{Hc} | OA, 0, \hat{n} \rangle_C = \frac{k}{2} \sum_{j=0}^{k/2} \chi_{j,0}(q) + E_k(-)^{\hat{n}/2} \chi_{k/4,k/2}(q) .$$

(4.29)

The first term has only $n = 0$ sector in contrast to the $U(1)$ case (3.23) because of the spectral flow identification (4.2). The last term exists since it is the fixed point of the spectral flow.

The Möbius strip amplitudes between A-branes and A-orientifolds can by obtained after some calculation as

$$M = C \langle A, \hat{j}, \tilde{n} | \tilde{q}^{Hc} | OA, 0, \hat{n}' \rangle_C = \sum_{j=0}^{k/2} \chi_{j,0}^{\hat{j}}(q) , \ Y_{j,0}^{\hat{j}} = (-1)^{2\hat{j}}(-1)^{\hat{n}} N_{j,j}^{\hat{n}} .$$

(4.30)

The following formula for $SU(2)$ $Y$ matrix is useful to show the above equation

$$Y_{i,k/2-j}^{k/2-l} = (-1)^{l-j-2i} Y_{i,j}^{l} .$$

(4.31)

From the above amplitudes (4.30), we can see that the mirror image of A-brane is $|A, \hat{j}, \tilde{n}' - \hat{n}\rangle$, in fact,

$$A = C \langle A, \hat{j}, \tilde{n} | \tilde{q}^{Hc} | A, \hat{j}, \tilde{n}' - \hat{n} \rangle_C = \sum_{j=0}^{k/2} \chi_{j,0}^{\hat{j}}(q) ,$$

(4.32)

where the spectrum is same as the Möbius strip amplitudes. We can identify the orientifolds as $O0$-planes at the boundary of the disk in appendix B by scattering with the closed string states (see also figure 4). This result can be seen by the other way as follows. In $SU(2)$ WZW models, this type of crosscap state describes $O0$-planes [21, 22, 23]. To get $SU(2)/U(1)$ coset WZW model, we have to identify by the shift of the coordinate $\chi + \varphi$ in the term of appendix B. In the D-brane case, the geometry of A-branes in parafermionic theory can be obtained by projecting $U(1)$ sector; D2-brane wrapping the conjugacy class becomes the line segment with the corresponding length and D0-brane becomes D0-brane again at the boundary of the disk. Therefore the crosscap state we are dealing with can be identified as $O0$-planes at the boundary of the disk.
Figure 4: The geometry of one family of orientifolds which are followed by the solution (2.15) in parafermionic theory. (1) A-type orientifolds of (2.15) describe $O_0$-planes at the boundary of the disk. (2) B-type orientifold can be obtained by $\mathbb{Z}_k$ orbifold and T-duality just as B-branes. It is $O_0$-plane at the center of the disk.

Next let us consider the Möbius strip amplitudes with B-type branes (4.16). In the construction of B-type branes, we have chosen the Ishibashi states with $n = 0$ by using the spectral flow (4.2), thus we have to care about this identification. By changing the domain of $PF(k)$, we can rewrite the crosscap state (4.28) as

$$|OA, 0, \hat{n}\rangle_C = \sum_{j=0}^{k/2} \sum_{n=-2j}^{k-2j-2} \left( \frac{P_0 j P_n^n}{\sqrt{S_{PF}^{j,n}}} |OA, j, n\rangle_I \right) + (-1)^{-j-n/2} \frac{P_0 j P_n^n}{\sqrt{S_{PF}^{j,n}}} |OA, k/2 - j, n + k\rangle_I \right) . \quad (4.33)$$

By using this form of the crosscap states, we can now calculate the amplitudes as

$$\mathcal{M} = C \langle B, \hat{j}|q^{H+}|OA, 0, \hat{n}\rangle_C = \sum_{j=0}^{k/2} Y_{j,0}^j \hat{\chi}^{n''}(q), \quad (4.34)$$

where $\hat{\chi}^{n''}(q)$ are defined by $P$ transformation of $\hat{\chi}^{n'}(q)$ (4.19) as

$$\hat{\chi}^{n'}_j(q) = \sum_{j'=0}^{k/2} P_j^{j'} \hat{\chi}^{n''}_{j'}(q), \quad \hat{\chi}^{n''}_j(q) = e^{\pi ik\tau/2} \chi^{SU(2)}_{j'}(\tau, -\tau)/\chi^{ND}(\tau). \quad (4.35)$$

We have already seen that we can construct the other type of boundary states, which are called as B-type boundary states and constructed by using $\mathbb{Z}_k$ orbifold procedure and T-duality. Just as the crosscap states of $U(1)$ WZW models, we can get the B-type crosscap
states in parafermionic theory by the same procedure. The invariant crosscap states under $\mathbb{Z}_k$ transformation can be obtained automatically by summing all images just as $U(1)$ case (3.30) as:

$$\frac{1}{\sqrt{k}} \sum_{m=0}^{k-1} |OA, 0, 2\hat{m}\rangle_C = \sqrt{k} \sum_{j=0}^{k/2} P_0^j P^{U(1), 0} \sqrt{S_{0,0}^{PF,j,0}} |OA, j, 0\rangle_I$$

$$= \left(\frac{k}{2}\right)^{ \frac{k}{2} } \sum_{j=0}^{k/2} \frac{P_0^j}{\sqrt{S_{0,0}^{PF,j,0}}} |OA, j, 0\rangle_I . \tag{4.36}$$

Since there is a subtlety of the spectral flow identification, we used the A-type crosscap states of the form (4.33). Applying T-duality to this state, we can get the B-type crosscap state as

$$|OB, 0\rangle_C = \left(\frac{k}{2}\right)^{ \frac{k}{2} } \sum_{j=0}^{k/2} \frac{P_0^j}{\sqrt{S_{0,0}^{PF,j,0}}} |OB, j, 0\rangle_I , \tag{4.37}$$

where B-type Ishibashi crosscap states are defined by the same way as B-type Ishibashi boundary states. The orientifold projection given by this crosscap state can be seen by the Klein bottle amplitude

$$\mathcal{K} = c\langle OB, 0|q^{H_c}|OB, 0\rangle_C = \sum_{(j,n)\in PF(k)} Y_{0,0}^j \hat{\chi}_{j,n}(q) . \tag{4.38}$$

The geometry of B-type orientifold can be seen by the sigma model description and the scattering with the closed string states given in appendix B. Both results show that this orientifold is $O_0$-plane located at the center of the disk as figure 4.

The Möbius strip amplitudes with the boundary states can be obtained just as A-type crosscap states. The amplitudes with B-type boundary states are given by

$$\mathcal{M} = c\langle B, \hat{j}|q^{H_c}|OB, 0\rangle_C = \sum_{(j,n)\in PF(k)} Y_{j,0}^j \hat{\chi}_{j,n}(q) , \tag{4.39}$$

and the amplitudes with A-type boundary states are obtained as

$$\mathcal{M} = c\langle A, \hat{n}|q^{H_c}|OB, 0\rangle_C = \sum_{j=0}^{k/2} Y_{j,0}^j \hat{\chi}''_j(q) , \tag{4.40}$$

where we use the character $\hat{\chi}''_j(q)$ defined above (4.35).

As we saw in section 2, there is the other type of crosscap states (2.22). In $SU(2)$ WZW model, this type of crosscap state corresponds to $O_2$-plane at the equator of $S^3$ [21, 22, 23].

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4We assume that $k$ is even. In odd $k$ case, we sum over odd $\hat{m}$ states just as $U(1)$ case.
Therefore this type of crosscap states should represent \( \mathcal{O}_1 \)-planes across the center of the disk. We can construct B-type crosscap state form this type of A-orientifolds by \( \mathbb{Z}_k \) orbifold and T-duality and this orientifold can be identified as \( \mathcal{O}_2 \)-plane wrapping the whole disk. From now on we will call this type of orientifolds obtained by using the simple current technique as A’-type orientifolds and B’-type orientifold.

We can obtain the crosscap states of (2.22) by using \( P \) matrix (4.26) as

\[
|OA, \frac{k}{2}, \hat{n} \rangle_C = \sum_{(j,n) \in PF(k)} \frac{P_{\hat{j}\hat{n}}^{PF}}{\sqrt{S_{PF_{00}}^{j,n}}} |OA, j, n \rangle_I.
\]  

(4.41)

The orientifold projection can be seen by the Klein bottle amplitudes

\[
\mathcal{K} = c \langle OA, \frac{k}{2}, \hat{n} | \tilde{q}^{Hc} | OA, \frac{k}{2}, \hat{n} \rangle_C = \sum_{j=0}^{k/2} \chi_{j,0}(q) + E_k(-1)^{(k+\hat{n})/2} \chi_{k/4, k/2}(q),
\]  

(4.42)

which are almost same as the previous ones of A-type orientifolds in spite of the difference of the geometry. The Möbius strip amplitudes with A-type branes can be obtained as

\[
\mathcal{M} = c \langle A, \hat{j}, \hat{n} | \tilde{q}^{Hc} | OA, \frac{k}{2}, \hat{n}' \rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{2}} \frac{\chi'_{j}^{\hat{j}}}{2\hat{n} - \hat{n}'}(q),
\]  

(4.43)

From this equation, we can see that this type of orientifolds reflect at the opposite boundary points of the disk, since the annulus amplitudes with image branes are given by

\[
\mathcal{A} = c \langle A, \hat{j}, \hat{n} | \tilde{q}^{Hc} | A, \hat{j}, k + \hat{n}' - \hat{n} \rangle_C = \sum_{j=0}^{k/2} N_{\frac{j}{2}} \chi_{j,2\hat{n} - \hat{n}'}(q).
\]  

(4.44)

In fact, the geometry of the A’-type orientifolds is determined in appendix B and we show that the orientifolds are \( \mathcal{O}_1 \)-planes connecting the opposite boundary points of the disk as figure 5. The Möbius strip amplitudes with B-type branes can be obtained as

\[
\mathcal{M} = c \langle B, \hat{j} | \tilde{q}^{Hc} | OA, \frac{k}{2}, \hat{n} \rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{2}} \chi_{j}^{\prime'}(q),
\]  

(4.45)

by using the character (4.35).

We can also obtain B’-type crosscap state by \( \mathbb{Z}_k \) orbifold procedure and T-duality and this state is given by

\[
|OB, \frac{k}{2} \rangle_C = \left( \frac{k}{2} \right)^{\frac{k}{2}} \sum_{j=0}^{k/2} \frac{P_{\frac{k}{2}}^{j}}{\sqrt{S_{00}^{j}}} |OB, j, 0 \rangle_I.
\]  

(4.46)
Figure 5: The geometry of the other family of orientifolds which are constructed by applying the simple currents (2.22) in parafermionic theory. (1) A'-type orientifolds of (2.22) are $O_1$-planes crossing the disk. (2) B'-type orientifold can be also obtained by $\mathbb{Z}_k$ orbifold and T-duality. It is $O_2$-plane wrapping the whole disk.

The Klein bottle amplitude between this crosscap state is

$$\mathcal{K} = c\langle OB, \frac{k}{2}|\tilde{q}H_c|OB, \frac{k}{2}\rangle_C = \sum_{(j,n) \in PF(k)} Y_{\frac{j}{k} \frac{k}{2}} \chi_{j,n}(q) . \tag{4.47}$$

The geometry of this orientifold can be also determined by scattering with the closed string states in appendix B. There we show that this orientifold is $O_2$-plane wrapping the whole disk as figure 5, which is sharply contrast to the previous B-type orientifold. The M"obius strip amplitudes with B-type branes are given by

$$\mathcal{M} = c\langle B, \hat{j}|\tilde{q}H_c|OB, \frac{k}{2}\rangle_C = \sum_{(j,n) \in PF(k)} Y_{\hat{j} \frac{k}{2}} \hat{\chi}_{j,n}(q) , \tag{4.48}$$

and the amplitudes with A-type branes are obtained as

$$\mathcal{M} = c\langle A, \hat{j}, \hat{n}|\tilde{q}H_c|OB, \frac{k}{2}\rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{k} \frac{k}{2}} \chi_j''(q) , \tag{4.49}$$

where we again use the character (4.35).

In summary, we have constructed four types of orientifolds. We have shown that A-type orientifolds are $O_0$-planes at the boundary of the disk and B-type one is $O_0$-plane at the center of the disk. A'-type ones are $O_1$-planes connecting the boundary points and B'-type one is $O_2$-plane wrapping the whole disk. In this section, we investigate all amplitudes with crosscap states in parafermionic theory and all spectrum in the direct channel have integer degeneracies. Just as the $U(1)$ case, this is the important consistency check.
5 Super $SU(2)/U(1)$ WZW Models and Orientifolds

To apply to superstring theories or to analyze the instability, we have to extend the previous analysis to the supersymmetric one. Although $\mathcal{N} = 1$ supersymmetry is included in super parafermionic theory, the supersymmetry is enhanced to $\mathcal{N} = 2$. We study the closed strings and D-branes in super parafermionic theory in the next section and the orientifolds in section 5.2. The construction of orientifolds can be proceeded just as the bosonic case and we get the similar results.

5.1 Super $SU(2)/U(1)$ WZW models and D-branes

This theory is known as $\mathcal{N} = 2$ minimal model and is given by $SU(2)_k \times U(1)_2/U(1)_{k+2}$ coset WZW models. The Hilbert space of super parafermionic theory is given by the decomposition as

$$\mathcal{H}^{SU(2)}_{j,n} \otimes \mathcal{H}^{U(1)}_{s} = \mathcal{H}^{SPF}_{j,n,s} \otimes \mathcal{H}^{U(1)}_{n} ,$$

where the labels run $j = 0, 1/2, \cdots , k/2$, $n \in \mathbb{Z}_{2k+4}$ and $s \in \mathbb{Z}_4$. The states with $s = 0, 2$ are in the NS sector and the ones with $s = 1, 3$ are in the R sector. Just as the bosonic case, there are the spectral flow identification

$$\mathcal{H}^{SPF}_{j,n,s} = \mathcal{H}^{SPF}_{-j,n+k+2,s+2} ,$$

and the restriction of the label $2j + n + s \in 2\mathbb{Z}$. We will call the distinct irreducible representation of super parafermionic theory as $SPF(k)$ and the characters as $\chi_{j,n,s}(q)$. The lowest conformal weights are given by

$$h_{j,n,s} = \begin{cases} \frac{j(j+1)}{k+2} - \frac{s^2}{4(k+2)} + \frac{n^2}{8} - 2j \leq n - s \leq 2j \\ \frac{j(j+1)}{k+2} - \frac{s^2}{4(k+2)} + \frac{n-s-2j}{2} + \frac{n-s-2j}{2} \end{cases} 2j \leq n - s \leq 2k - 2j ,$$

and the conformal weights of the fields outside this region can be obtained by making use of the spectral flow. The modular transformation matrix is given by

$$S^{SPF}_{j,n,s} = \frac{1}{k+2} \sin \left( \frac{\pi(2j+1)(2j'+1)}{k+2} \right) e^{\pi in'(k+2)} e^{-\pi is'/2} ,$$

and the fusion coefficients are

$$N^{SPF}_{(j,n,s),(j',n',s')} = N_j^{j''} \delta^{(2k+4)}_{n+n'-n''} \delta^{(4)}_{s+s'-s''} + N_{j,j'}^{k/2-j''} \delta^{(2k+4)}_{n+n'-n''-k-2} \delta^{(4)}_{s+s'-s''-2} .$$
The diagonal modular invariant is given by
\[
\mathcal{T} = \sum_{(j,n,s) \in SPF(k)} |X_{j,n,s}(q)|^2 ,
\]
and this theory has the global symmetries which are generated by
\[
g_1 \Phi_{j,n,s} = e^{2\pi i (n/(2k+4) - s/4)} \Phi_{j,n,s}
g_2 \Phi_{j,n,s} = e^{\pi is} \Phi_{j,n,s} .
\]

We will use the orbifold procedure as \( \mathbb{Z}_{k+2} \times \mathbb{Z}_2 \) symmetries which is generated by \( g_1^2 g_2 \times g_2 \).

Let us study D-branes in super parafermionic theory. A-type Cardy states can be constructed by the usual way as
\[
|A, \hat{j}, \hat{n}, \hat{s} >_C = \sum_{(j,n,s) \in SPF(k)} \sum_{SPF} S_{SPF}^{j,n,s \hat{j}, \hat{n}, \hat{s}} \sqrt{N_{SPF}^{j,n,s}} |A, j, n, s >_I ,
\]
where \(|A, j, n, s >_I \) are Ishibashi states in super parafermionic theory. The geometry of A-branes can be seen as follows. The target space of super parafermionic theory is the disk with fermions and it is convenient to use the disk with \( 2k + 4 \) special points by making use of the bosonized fermions [33]. Then the geometry of D-brane described by \(|A, \hat{j}, \hat{n}, \hat{s} >\) can be determined as D1-brane, which is the straight line connecting \( 4\hat{j} + 2 \) separated points. The label \( \hat{n} \) can be changed by the \( \mathbb{Z}_{k+2} \) rotation of the disk. The label \( \hat{s} \) means the spin structure and the states with \( \hat{s} \) and \( \hat{s} + 2 \) describes the D1-branes with opposite orientation.

The annulus amplitudes are given by
\[
\mathcal{A} = C \langle A, \hat{j}, \hat{n}, \hat{s} | q^{H_c} | A, \hat{j}', \hat{n}', \hat{s}' >_C = \sum_{(j,n,s) \in SPF(k)} N_{SPF}^{(j,n,s)} N_{SPF}^{j,n,s} \chi_{j,n,s}(q) = \sum_{j=0}^{k/2} N_{j,j'} \chi_{j,n-n', \hat{s}-\hat{s}'}(q) ,
\]
where we use the fact that the bra state \( C \langle A, \hat{j}, \hat{n}, \hat{s} + 2 | \) has the same orientation with \(|A, \hat{j}, \hat{n}, \hat{s} >_C \). We should note that RR states in the closed channel have the factor \((-1)^s\) in the annulus amplitudes between the same branes. We can see that there is no tachyon in the open string spectrum of the annulus amplitudes between the same branes.

B-type Cardy states are similarly constructed by using \( \mathbb{Z}_{k+2} \times \mathbb{Z}_2 \) orbifold and T-duality. The orbifold procedure restricts the labels of Ishibashi states to \( n = 0, k+2 \) and \( s = 0, 2 \) and we will only use \( n = 0 \) states by using the spectral flow identification (5.2). T-duality converts
from A-type Ishibashi states to B-type ones. The A-type Ishibashi states can be obtained by the decomposition

$$\langle j \rangle_I^{SU(2)_k} \otimes |A, s \rangle_I^{U(1)_2} = \sum_{n=0}^{2k+3} |A, j, n, s \rangle_{I}^{SFP} \otimes |A, n \rangle_I^{U(1)_k+2} , \quad (5.10)$$

and we can apply T-duality just as the bosonic case by making use of the operator \(\exp(\pi i J_0^2)\) as

$$\langle 1 \otimes e^{\pi i J_0^2} \rangle_I^{SU(2)_k} \otimes |B, s, -s \rangle_I^{U(1)_2} = \sum_{n=0}^{2k+3} |B, j, n, -n, s, -s \rangle_{I}^{SFP} \otimes |B, n, -n \rangle_I^{U(1)_k+2} , \quad (5.11)$$

where \(|B, r, -r \rangle_I^{U(1)}\) are defined by (3.10).

Now we can construct B-type Cardy crosscap states as

$$|B, \hat{j}, \hat{n}, \hat{s}\rangle_C = \sqrt{2(k+1)} \sum_{j \in \mathbb{Z}} \sum_{s = 0, 2} \sqrt{S^{j, \hat{n}, \hat{s}}_{0,0,s}} |B, j, 0, 0, s, -s \rangle_I , \quad (5.12)$$

and these states represent D2-branes at the center of the disk. Since we sum over only \(s = 0, 2\), the label of the distinct states can be given by \(\hat{s} = 0, 1\), which corresponds to the spin structure. The label \(\hat{n}\) is only used to be survived by the selection rule \(2\hat{j} + \hat{n} + \hat{s} < 2\mathbb{Z}\). Therefore there are \(\hat{j} = 0, 1/2, \ldots, (k-1)/4\) states with odd \(k\). In even \(k\) case, there is \(\hat{j} = k/4\) state in addition to \(\hat{j} = 0, 1/2, \ldots, (k-2)/4\) states. The general \(\hat{j}\) states are non-orientable and do not have RR charges. In fact they are unstable because there are tachyons in the open string spectrum of the annulus amplitudes between B-type Crosscap states

$$\mathcal{A} = C(B, \hat{j}, \hat{n}, \hat{s}|\tilde{q}^{H_c}|B, j', \hat{n}', \hat{s}')_C = \sum_{(j,n,s) \in SPP(k)} E_{\hat{s}', -\hat{s}, -s}(N_{\hat{j}, j', k/2-j} + N_{\hat{j}, \hat{j}, 0}) \chi_{\hat{j}, n, s}(q) . \quad (5.13)$$

On the contrary, the state with \(\hat{j} = k/4\) is orientable and has RR charge, moreover they are free of tachyons in the open string amplitudes between this boundary state. The detailed analysis was given in [33], however we do not deal with this states for simplicity.

The annulus amplitudes between A-type and B-type branes can be obtained by using the overlaps between A-B Ishibashi states as

$$\langle I | A, j, 0, 0 | q^{H_c} | B, j', 0, 0 \rangle_I = \delta_{j,j'} \chi^{SU(2)}_j(-1/\tau, 1/2) . \quad (5.14)$$

The character \(\chi_{ND}\) as in (4.19) does not appear since there are the \(U(1)\) sectors in the both sides of the equation (5.10). Thus the amplitudes are given by

$$\mathcal{A} = C(A, \hat{j}, \hat{n}, \hat{s}|\tilde{q}^{H_c}|B, j', \hat{n}', \hat{s}')_C = \sum_{j=0}^{k/2} N_{\hat{j}, j} \langle j \hat{j}, \hat{n}, \hat{s} | q^{H_c} | j', \hat{n}', \hat{s}' \rangle \chi^{SU(2)}_j(\tau, \tau/2) , \quad (5.15)$$

which are independent with the labels \(\hat{n}, \hat{n}', \hat{s}\) and \(\hat{s}'\).
5.2 Crosscap states in Super $SU(2)/U(1)$ WZW Models

The crosscap states in super parafermionic theory can be constructed just like the bosonic case by using the $P$ matrix

$$P_{j,n,s}^{SPF} j',n',s' = P_j j' P_n n' P_s s' + (-1)^{(2j-n+s)/2} P_{j,n+k+2}^{k/2-j'} P_n n'+k+2 P_{s'} s'+2 ,$$

(5.16)

where we used the $U(1)$ $P$ matrices as

$$P_{n,n'} = \frac{1}{\sqrt{k+2}} e^{\pi i n n'/2(k+2)} E_n + n',$$

(5.17)

The last term of (5.16) is included because of the spectral flow identification just as (4.26). A-type orientifolds can be obtained by the solution (2.15) as

$$|OA, 0, \hat{n}, \hat{s}\rangle_C = \sum_{(j,n,s) \in SPF(k)} P_{0,0,0,n,s}^{SPF} |OA, j, n, s\rangle_I ,$$

(5.18)

where $|OA, j, n, s\rangle_I$ are Ishibashi crosscap states in super parafermionic theory. These states represent $O0$-planes at the opposite points of the boundary. We can calculate the Klein bottle amplitudes as

$$K = c\langle OA, 0, \hat{n}, \hat{s} + 4 | \tilde{q} Hc | OA, 0, \hat{n}, \hat{s}\rangle_C$$

$$= \sum_{j=0}^{k/2} (\chi_{j,0,2}(q) + (-1)^{\hat{s}} \chi_{j,0,0}(q))$$

$$+ E_k(-1)^{(\hat{n}-\hat{s})/2}(\chi_{j,\frac{k+2}{2},3}(q) + (-1)^{\hat{s}} \chi_{j,\frac{k+2}{2},1}(q)) .$$

(5.19)

We should note here that the bra state $\langle A, \hat{j}, \hat{n}, \hat{s} + 4 | C$ corresponds to the same brane described by the ket state $|A, \hat{j}, \hat{n}, \hat{s}\rangle_C$. If we apply to superstring theories, one might be able to remove tachyons by combining the states with different $\hat{s}$.

The Möbius strip amplitudes with A-type branes are given by

$$\mathcal{M} = c\langle A, \hat{j}, \hat{n}, \hat{s} | \tilde{q} Hc | OA, 0, \hat{n}', \hat{s}'\rangle_C = \sum_{j=0}^{k/2} Y_{j,0}' \hat{j} \tilde{j}_{j,2\hat{n}-\hat{n}',2\hat{s}-\hat{s}'}(q) ,$$

(5.20)

and the amplitudes with B-branes are obtained as

$$\mathcal{M} = c\langle B, \hat{j}, \hat{n}, \hat{s} | \tilde{q} Hc | OA, 0, \hat{n}', \hat{s}'\rangle_C = \sum_{j=0}^{k/2} Y_{j,0}' \hat{j} \tilde{j}_{j}''(q) ,$$

(5.21)

where the characters are defined just like the bosonic case (4.35) as

$$\chi_{j}''(q) = e^{\pi i k \tau/2} \chi_{j}^{SU(2)}(\tau, -\tau) .$$

(5.22)
The construction of B-type orientifolds can be proceeded by $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ orbifold and T-duality as before. The orbifold procedure restricts the labels of Ishibashi states as $(n, s) = (0, 0)$. T-duality makes B-type Ishibashi states from A-type Ishibashi states just as the B-type boundary states. The result is given by

$$|OB, 0\rangle_C = (2k + 4)^{1/4} \sum_{j=0}^{k/2} \frac{P_0^j}{S_0^j} |OB, j, 0, 0\rangle_I ,$$

which is the $\mathcal{O}0$-plane at the center of the disk. The orientifold projection is given by taking the right-mover of the diagonal modular invariant (5.6) with phases as

$$K = C\langle OB, 0|q^{H_c}|OB, 0\rangle_C = \sum_{(j, n, s) \in SPF(k)} Y_{0,0}^j \chi_{j,n,s}(q) .$$

There are tachyons in the direct channel spectrum of the amplitude. These tachyons are related to the fact that we are dealing with the system with the diagonal modular invariant (5.6). Therefore if we apply the correct GSO projections we can remove these tachyons and make the system stable.

Furthermore we can calculate the Möbius strip amplitudes with the boundary states in super parafermionic theory. The Möbius strip amplitudes with B-type branes can be given by

$$\mathcal{M} = C\langle B, \hat{j}, \hat{n}, \hat{s}|q^{H_c}|OB, 0\rangle_C = \sum_{(j, n, s) \in SPF(k)} Y_{j,0}^j \hat{\chi}_{j,n,s}(q) ,$$

and the amplitudes with A-type branes are obtained by using the characters (5.22) as

$$\mathcal{M} = C\langle A, \hat{j}, \hat{n}, \hat{s}|q^{H_c}|OB, 0\rangle_C = \sum_{j=0}^{k/2} Y_{j,0}^j \hat{\chi}'''(q) .$$

A'-type orientifolds which are constructed by applying the simple current technique are given by the solution (2.22) as

$$|OA, \frac{k}{2}, \hat{n}, \hat{s}\rangle_C = \sum_{(j, n, s) \in SPF(k)} \frac{P^{SPF, j,n,s}}{\sqrt{P^{SPF, j,n,s}_{0,0,0}}} |OA, j, n, s\rangle_I ,$$

and these states represent $\mathcal{O}1$-planes connecting the opposite points of the boundary. The Klein bottle amplitudes can be computed as

$$K = C\langle OA, \hat{j}, \hat{n}, \hat{s} + 4|q^{H_c}|OA, \frac{k}{2}, \hat{n}, \hat{s}\rangle_C$$

$$= \sum_{j=0}^{k/2} \left( \chi_{j,0,2}(q) + (-1)^S \chi_{j,0,0}(q) \right)$$

$$+ E_k(-1)^{(k+n-s)/2}(\chi_{\frac{k}{2}, \frac{k+2}{2}, 3}(q) + (-1)^S \chi_{\frac{k}{2}, \frac{k+2}{2}, 1}(q)) .$$
The Möbius strip amplitudes are given as follows. The amplitudes with A-type branes are given by
\[
\mathcal{M} = C \langle A, \hat{j}, \hat{n}, \hat{s} | \tilde{q}^{H_c} | OA, \frac{k}{2}, \hat{n}', \hat{s}' \rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{k}} \hat{\chi}_{j,2n-2n',2s-s'}(q), \tag{5.29}
\]
and the amplitudes with B-branes are obtained as
\[
\mathcal{M} = C \langle B, \hat{j}, \hat{n}, \hat{s} | \tilde{q}^{H_c} | OA, \frac{k}{2}, \hat{n}', \hat{s}' \rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{k}} \hat{\chi}_{j,2n-2n',2s-s'}(q). \tag{5.30}
\]

B'-type orientifolds can be constructed by \( \mathbb{Z}_{k+2} \times \mathbb{Z}_2 \) orbifold and T-duality and we obtain
\[
|OB, \frac{k}{2} \rangle_C = (2k + 4)^{1/4} \sum_{j=0}^{k/2} \frac{P_{\frac{j}{k}}}{\sqrt{S_0^j}} |OB, j, 0, 0 \rangle_I, \tag{5.31}
\]
which is the \( O2 \)-plane wrapping the disk. The orientifold projection is given by
\[
\mathcal{K} = C \langle OB, \frac{k}{2} | \tilde{q}^{H_c} | OB, \frac{k}{2} \rangle_C = \sum_{(j,n,s) \in SPF(k)} Y_{\frac{j}{k}} \hat{\chi}_{j,n,s}(q), \tag{5.32}
\]
which is the right-mover of the diagonal modular invariant (5.6) and this may be the most natural orbifold projection. The Möbius strip amplitudes with B-type branes can be given by
\[
\mathcal{M} = C \langle B, \hat{j}, \hat{n}, \hat{s} | \tilde{q}^{H_c} | OB, \frac{k}{2} \rangle_C = \sum_{(j,n,s) \in SPF(k)} Y_{\frac{j}{k}} \hat{\chi}_{j,n,s}(q), \tag{5.33}
\]
and the amplitudes with A-type branes are obtained as
\[
\mathcal{M} = C \langle A, \hat{j}, \hat{n}, \hat{s} | \tilde{q}^{H_c} | OB, \frac{k}{2} \rangle_C = \sum_{j=0}^{k/2} Y_{\frac{j}{k}} \hat{\chi}_{j}(q). \tag{5.34}
\]

Finally we again emphasize that the spectrum in the direct channel of every amplitude we calculated has only integer degeneracies. This is the important consistency condition and all orientifolds we constructed satisfy this condition.

6 Conclusion

In this paper, the orientifolds of parafermionic theories are investigated. The D-branes in parafermionic theories are analyzed in [33] and they determine the geometry of D-branes by scattering with the closed string states. The geometry of orientifolds can be also determined
by scattering the closed string states and we summarize in appendix B. The paper [33] also
construct the new types of branes which are called as B-type branes. In $SU(2)_k/U(1)_k$ WZW
model, it can be seen that the $\mathbb{Z}_k$ orbifold is T-dual to the original theory. Therefore we can
construct new type branes from the previously known branes which are called as Cardy type
branes or A-type branes.

The same method can be applied to the orientifolds of parafermionic theory. The Cardy
type orientifolds, which are called as A-type orientifolds, are constructed in [15, 16, 17, 18]
and expressed in terms of the simple currents [19, 20]. In our case there are A-type crosscap
states (4.28) and A'-type crosscap states (4.41), which are the solutions (2.15) and (2.22),
respectively. We can see that these orientifolds are $O_0$-planes at the boundary of the disk
and $O_1$-planes connecting the boundary points. Using $\mathbb{Z}_k$ orbifold and T-duality, we can
construct B-type and B'-type orientifolds. B-type crosscap state is given by (4.37) and it
represents $O_0$-plane at the center of the disk. B'-type crosscap state is given by (4.46) and it
represents $O_2$-plane wrapping the whole disk. The extension to the supersymmetric case is
also discussed in section 5.

As we said in introduction, D2-branes in $SU(2)$ WZW models can be thought of the bound
states of D0-branes [9, 10, 11, 12, 43, 44]. This can be used to define the D-brane charges
and they can be classified by twisted K-theory [45, 46, 47, 48, 49]. Therefore it is interesting
to calculate the K-theoretic D-brane charges in the system with orientifolds of WZW models
just like flat space analysis [50]. The extension to the orientifolds of other backgrounds are
also interesting, for example, more general coset WZW models. The study of orientifolds of
$AdS_3$ space seems important because the orientifolds wrap the twisted conjugacy classes as
pointed out in [23]. We might be able to find the orientifolds wrapping the twisted conjugacy
classes in $SU(N)$ WZW models.

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Appendix A  Modular Transformations of Several Functions

The Dedekind $\eta$ function is given by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (A.1)$$

where $q = \exp(2\pi i \tau)$ and its modular transformation is

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau), \quad \eta(-\tau) = \sqrt{-i\tau} \eta(\tau). \quad (A.2)$$

The theta functions are defined as

$$\Theta_{n,k}(\tau, z) = \sum_{l \in \mathbb{Z}} e^{2\pi i \tau k (l + \frac{n}{2})^2} e^{2\pi i k (l + \frac{n}{2})} e^{2\pi i k^2 z^2/4\tau}, \quad (A.3)$$

and their modular transformations are

$$\Theta_{n,k}(\tau + 1, z) = e^{\pi i n^2/2k} \Theta_{n,k}(\tau, z)$$

$$\Theta_{n,k}(-\frac{1}{\tau}, -\frac{z}{\tau}) = \sqrt{-i\tau} e^{2\pi i k z^2/4\tau} \sum_{n'} e^{-i\pi n' n/k} \Theta_{n',k}(\tau, z). \quad (A.4)$$

The characters of $SU(2)_k$ current algebra are written by using theta functions as

$$\chi^{SU(2)}_j(\tau, z) = \frac{\Theta_{2j+1,k+2} - \Theta_{-(2j+1),k+2}}{\Theta_{1,2} - \Theta_{-1,2}}(\tau, z), \quad (A.5)$$

and their modular transformations are

$$\chi^{SU(2)}_j(\tau + 1, z) = e^{2\pi i(\frac{j(j+1)}{k+2} - \frac{k}{4k+8})} \chi^{SU(2)}_j(\tau, z)$$

$$\chi^{SU(2)}_j(-\frac{1}{\tau}, -\frac{z}{\tau}) = e^{2\pi i k z^2/4\tau} \sum_{j' = 0}^{k/2} S_{j,j'} \chi^{SU(2)}_{j'}(\tau, z), \quad (A.6)$$

where $S$ matrix of $SU(2)$ character are given as

$$S_{j,j'} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi (2j+1)(2j'+1)}{k+2} \right). \quad (A.7)$$
Appendix B  

Geometry of Orientifolds

Let us first examine the geometry of $SU(2)_k$ WZW model, whose target space is $S^3$. The metric of $S^3$ can be parametrized by the term of Euler angle

$$g = e^{i\chi \frac{\sigma_3}{2}} e^{i\theta \frac{\sigma_1}{2}} e^{i\phi \frac{\sigma_3}{2}},$$

(B.1)

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ and $0 \leq \chi \leq 4\pi$. When analyzing D-branes in $SU(2)_k$ WZW models, it is convenient to use the metric

$$ds^2 = d\psi^2 + \sin^2 \psi d^2 S^2,$$

(B.2)

because the maximally symmetric D-branes are located at $\psi = \text{const}$. The parafermionic theory can be obtained by gauging $U(1)$ sector of $SU(2)$ currents. In the term of Euler angle (B.1), this $U(1)$ sector corresponds to the shift of $\chi + \varphi$. Using $\rho = \sin \frac{\theta}{2}$ and $\phi = \frac{1}{2}(\chi - \varphi)$, the target space of parafermionic theory can be described as the disk with the radius $R = 1$

$$ds^2 = \frac{k}{1 - \rho^2}(d\rho^2 + \rho^2 d\phi^2)$$

$$g_s = g_s(0)(1 - \rho^2)^{-\frac{1}{2}}. $$

(B.3)

T-duality can be performed on the $U(1)$ isometry and we get

$$ds^2 = \frac{k}{1 - \rho^2}(d\rho^2 + \rho^2 d\phi'^2)$$

$$g_s = \frac{g_s(0)}{\sqrt{k}}(1 - \rho^2)^{-\frac{1}{2}}, $$

(B.4)

where $\rho^2 = (1 - \rho^2)$ and $\phi' \sim \phi' + 2\pi/k$. Thus we can see that the original theory is T-dual to $\mathbb{Z}_k$ orbifold and the center of the disk of the original theory corresponds to the boundary of the disk.

The geometry of D-branes and orientifolds can be determined by scattering with the closed strings [7, 33, 21] in the large $k$ limit. Since the shape of D-branes in parafermionic theory was investigated in [33], we will concentrate on the shape of orientifolds. By using the matrix representation

$$D^j_{mm'}(g) = \langle j, m| e^{i\chi m\frac{\sigma_3}{2}} e^{i\theta m\frac{\sigma_1}{2}} e^{i\phi m\frac{\sigma_3}{2}} |j, m'\rangle, \quad \langle j, m| j, m'\rangle = \delta_{m,m'}, $$

(B.5)

we define the closed string states $|g\rangle$ which have the properties as

$$D^j_{m,m'}(g) \sim \frac{1}{\sqrt{2j + 1}} \langle g|j, m, m'\rangle. $$

(B.6)
The closed strings are well localized if we use \( j \leq k/2 \). As we said above, the geometry of parafermionic theory can be obtained by gauging \( \chi + \varphi \). Thus we scatter the closed string states \(|\theta, \phi\rangle\), which satisfy

\[
e^{i\phi m} \langle j, m|e^{i\theta J_0}|j, -m\rangle = \frac{1}{\sqrt{2j + 1}}(\theta, \phi|j, m, m\rangle.
\] (B.7)

Now we can examine the geometry of orientifolds. We begin with A-type orientifolds constructed (4.28) in parafermionic theory. In the large \( k \) limit, we can neglect the second term of the crosscap states (4.33). We will use the states with \( \hat{n} = 0 \) for simplicity and the other states can be obtained by \( \mathbb{Z}_k \) rotation of the disk. The scattering amplitudes between the A-type crosscap states and the closed string states are given by

\[
c_{\langle OA, 0|\theta, \phi\rangle} \sim \sum_{j} \sum_{m=−2j}^{2j} P^0_j \langle j, m, m|\theta, \phi\rangle
\]

\[
\sim \sum_{j} \sum_{m=−2j}^{2j} \sin[\hat{\psi}(2j + 1)]e^{i\phi m} \langle j, m, m|e^{i\theta J_0}|j, -m\rangle
\]

\[
= \sum_{j} \sin[\hat{\psi}(2j + 1)] \frac{\sin(2j + 1)\psi}{\sin \psi},
\] (B.8)

where we define

\[
\hat{\psi} = \frac{\pi}{2(k + 1)}, \quad \cos \psi(\theta, \phi) = \cos \phi \sin \frac{\theta}{2}.
\] (B.9)

Evaluating the summation for small \( j \), we get

\[
c_{\langle OA, 0|\theta, \phi\rangle} \sim \delta(\hat{\psi} - \psi(\theta, \phi)),
\] (B.10)

therefore we can see that the crosscap state describes \( O0 \)-planes at \( \phi = 0 \) and \( \phi = \pi \) in the large \( k \) limit, see figure 4. The D-branes in the compact WZW models with finite \( k \) are not exactly localized and are smeared out [7]. However the orientifolds are obtained by the combination of \( \mathbb{Z}_2 \) orbifold and worldsheet parity transformation, hence the shape of orientifolds should be sharply localized even in finite \( k \). This is the remarkable contrast to the shape of D-branes in finite \( k \). The shape of orientifolds represented by A’-type crosscap states (4.41) can be determined by replacing \( P^0_j \) with \( P^k_j \), then we get the same result (B.10) except for \( \hat{\psi} = \frac{\pi(k + 1)}{2(k + 2)} \to \frac{\pi}{2} \). Thus the geometry of orientifold can be identified with the line segment between \( \phi = \frac{\pi}{2} \) and \( \phi = \frac{3\pi}{2} \), see figure 5.

The geometry of B-type orientifolds can be seen by the sigma model description and they are obtained by applying \( \mathbb{Z}_k \) orbifold procedure and T-duality to A-type orientifolds. Since
A-type orientifolds are at the boundary of the disk, after summing the image branes and applying T-duality, we get the orientifolds at the center of the disk, see figure 4. On the other hand, A'-type orientifolds are the straight line across the center of the disk, we get the orientifolds wrapping the whole disk after $\mathbb{Z}_k$ orbifold procedure and T-duality as figure 5. The same results can be obtained by scattering with the closed string states just as the case of A-type orientifolds. First we investigate B-type crosscap state (4.37) which is constructed by A-type orientifold (4.28). Using the Legendre polynomials $P_j(\cos \theta)$ as

$$D_{00}^j = \langle j, 0 | e^{i\theta J_0^0} | j, 0 \rangle = P_j(\cos \theta), \quad (B.11)$$

we can get

$$\langle OB, 0 | \theta, \phi \rangle \sim \sum_j D_{00}^j P_0^j \sim -ie^{i\hat{\psi}} \sum_j P_j(\cos \theta)e^{in\hat{\psi}} + \text{c.c.}$$

$$\sim \frac{\Theta(\cos \theta - \cos 2\hat{\psi})}{\sqrt{\cos \theta - \cos 2\phi}}, \quad (B.12)$$

where $\hat{\psi} = \frac{\pi}{2(k+1)}$ and c.c. means complex conjugate. We also used $\Theta(z)$ as the usual theta function (1 for $z \geq 0$, 0 for others). The last line in the above equation can be obtained by using the generating function of Legendre polynomials

$$\sum_{n=0}^\infty t^n P_n(x) = \frac{1}{\sqrt{1 - 2tx + t^2}}. \quad (B.13)$$

In the large $k$ limit, $\hat{\psi}$ goes to zero, thus we can conclude that the B-type orientifold (4.37) is $O_0$-plane located at the center of the disk ($\rho = \sin \frac{\theta}{2} = 0$) as figure 4. The shape of B'-orientifold can be determined by the same way. The only thing we have to do is replacing $P_0^j$ with $P_{\frac{k}{2}}^j$, hence we get (B.12) with $\hat{\psi} = \frac{\pi(k+1)}{2(k+1)} \rightarrow \frac{\pi}{2}$. Thus the B'-type orientifold (4.46) can be identified as $O_2$-plane wrapping the whole disk as figure 5.

References


