Two families of superintegrable and isospectral potentials in two dimensions

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Abstract

As an extension of the intertwining operator idea, an algebraic method which provides a link between supersymmetric quantum mechanics and quantum (super)integrability is introduced. By realization of the method in two dimensions, two infinite families of superintegrable and isospectral stationary potentials are generated. The method makes it possible to perform Darboux transformations in such a way that, in addition to the isospectral property, they acquire the superintegrability preserving property. Symmetry generators are second and fourth order in derivatives and all potentials are isospectral with one of the Smorodinsky-Winternitz potentials. Explicit expressions of the potentials, their dynamical symmetry generators and the algebra they obey as well as their degenerate spectra and corresponding normalizable states are presented.

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A Hamiltonian system of $N$ degrees of freedom is said to be completely integrable, in the Liouville-Arnold sense, if it possesses functionally independent globally defined and single-valued $N$ integrals of motion in involution \[1,2\]. It is called to be superintegrable if it admits more than $N$ integrals of motion. Not all the integrals of superintegrable system can be in involution, but they must be functionally independent otherwise the extra invariants are trivial. In analogy to the classical mechanics, a quantum mechanical system described in $N$-dimensional (ND) Euclidean space by a stationary Hamiltonian operator $H$ is called to be completely integrable if there exists a set of $N - 1$ (together with $H, N$) algebraically independent linear operators $X_i, i = 1, 2, ..., N - 1$ commuting with $H$ and among each other \[3–11\]. If there exist $k$ additional operators $Y_j, j = 1, 2, ..., k$ where $0 < k \leq N - 1$, commuting with $H$ it is said to be superintegrable. The superintegrability is said to be minimal if $k = 1$ and maximal if $k = N - 1$.

Classical and quantum mechanical examples of the maximally superintegrable systems for any finite $N$ are; the Kepler-Coulomb problem, the harmonic oscillator with rational frequency ratio, the Calogero-Moser system in a harmonic well and the Winternitz (or, Smorodinsky-Winternitz) system. The first two are known, for $N = 3$, since the time of Laplace and the superintegrability of the last two systems were established for the first time, respectively, by Wojciechowski [5] and by Evans [6]. The first systematic search for other possible superintegrable systems was begun by Winternitz and coworkers. They firstly found four independent 2D potentials that are separable in more than one coordinate system [3], and then they extended this to $N = 3$ [4]. This approach is based on two assumptions; (1) Hamiltonians are of potential form. (2) Integrals of motion are at most quadratic in momenta (or, in derivatives). The Winternitz program has been completed in Ref. [7] where a complete list consisting, up to the equivalence of linear transformations, thirteen different 3D potentials with four or five independent integrals of motion is given. Winternitz potentials have also been considered by different formulations such as path integral formulation [8], Lagrangian formalism [9] and evolutionary vector fields formalism [10].

In this paper we report an infinite family of 2D potentials which are not only superintegrable, but at the same time isospectral. We shall give explicit expressions of the potentials, their dynamical symmetry generators and the algebra they obey as well as their degenerate spectrum and corresponding normalizable states. We achieve this goal by following an algebraic method which is based on and, in fact, is an extension of intertwining operator idea. This is closely connected with the supersymmetric (SUSY) methods such as the Darboux transformation, and Schrödinger factorization which deal with pairs of Hamiltonians having the same energy spectra but different eigenstates \[12–15\]. It turns out that each member of this infinite family is a triplet of potentials one of which is the same for entire family and the other two change from member to member. Hence, we have, in fact, two different infinite families of superintegrable and isospectral potentials. The fixed potential turns out to be one of 2D Winternitz potentials and determines the spectra of both families and the other two are intertwined to it by Darboux type transformations. The generators of these transformations depend on eigenfunctions of two associated solvable 1D problems that result from the separation of the Winternitz potential in different coordinates. We should emphasize that our approach makes it possible to apply Darboux transformations simultaneously
Formal aspects of our method together with a brief review of the main points of the intertwining operator idea will be given in the next section. Secs. III-V are devoted to explicit realization of our method. In Sec. IV we present the most general form of 2D integrable and isospectral potentials in the plane polar coordinates. Two subfamilies of superintegrable and isospectral potentials and then their general forms are presented in Secs. VI and VII. Sec. VIII contains a review of bound states of the associated 1D problems and the above mentioned Winternitz potential. After investigating the symmetry generators and their algebra in Sec. IX, the normalizable states of the generated superintegrable and isospectral potentials are given in Sec. X.

II. MULTIPLE INTERTWINING METHOD

The object of the intertwining method is to construct a linear differential operator $\mathcal{L}$ which intertwines two Hamiltonian operators $H_0$ and $H_1$ such that $\mathcal{L}H_0 = H_1\mathcal{L}$. Two important facts that immediately follow from this relation are; (i) If $\psi^0$ is an eigenfunction of $H_0$ with eigenvalue of $E^0$ then $\psi^1 = \mathcal{L}\psi^0$ is an (unnormalized) eigenfunction of $H_1$ with the same eigenvalue $E^0$. (ii) When $H_0$ and $H_1$ are self-adjoint (on some common function space) $\mathcal{L}^\dagger$ intertwines in the other direction $H_0\mathcal{L}^\dagger = \mathcal{L}^\dagger H_1$ and this in turn implies that $[H_0, \mathcal{L}^\dagger\mathcal{L}] = 0 = [\mathcal{L}\mathcal{L}^\dagger, H_1]$, where $^\dagger$ and $[\cdot,\cdot]$ stand for Hermitian conjugation and commutator. The first property shows that $\mathcal{L}$ transforms one solvable problem into another, and the second one means that two hidden dynamical symmetries of $H_0$ and $H_1$ are immediately constructed in terms of $\mathcal{L}$. These are dimension and form independent general properties of this method [16,17]. In the context of 1D systems where $\mathcal{L}$ is taken to be the first order differential operator and Hamiltonians are of the potential forms two additional properties arise; (i) Every eigenfunction of $H_0$ (without regard to boundary conditions or normalizability) can be used to generate a transformation to a new solvable problem. (ii) A direct connection to a SUSY algebra can be established [18]. The first property is a manifestation of the celebrated Darboux transformation and its generalization (Crum transformation). The second property enables us to express in a compact algebraic form of the spectral equivalence of the intertwined systems.

Now suppose that there are three self-adjoint Hamiltonian operators $H_0, H_1, H_2$ which are intertwined as

\[ L_{10}H_0 = H_1L_{10}, \quad L_{21}H_1 = H_2L_{21}. \]  

(1)

The subscripts of the intertwining operators are used to distinguish them and to denote the intertwined Hamiltonians. Eqs. (1) immediately imply that $L_{20} \equiv L_{21}L_{10}$ will intertwine $H_0$ and $H_2$ as follows,

\[ L_{20}H_0 = H_2L_{20}. \]  

(2)

Eqs. (1) and (2) can be unified into the following diagram

\[ H_0 \rightarrow H_1 \quad \text{↓} \quad H_2 \]  

(3)
which must be understood in the sense described by (1) and (2).

Adjoints of (1) and (2) yield
\[ \mathcal{L}^\dagger_{21} H_2 = H_1 \mathcal{L}^\dagger_{21}, \quad \mathcal{L}^\dagger_{10} H_1 = H_0 \mathcal{L}^\dagger_{10}, \quad \mathcal{L}^\dagger_{20} H_2 = H_0 \mathcal{L}^\dagger_{20}. \] (4)

That is, the adjoints of the intertwining operators will intertwine in the reverse directions and this can be represented by a diagram the same as (3) with reversed directions of arrows. Making use of (1-4) it is easy to show that each of \( H_0, H_1, H_2 \) has two dynamical symmetry generators respectively given by;
\[
X_0 = \mathcal{L}^\dagger_{10} \mathcal{L}_{10}, \quad Y_0 = \mathcal{L}^\dagger_{20} \mathcal{L}_{20}, \\
X_1 = \mathcal{L}_{10} \mathcal{L}^\dagger_{10}, \quad Y_1 = \mathcal{L}^\dagger_{21} \mathcal{L}_{21}, \\
X_2 = \mathcal{L}_{21} \mathcal{L}^\dagger_{21}, \quad Y_2 = \mathcal{L}^\dagger_{20} \mathcal{L}_{20}. \] (5)

The subscripts of \( X_j, Y_j \) indicate the Hamiltonians they belong to. Throughout this paper we assume that the domains of definition of Hamiltonians and intertwining operators are some linear subspaces of a common Hilbert space \( \mathcal{H} = L^2(\Omega) \) with the standard sesquilinear inner product. \( L^2(\Omega) \) is the space of all square-integrable functions (and distributions) defined on a subspace \( \Omega \) of ND Euclidean space \( R^N \) [19–21].

For all \( N \geq 2 \), the diagram (3) implies a triplet of isospectral Hamiltonians such that each has two dynamical symmetries. By construction, all the symmetry operators obtained in this manner will be factorized, and have even orders depending on the order of intertwining operators. They will be of the same order only for \( H_1 \). According to the von Neumann theorem (see Ref. [19], pp.141 and Ref. [20], pp.275) \( \mathcal{L}_{ij} \mathcal{L}^\dagger_{ij} \) (and \( \mathcal{L}^\dagger_{ij} \mathcal{L}_{ij} \)) are self-adjoint and nonnegative if \( \mathcal{L}_{ij} \) are closed with dense domains of definition. Otherwise there may exist states in which they have negative expectation values (see Sec. X). If \( \mathcal{L}_{10} \) and \( \mathcal{L}_{21} \) are taken to be algebraically independent, the independence of \( X_i, Y_i \) pairs will be guaranteed from the outset. Extensions of these ideas to higher dimensions will be, generically, called multiple intertwining method. A simple observation that this work initiated from is that, in the particular case of \( N = 2 \) the diagram (3) guarantees the superintegrability of the three Hamiltonians. In the case of \( N = 3 \) such a diagram will imply, provided that symmetry generators are commutative, the integrability of the potentials.

The rest of the paper is devoted to explicit realization of these formal observations for 2D systems. Firstly we will determine the most general form of the potentials and the first order intertwining operator for two Hamiltonians. We then construct the intertwinings \( H_0 \rightarrow H_1 \) and \( H_1 \rightarrow H_2 \) by special forms of the intertwining operator. We end this section by explaining our use of the adjective “isospectral”. Two Hamiltonians are said to be isospectral if they have the same eigenvalue spectrum \([18,22,23]\). In this sense two linearly intertwined Hamiltonians are always formally isospectral except the eigenvalues corresponding to the kernel of the intertwining operator. Even for these exceptional cases one can construct eigenfunctions corresponding to these eigenvalues at least for 1D and 2D systems by appealing to the Liouville formula and its 2D version \([14]\). However, due to physical requirements, in the case of the bound states mainly due to normalizability conditions, some eigenvalues of one of the partner potentials are to be discarded. For higher dimensional systems also the degree of degeneracy of a common eigenvalue may be different (see Sec. X). These will just mean that a finite number of eigenvalues are to be disregarded for they are not physically admissible.
We start by considering a pair of 2D one particle systems characterized by the Hamiltonian operators of potential form,

\[ H_i = -\nabla^2 + V_i, \quad H_f = -\nabla^2 + V_f, \]

where the potentials \( V_i, V_f \) (and eigenvalues of \( H_i, H_f \)) are expressed in terms of \( 2m/\hbar^2 \) and

\[ \nabla^2 = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \]

is the Laplace operator in the plane polar coordinates \((r, \theta)\). \( m \) is the mass of the particle and \( \hbar \) denotes the Planck constant. Here and hereafter we use the notation \( \partial_x \) for partial derivative \( \partial/\partial x \) and the subindexes \( "i", "f" \) as the shorthands for the “initial” and “final”.

We suppose that the Hamiltonians are intertwined by

\[ L_{fi} H_i = H_f L_{fi} \]  

and propose the ansatz that \( L_{fi} \) is the most general first order linear operator

\[ L_{fi} = L_0 + L_d = L_0 + L_1 \partial_r + L_2 \partial_\theta, \]

where \( L_d = L_1 \partial_r + L_2 \partial_\theta \) will be referred to as the differential part of \( L_{fi} \). The potentials and \( L_0, L_1, L_2 \) are some functions of \((r, \theta)\) which are to be determined from consistency equations of the intertwining relation (7).

In view of (6) and (8) the relation (7) explicitly reads as

\[ [\nabla^2, L_d] = -[\nabla^2, L_0] + [V_i, L_d] + P L_{fi}, \]

where \( P = V_f - V_i \). The second order derivatives come, together with some first order derivatives, only from

\[ [\nabla^2, L_d] = (\nabla^2 L_1 + \frac{1}{r^2} L_1) \partial_r + (\nabla^2 L_2) \partial_\theta + \]

\[ 2 (\frac{1}{r^2} \partial_\theta L_1 + \partial_r L_2) \partial_\theta \partial_r + 2 (\partial_r L_1) \partial_r^2 + \frac{2}{r^3} (L_1 + r \partial_\theta L_2) \partial_\theta^2, \]

and by setting their coefficients to zero we obtain;

\[ \partial_\theta L_1 + r^2 \partial_r L_2 = 0, \quad \partial_r L_1 = 0, \quad L_1 + r \partial_\theta L_2 = 0. \]

It is straightforward to show that the general solutions of these equations are

\[ L_1 = A \sin(\theta + \phi), \quad L_2 = B + \frac{A}{r} \cos(\theta + \phi), \]

where \( A, B \) and \( \phi \) are integration constants. Since \( \nabla^2 L_1 = -L_1/r^2, \) and \( \nabla^2 L_2 = 0 \), we have from (10) \([\nabla^2, L_d] = 0\). As a result of this the relation (9) simplifies to

\[ [\nabla^2, L_0] = -L_1 \partial_r V_i - L_2 \partial_\theta V_i + P (L_0 + L_d). \]
By substituting
\[ [\nabla^2, L_0] = \nabla^2 L_0 + 2(\partial_r L_0)\partial_r + \frac{2}{r^2}(\partial_\theta L_0)\partial_\theta, \]
into (12), and then by equating the coefficients of the first and zeroth powers of derivatives we obtain
\[ 2\partial_r L_0 = PL_1, \quad (13) \]
\[ 2\partial_\theta L_0 = r^2 PL_2, \quad (14) \]
\[ (-\nabla^2 + P)L_0 = L_1 \partial_r V_i + L_2 \partial_\theta V_i. \quad (15) \]

These three partial differential equations, the first two of which are linear and the third is nonlinear, constitute a reduced form of the consistency conditions for three unknown functions \( L_0, V_i \) and \( V_f \).

IV. GENERAL FORM OF 2D INTEGRABLE ISOSPECTRAL POTENTIALS IN POLAR COORDINATES

Eqs. (11), (13-14) and the compatibility condition \( \partial_r \partial_\theta L_0 = \partial_\theta \partial_r L_0 \) imply that
\[ 2\nabla^2 L_0 = L_d P, \quad ZL_0 = 0, \quad ZP = 2BrP, \]
where \( Z = L_1 \partial_\theta - r^2 L_2 \partial_r \). From the second and third of these equations (or, from (13) and (14)) we have \( L_0 = f(w) \), and \( P = -2A^2 f'(w)/r^2 L_1^2 \), where \( f \) is an arbitrary function of
\[ w = B \cot(\theta + \phi) + \frac{A}{r \sin(\theta + \phi)}. \]

Prime stands for derivative with respect to the argument and when there is no risk of confusion the argument will be suppressed. By combining \( 2\nabla^2 L_0 = L_d P \) with (15) and using the found \( L_0 \) and \( P \) we obtain an inhomogeneous equation from the general solution of which the general form of potentials are found to be
\[ V_i = h(\kappa) + \frac{V_-(w)}{\kappa^2}, \quad V_f = h(\kappa) + \frac{V_+(w)}{\kappa^2}. \quad (16) \]

Here \( h \) is an arbitrary function of \( \kappa = [A^2 + B^2 r^2 + 2AB \cos(\theta + \phi)]^{1/2} \) such that \( L_d h = 0 \) and
\[ V_{\pm}(w) = f^2(w) \mp (w^2 + B^2) f'(w). \quad (17) \]

Eqs. (16) represent the most general form of 2D integrable and isospectral potentials in polar coordinates.

Let us define the operators
\[ T_1 = \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta, \quad T_2 = \sin \theta \partial_r + \frac{1}{r} \cos \theta \partial_\theta, \quad J = \partial_\theta, \quad (18) \]
which close in the defining relations of the Euclidean Lie algebra \( e(2) \) in two dimensions

\[
[J, T_1] = -T_2, \quad [J, T_2] = T_1, \quad [T_1, T_2] = 0.
\]

(19)

Now \( L_d \) can be rewritten as

\[
L_d = A \sin \phi T_1 + A \cos \phi T_2 + BJ,
\]

(20)

which shows that the differential part of \( L_{fi} \) is an element of \( e(2) \). In terms of the Cartesian coordinates \( x = r \cos \theta, y = r \sin \theta \) we have \( T_1 = \partial_x, T_2 = \partial_y, J = x \partial_y - y \partial_x \) and \( T_i \dagger = -T_i, J\dagger = -J \). These relations can also be verified from (18) by noting that \((\partial_r)\dagger = -(r^{-1} + \partial_r), (\partial_\theta)\dagger = -\partial_\theta \). Now from (5) and (17) the symmetry generators of \( H_i \) and \( H_f \) are

\[
\mathcal{L}_{fi} \mathcal{L}_{fi}^\dagger = \mathcal{V}_- - L^2_d, \quad \mathcal{L}_{fi}^\dagger \mathcal{L}_{fi} = \mathcal{V}_+ - L^2_d,
\]

where \( L^2_d \) is at most quadratic operator in generators of \( e(2) \).

V. CONSTRUCTION OF THE INTERTWINING OPERATORS

We shall construct the legs of the diagram (3) by adopting particular forms of (20) as the differential parts of \( \mathcal{L}_{10} \) and \( \mathcal{L}_{21} \). In doing that we shall make use of the orbit structure of \( e(2) \) under the adjoint action of the Euclidean group \( E(2) \) in two dimensions [24].

Under a unitary similarity transformation, generated by

\[
U = e^{a_0 J} e^{a_1 T_1 + a_2 T_2}, \quad U^\dagger = U^{-1} = e^{-(a_1 T_1 + a_2 T_2)} e^{-a_0 J},
\]

(21)

where \( a_i \)'s are real parameters and \( U^{-1} \) stands for the inverse of \( U \in E(2) \), the relation (7) transforms into \( \tilde{\mathcal{L}}_{fi} \tilde{H}_i = \tilde{H}_f \tilde{\mathcal{L}}_{fi} \), where \( \tilde{X} = UXU^\dagger \). Since \( \nabla^2 = T_1^2 + T_2^2 \) is the Casimir invariant of \( e(2) \), only \( V_i, V_f \) and \( \mathcal{L}_{fi} \) will change under this \( E(2) \) action. Now suppose that \( L_d \) is of the form (20). Making use of the well known operator identity

\[
e^{bK} M e^{-bK} = M + b[K, M] + \frac{b^2}{2!} [K, [K, M]] + \cdots,
\]

where \( b \) is a constant and \( K, M \) are two arbitrary operators, one can easily show that

\[
\tilde{L}_d = BJ + e^{a_0 J} [T_1 (A \sin \phi - a_2 B) + T_2 (A \cos \phi + a_1 B)] e^{-a_0 J}.
\]

Hence, if \( B \neq 0 \) we can take \( \tilde{L}_d = BJ \) by choosing \( a_1 = -A \cos \phi / B, a_2 = A \sin \phi / B \). On the other hand, if \( B = 0, A \neq 0 \) we get \( \tilde{L}_d = AT_1 \) (or, \( \tilde{L}_d = AT_2 \)) for the choice \( a_0 = \phi \) (or, \( a_0 = -\phi \)). Therefore, under the adjoint action of \( E(2) \), \( e(2) \) has two orbits represented by \( J \) and \( T_2 \). Since \( L_d \) and \( cL_d \) belong to the same orbit for \( c \neq 0 \), we can choose \( L_d = J \) for \( \mathcal{L}_{10} \) and \( L_d = T_2 \) for \( \mathcal{L}_{21} \). In such a case the potentials and \( L_0 \) will be specified up to the adjoint action of \( E(2) \).

For the first leg \( H_0 \rightarrow H_1 \) of (3) we take \( A = 0, B = 1 \) in Eq. (11) and redefine the Hamiltonians as \( H_i = H_0 \) and \( H_f = H_1 \). Hence \( L_1 = 0, L_2 = 1 \) and Eqs. (13-14) imply that \( L_0 = f(\theta) \) and

\[
L_d = A \sin \phi T_1 + A \cos \phi T_2 + BJ,
\]

(20)
\[ \mathcal{L}_{10} = f(\theta) + \partial_\theta, \quad P = V_1 - V_0 = \frac{2}{r^2} f'(\theta), \]  
(22)

where \( f \) is an arbitrary differentiable function of \( \theta \). Noting that \( \nabla^2 L_0 = f''(\theta)/r^2 \) we obtain from (15) and (22)

\[ V_0 = h(r) + \frac{V_-(\theta)}{r^2}, \quad V_1 = h(r) + \frac{V_+(\theta)}{r^2}, \]  
(23)

where \( h \) is an arbitrary differentiable function of \( r \) and

\[ V_{\pm}(\theta) = f^2(\theta) \pm f'(\theta). \]  
(24)

As a result the first \( H_0 \to H_1 \) leg of the diagram (3) has been constructed.

For the second leg we take \( B = 0, A = 1 \), fix the form of \( H_1 \) and denote it as \( H_i = H_1 \). We then look for \( H_f = H_2 \) such that \( \mathcal{L}_{21} H_1 = H_2 \mathcal{L}_{21} \) and \( \mathcal{L}_{21} = L_0 + \sin \phi T_1 + \cos \phi T_2 \). In that case from Eqs. (13-14) we get

\[ L_21 = g(u) \]  
and

\[ P = V_2 - V_1 = 2g'(u), \]  
(26)

where \( g \) is an arbitrary differentiable function of \( u = r \sin(\theta + \phi) \). It only remains to solve the nonlinear equation (15) which now takes the form

\[ \partial_u [g^2(u) - g'(u)] = \sin(\theta + \phi)h'(r) + \frac{1}{r^2} [\cos(\theta + \phi) V'_+(\theta) - 2 \sin(\theta + \phi) V_+(\theta)], \]  
(27)

where we have made use of \( \nabla^2 L_0 = g''(u) \) and of the second equation of (23). Note that we could have chosen \( \phi = 0 \), but since it costs almost nothing we keep \( \phi \) in our formulae in order to see that action of \( E(2) \).

Since it further restricts the three arbitrary functions specifying the potentials, Eq. (27) is the main equation which determines the final form of the potentials. As a consistency condition the right hand side of Eq. (27) must be only a function of \( u \). Nevertheless this requirement provides us with many possibilities for \( f, g \) and \( h \), which are investigated in the next two sections. Note that for any solutions of Eq. (27) the potentials will be connected to each other as follows:

\[ V_0 = V_1 - \frac{2}{r^2} f'(\theta), \quad V_2 = V_1 + 2g'(u), \quad V_0 = V_2 - 2[g'(u) + \frac{f'(\theta)}{r^2}]. \]  
(28)

VI. TWO SUBFAMILIES OF POTENTIALS

We construct the simplest family of potentials by taking, in (24) and (27) \( h = (\lambda_1/r^2) + a, V_+ = -\lambda_1 \). These lead us to

\[ f^2 + f' = -\lambda_1, \quad g^2 - g' = -\lambda_2, \]  
(29)

where \( a, \lambda_1, \lambda_2 \) are some arbitrary constants. Then, by Eqs. (23-24) and (26), we obtain
The general solution of \( g^2 - g' = -\lambda_2 \) is

\[
g = \begin{cases} 
\lambda_2^{1/2} \tan(\lambda_2^{1/2}u + a_1); & \text{for } \lambda_2 > 0, \\
-\frac{1}{u + a_1}; & \text{for } \lambda_2 = 0, \\
(-\lambda_2)^{1/2} \tanh[(-\lambda_2)^{1/2}u + a_1]; & \text{for } \lambda_2 < 0,
\end{cases}
\]

where \( a_1 \) is a constant. The solution of \( f^2 + f' = -\lambda_1 \) can be directly read from the above relation after the replacement \((g, u, \lambda_2) \rightarrow (f, -\theta, \lambda_1)\).

An important point is that, by the usual linearization of the Riccati equation, if we substitute \( f(\theta) = \frac{\psi'(\theta)}{\psi(\theta)} \) and \( g(u) = -\frac{\Psi'(u)}{\Psi(u)} \)

into (29) we arrive at two 1D Schrödinger equations

\[
-\psi''(\theta) = \lambda_1 \psi(\theta), \quad -\Psi''(u) = \lambda_2 \Psi(u).
\]

While the second one can be considered as a free motion, this is not the case for the first since \( 0 \leq \theta < 2\pi \). An appealing case is to consider one, or, both of them as infinite square-well problem. Normalized eigenfunctions subjected to boundary conditions, say, \( \psi(0) = 0 = \psi(2\pi) \) and corresponding eigenvalues are

\[
\psi_k(\theta) = \pi^{-1/2} \sin\left(\frac{1}{2}k\theta\right), \quad \lambda_{1,k} = \frac{k^2}{2}, \quad k = 1, 2, \ldots
\]

Hence \( f_k = (k/2) \cot(k\theta/2) \) and by virtue of Eqs. (22) and (24) we have

\[
V_0^{(k)} = \frac{k^2}{2r^2 \sin^2 \frac{1}{2}k\theta} + a, \quad L^{(k)}_{10} = \frac{k}{2} \cot\left(\frac{1}{2}k\theta\right) + \partial_\theta.
\]

To distinguish the resulting potentials, corresponding intertwining operators and the parameter \( \lambda_1 \) we have labelled them by the quantum number \( k \). The \( u \)-problem can be treated in a similar way. In any case, the potentials and transformations among them are generated by solutions of these two auxiliary 1D problems. The existence of \( V_1 = a \) explicitly shows that the member potentials are isospectral to a 2D free motion. As a result we have found a five parameter \((a, a_1, \lambda_1, \lambda_2, \phi)\) family of 2D potentials that are generated, in a nontrivial way, by two 1D problems.

We specify a second subfamily of potentials by taking, in (23-24) and (27)

\[
h = \frac{\lambda_1}{r^2} + \frac{1}{2} \alpha r^2 + a,
\]

and \( V_+ = -\lambda_1 \). These lead us to the same equation as in (29) for \( f \) and to the Riccati’s equation

\[
g^2 - g' - \frac{1}{2} \alpha u^2 + \lambda_2 = 0,
\]

(36)
for $g$. By Eqs. (23-24) and (26) the member potentials are found to be

$$V_0 = \frac{1}{2} \alpha r^2 + \frac{2(f^2 + \lambda_1)}{r^2} + a,$$

$$V_1 = \frac{1}{2} \alpha r^2 + a,$$

$$V_2 = \frac{1}{2} \alpha r^2 \cos(\theta + \phi) + 2g^2(u) + (a + 2\lambda_2),$$

where $g$ is any solution of (36) and $f$ is any solution of $f^2 + f' = -\lambda_1$.

Now the ansatz (31) for $g$ transforms (36) into

$$-\Psi''(u) + \frac{1}{2} \alpha u^2 \Psi(u) = \lambda_2 \Psi(u),$$

which is the well known Schrödinger equation for the 1D harmonic oscillator. In that case the entire family will have 2D isotropic harmonic oscillator spectrum given by the eigenvalues

$$E_{\ell}^{(1)} = \hbar \omega (\ell + 1), \quad \ell = 0, 1, 2, \ldots,$$

which are $\ell + 1$ times degenerate for a given $\ell$. For concrete examples we recall the normalized eigenfunctions and corresponding eigenvalues of the 1D harmonic oscillator:

$$\Psi_n(u) = N_n e^{-\beta^2 u^2 / 2} H_n(\beta u), \quad E_n = \frac{\hbar^2}{2m} \lambda_{2,n} = \hbar \omega (n + \frac{1}{2}), \quad n = 0, 1, 2, \ldots,$$

where $N_n$ is the normalization constant, $H_n$ denote the Hermite polynomials and

$$\beta = \left(\frac{m \omega}{\hbar}\right)^{1/2} = \left(\frac{\alpha}{2}\right)^{1/4}, \quad N_n = \left(\frac{\beta}{\pi^{1/2} 2^n n!}\right)^{1/2}.$$

In writing Eqs. (39-41) we have restored $2m/\hbar^2$ into our notation in which the dimension of $\beta$ is $(\text{length})^{-1}$. Like $\lambda_2$, also $V_2, L_{21}$ and the function $g$ must be labelled by the quantum number $n$:

$$g_n(u) = -\frac{\Psi'_n(u)}{\Psi_n(u)} = \beta^2 u - \partial_u \ln[H_n(\beta u)],$$

$$V_2^{(n)} = \beta^4 r^2 \cos(\theta + \phi) + 2g_n^2(u) + a + 4\beta^2(n + \frac{1}{2}),$$

$$L_{21}^{(n)} = g_n(u) + \sin(\theta + \phi) \partial_r + \frac{1}{r} \cos(\theta + \phi) \partial_\theta.$$  

For the first three Hermite polynomials $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2$ we have

$$g_0 = \beta^2 u, \quad g_1 = \beta^2 u - \frac{1}{u}, \quad g_2 = \beta^2 \frac{2\beta^2 u^2 - 5}{2\beta^2 u^2 - 1} u.$$  

Considering the $f$-problem as above $L_{10}^{(k)}$ is given by (34) and $V_0$ is

$$V_0^{(k)} = \frac{1}{2} \alpha r^2 + \frac{k^2}{2r^2 \sin^2(\frac{1}{2}k \theta)} + a.$$  

(43)
VII. GENERAL FORM OF THE POTENTIALS

Returning the general discussion of Sec. V, the most general potentials are obtained by choosing, in Eq. (27), $h$ as in (35) and by postulating the equation

$$\cos(\theta + \phi)V_+'(\theta) - 2\sin(\theta + \phi)V_+ (\theta) = 2\lambda_1 \sin(\theta + \phi) - \frac{2c}{\sin^3(\theta + \phi)},$$

(44)

for $V_+$. It is not hard to check that (35) and (44) are the most general relations which make the right hand side of Eq. (27) only a function of the $u$ variable. The general solution of Eq. (44) is

$$V_+ = f^2(\theta) + f'(\theta) = \frac{b}{\cos^2(\theta + \phi)} + \frac{c}{\sin^2(\theta + \phi)} - \lambda_1,$$

(45)

where $\lambda_1, b$ and $c$ are some constants. When (35) and (44) are inserted into (27) we obtain a new Riccati’s equation for $g(u)$

$$g^2 - g' = \frac{1}{2}\alpha u^2 + \frac{c}{u^2} - \lambda_2.$$  

(46)

By virtue of (23), (28), (35), (45) and (46) the corresponding potentials can be written as

$$V_0 = \frac{1}{2}\alpha r^2 - \frac{1}{r^2}\left[\frac{b}{\cos^2(\theta + \phi)} + \frac{c}{\sin^2(\theta + \phi)} + \frac{2(f^2 + \lambda_1)}{r^2}\right] + a,$$

$$V_1 = \frac{1}{2}\alpha r^2 + \frac{1}{r^2}\left[\frac{b}{\cos^2(\theta + \phi)} + \frac{c}{\sin^2(\theta + \phi)}\right] + a,$$

$$V_2 = \frac{1}{2}\alpha r^2 \cos 2(\theta + \phi) + \frac{1}{r^2}\left[\frac{b}{\cos^2(\theta + \phi)} - \frac{c}{\sin^2(\theta + \phi)}\right] + 2(g^2 + \lambda_2 + \frac{a}{2}).$$

(47)

$V_1$ is immediately recognized as one of 2D Smorodinsky-Winternitz potentials which accepts separation of variables in the Cartesian, polar and elliptic coordinates. Being fixed in the whole family it determines the structure of spectrum of all potentials. While $V_0$ is separable in the plane polar coordinates $V_2$ is separable only in the Cartesian coordinates. $V_0$ and $V_2$ represent two families of the superintegrable and isospectral potentials generated by the functions $f$ and $g$ which are subjected to Eqs. (45) and (46). The normalized eigenfunctions, corresponding eigenvalues and the symmetry generators will be the subject of the next three sections.

Having specified the most general forms of the potentials we now show how to develop a hierarchy of the potentials.

On substituting (31) into (45) and (46) we arrive at the following two 1D problems

$$H_{PT}\psi_k(\theta) = \lambda_{1,k}\psi_k(\theta), \quad H_{SO}\Psi_n(u) = \lambda_{2,n}\Psi_n(u),$$

(48)

where $k$ and $n$ are possible quantum numbers and

$$H_{PT} = -\frac{d^2}{d\theta^2} + V_{PT}, \quad V_{PT} = \frac{b}{\cos^2(\theta + \phi)} + \frac{c}{\sin^2(\theta + \phi)},$$

$$H_{SO} = -\frac{d^2}{du^2} + V_{SO}, \quad V_{SO} = \frac{1}{2}\alpha u^2 + \frac{c}{u^2}.$$  

(49)
These are the well-known generalized Pöschl-Teller (PT) and singular oscillator (hence the subscript SO), or the radial oscillator potentials. By virtue of (28) and (31) the potentials can be rewritten as

\[
V^{(k)}_0 = V_1 - \frac{2}{r^2} \partial^2_{\theta} \ln \psi_k(\theta), \quad V^{(n)}_2 = V_1 - 2 \partial^2_u \ln \Psi_n(u). \tag{51}
\]

Here and here after we label the potentials by the quantum numbers of the associated 1D problems that generate them. (51) explicitly shows that \(V^{(k)}_0\) and \(V^{(n)}_2\) are generated from \(V_1\) by the Darboux type transformations. The functions that generate these transformations are the eigenfunctions of the associated 1D problems. This constitutes an extension of Darboux transformations for 2D problems. Another point worth emphasizing is that any solution of these 1D problems can be used in generating the potentials. But, as easily accessible results from the literature, only normalizable solutions of these problems will be presented below. From now on we take \(\phi = 0\) and in Secs. VIII and X we include \(2m/\hbar^2\) into our notation.

VIII. BOUND STATES OF THE ASSOCIATED PROBLEMS AND \(V_1\)

Provided that \(c \geq -1/4\), the bound states of \(H_{SO}\) belonging to the Hilbert space \(L^2(0, \infty)\) are given as follows [3,26–28]

\[
\Psi^\varepsilon_n(u) = N^\varepsilon_n u^{1/2 + \varepsilon} e^{-\beta^2 u^2/2} L_\nu^\varepsilon(u^2),
E_n^\varepsilon = \frac{\hbar^2}{2m} \lambda_{2n}^\varepsilon = \hbar \omega(2n + \varepsilon + 1), \quad n = 0, 1, 2, \ldots,
N^\varepsilon_n = \frac{n!2\beta^{2(1+\varepsilon)}}{\Gamma(n+\varepsilon+1)}^{1/2}, \quad \nu = \frac{1}{2}(1+4c)^{1/2},
\tag{52}
\]

where \(N^\varepsilon_n\) is the normalization constant, \(L_n^\varepsilon(z)\) are the generalized Laguerre polynomials, \(\beta\) is defined by Eq. (41), \(\Gamma\) stands for the Gamma function and \(\varepsilon = \pm\). \(\Psi^\varepsilon_n(u)\)'s satisfy the orthogonality relation [29]

\[
\int_0^\infty \Psi^\varepsilon_n(u) \Psi^\varepsilon_{n'}(u) du = \delta_{nn'},
\tag{53}
\]

which is valid for \(\varepsilon \nu > -1\). This implies that for \(c \in I = [-1/4, 3/4)\) (that is for \(-1/4 \leq c < 3/4\)) both values of \(\varepsilon = \pm\), and for \(c \geq 3/4\) only \(\varepsilon = +\) can be used for each \(n\). Although the generated potentials do not depend on the normalization constants of the associated 1D problems we write them for completeness.

From the most general point of view and in accordance with the fact that \(H_{SO}\) is parity invariant, defined parity states of \(H_{SO}\) belonging to the Hilbert space \(L^2(-\infty, \infty)\) can be given as follows [27]

\[
\Psi^\varepsilon_n(u) = \frac{1}{2^{1/2} \lambda_{n}^\varepsilon} \begin{cases} 
|u|^{1/2 + \varepsilon} e^{-\beta^2 u^2/2} L_\nu^\varepsilon(u^2); & \text{for } u \geq 0, \\
-\varepsilon |u|^{1/2 + \varepsilon} e^{-\beta^2 u^2/2} L_\nu^\varepsilon(u^2); & \text{for } u < 0.
\end{cases}
\tag{54}
\]

These obey the following orthogonality relation
\[ \int_{-\infty}^{\infty} \Psi_n^\varepsilon(u) \Psi_{n'}^\varepsilon(u) du = \delta_{nn'} \delta_{\varepsilon \bar{\varepsilon}}, \quad (55) \]

where \( \varepsilon, \bar{\varepsilon} \) may equal \( \pm \). For \( \varepsilon = \bar{\varepsilon} \) (55) follows from the orthogonality of the generalized Laguerre polynomials [29] and for \( \varepsilon \neq \bar{\varepsilon} \) from the parity reasons as can be verified directly from (54). The corresponding energy eigenvalues are given by (52). For \( c < -1/4 \) the energy spectrum is not bounded from below which implies “falling of the particle to the center” and physical interpretation is lost [27,28]. As \( c \to 0, \nu \to 1/2 \) and \( \Psi_n^\varepsilon(u) \)’s go over, for \( \varepsilon = + \) to odd parity and for \( \varepsilon = - \) to even parity harmonic oscillator wave functions. This follows from the relations between the Hermite and Laguerre polynomials [27,29]. The corresponding limits of the energy eigenvalues are obvious from (52).

The normalized eigenfunctions and corresponding eigenvalues of \( V_1 \) can now be written as

\[
\Psi_{\ell \varepsilon}^{(1)}(x,y) = \Psi_{n_1}^\varepsilon(x) \Psi_{n_2}^\varepsilon(y), \\
E_{\ell \varepsilon} = E_{n_1}^\varepsilon + E_{n_2}^\varepsilon = \hbar \omega (2\ell + \varepsilon \bar{\varepsilon} + \varepsilon \nu + 1), \\
\ell = n_1 + n_2, \quad \ell, n_1, n_2 = 0, 1, 2, \ldots
\]

where \( \bar{\nu} = (1 + 4b)^{1/2}/2 \). \( \Psi_{n_1}^\varepsilon(x), \Psi_{n_2}^\varepsilon(y) \) and \( E_{n_1}^\varepsilon, E_{n_2}^\varepsilon \) are given by (52) (or, (54)) with suitable replacements of the parameters and variables. It follows that bound states of \( V_1 \) exist for \( b, c \geq -1/4 \). For \( b, c \in I \) there are four different states for each value of \( \ell \). In the case of \( b \in I, c \geq 3/4 \), or \( c \in I, b \geq 3/4 \) there are two different states for each value of \( \ell \), and one state in the case of \( b, c \geq 3/4 \). In each case, for a given value of \( \ell \) the state with energy \( E_{\ell \varepsilon} \) is \( (\ell + 1) \)-fold degenerate. We should also note that if we require the wavefunctions to be separately continuous at the origin the interval \( I = [-1/4, 3/4] \) and the conditions \( b, c \geq 3/4 \) must be replaced as \( I = [-1/4, 0] \) and as \( b, c \geq 0 \).

The singular oscillator problem has the spectrum generating algebra \( su(1,1) = \{ J_0, J_\pm : [J_0, J_\pm] = \pm J_\pm, [J_+, J_-] = -2J_0 \} \) realized as [26,30]

\[
J_0 = \frac{H_{SO}}{4\beta^2}, \quad J_\pm = -\frac{1}{4}[\beta^2 (u \mp \beta^{-2} \partial_u)^2 - \frac{c}{\beta^2 u^2}], \quad (57)
\]

with the Casimir invariant \( C^2 = -J_+ J_- + J_0^2 - J_0 = (4c - 3)/16 \). Therefore, as will be shown in the next section, the symmetry algebra of the \( H_1 \)-problem is closely connected with this kind two commuting copies of \( su(1,1) \) algebra.

For later use it will be convenient to consider \( H_{PT} \)-problem in relation with the solution of \( H_1 \)-problem in the polar coordinates. In this case the eigenvalue equation of \( H_1 \) separates, by taking \( \Psi^{(1)}(r, \theta) = R_{k_1}(r) \psi_k(\theta) \), into the Pöschl-Teller problem given by (48) and into the radial equation

\[
[-(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho}) + \rho^2 + \frac{\lambda_{1,k}}{\rho^2}] R_{k_1}(\rho) = \lambda R_{k_1}(\rho), \quad (58)
\]

where \( \rho = \beta r \) and \( \lambda = E/\beta^2 \). In terms of \( v = \sin^2 \theta \), and \( \psi_k(\theta) = v^{\frac{1}{4}(1+\varepsilon \nu)}(1-v)^{\frac{1}{4}(1+\varepsilon \bar{\varepsilon})} F(v) \), the eigenvalue equation of \( H_{PT} \) leads us to the hypergeometric equation for \( F(v) \):

\[
v(1-v)\frac{d^2 F}{dv^2} + [\zeta - v(\gamma + \eta + 1)] \frac{dF}{dv} - \gamma \eta F = 0.
\]
The general solution of this equation is
\[ F(v) = A_2 F_1(\gamma, \eta; \zeta; v) + B v^{1-\zeta} F_1(\gamma - \zeta + 1, \eta - \zeta + 1; 2 - \zeta; v), \]
where \( A \) and \( B \) are arbitrary constants, \( F_\zeta \) denotes the hypergeometric function and
\[
\gamma = \frac{1}{2}(1 + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + \sqrt{\lambda_{1,k}}), \quad \eta = \frac{1}{2}(1 + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} - \sqrt{\lambda_{1,k}}), \quad \zeta = 1 + \varepsilon \nu.
\]

For normalizable solutions \( B \) must be zero and \( \gamma \) (or \( \eta \)) must be a negative integer, say, \(-k\). In that case the hypergeometric function goes over to Jacobi polynomials \( P^{(e\nu, e\bar{\nu})}_\mu(1 - 2v) \) and the resulting eigenfunctions and eigenvalues can be written as follows [7,31]
\[
\psi_k(\theta) = N_k^{PT} \sin^{\frac{1}{2} + \varepsilon \nu} \theta \cos^{\frac{1}{2} + \bar{\varepsilon} \bar{\nu}} \theta P^{(e\nu, e\bar{\nu})}_\mu(\cos 2\theta),
\]
\[
E_k = \frac{\hbar^2}{2m} \lambda_{1,k} = \frac{\hbar^2}{2m}(2k + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1)^2, \quad k = 0, 1, 2, \ldots,
\]
\[
N_k^{PT} = \left[ \frac{2(k + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1)\Gamma(k + 1)\Gamma(k + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1)}{\Gamma(k + \varepsilon \nu + 1)\Gamma(k + \varepsilon \nu + 1)} \right]^{1/2}.
\]

Substituting \( \lambda_{1,k} = (2k + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1)^2 \) into Eq. (58) and trying the solution \( R_{k_1}(\rho) = \rho^\mu e^{-\rho^2/2} G_{k_1}(\rho) \) we end up, for \( \mu = \sqrt{\lambda_{1,k}} = (2k + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1) \), with the equation
\[
z \frac{d^2 G_{k_1}}{dz^2} + (\mu + 1 - z) \frac{dG_{k_1}}{dz} - \frac{1}{4}[2(\mu + 1) - \lambda]G_{k_1} = 0,
\]
where \( z = \rho^2 \). Provided that \(-2(\mu + 1) - \lambda/4\) is an integer, say \( k_1 = 0, 1, 2, \ldots\), the solutions of (60) are the generalized Laguerre polynomials. Hence, the radial solutions are
\[
R_{k_1}(\rho) = N_{k_1} \rho^\mu e^{-\rho^2/2} \ell^\mu L^\mu_{k_1}(\rho^2), \quad N_{k_1} = \left[ \frac{2\Gamma(k_1 + 1)}{\Gamma(k_1 + \mu + 1)} \right]^{1/2}.
\]

One can easily verify that \( \psi_k(\theta) \)'s and \( R_{k_1}(\rho) \)'s obey the following orthogonality relations
\[
\int_0^\infty R_{k_1}(\rho) R_{k'_1}(\rho) r dr = \delta_{k_1 k'_1}, \quad \int_0^{\pi/2} \psi_k(\theta) \psi_{k'}(\theta) d\theta = \delta_{kk'}.
\]

As a result the eigenfunctions of \( H_1 \) can be written in polar coordinates as follows
\[
\Psi^{(1)\varepsilon}_{\ell}(r, \theta) = N_{k_1} N_{k_2}^{PT} (\beta \rho^\mu e^{-\beta \rho^2/2} \sin^{\frac{1}{2} + \varepsilon \nu} \theta \cos^{\frac{1}{2} + \bar{\varepsilon} \bar{\nu}} \theta L^\mu_{k_1}(\beta^2 r^2) P^{(e\nu, e\bar{\nu})}_{k_2}(\cos 2\theta),
\]
with \( \ell = k_1 + k_2; \ell, k_1, k_2 = 0, 1, 2, \ldots \), and \( \mu = (2k_2 + \varepsilon \nu + \bar{\varepsilon} \bar{\nu} + 1) \). Since \( \psi_k(\theta) \) given by (59) will be used in generating \( V^{(k)}_0 \) potentials, in writing (63) we have changed the quantum number \( k \) as \( k_2 \). Observe that a similar change \((n \to n_2)\) has been made in writing (56). Note also that the condition \(-2(\mu + 1) - \lambda/4 = k_1 \) gives the eigenvalue (56) for the \( V_1 \)-problem, with \( \ell = k_1 + k_2 \). We should also note that, as has been done in Eq. (54), the solutions (63) may be extended to all \( xy \)-plane such that they have definite parity under 2D parity transformation: \((r, \theta) \to (r, \theta + \pi)\).

Inserting \( \psi_k \) and \( \Psi_n \) into (51) explicit expressions of the potentials labelled by the quantum numbers \( k \) and \( n \) become available. Besides that presented in the Sec. VI several more special subfamilies can be identified. In doing that one should take care of the range of the parameters and the domain of definition for potentials. The bound states of \( V^{(k)}_0 \) and \( V^{(n)}_2 \) will be taken up in Sec. X after considering the symmetry generators in the next section.
As is apparent from previous two sections, the intertwining operators, symmetry generators, and the Hamiltonians $H_0, H_2$ must be labelled by the quantum numbers $(k, n)$ of the associated potentials. In terms of $e(2)$ generators the labelled intertwining operators are

$$L^{(k)}_{10} = f_k(\theta) + J, \quad L^{(k)\dagger}_{10} = f_k(\theta) - J,$$
$$L^{(n)}_{21} = g_n(u) + T_2, \quad L^{(n)\dagger}_{21} = g_n(u) - T_2. \tag{64}$$

It is easy to verify that they obey the following commutators

$$[L^{(k)}_{10}, L^{(k)\dagger}_{10}] = 2f'_k(\theta), \quad [L^{(n)}_{21}, L^{(n)\dagger}_{21}] = 2g'_n(u),$$
$$[L^{(k)}_{10}, L^{(n)}_{21}] = K^{(k,n)} - T_1, \quad [L^{(k)\dagger}_{10}, L^{(n)\dagger}_{21}] = -K^{(k,n)} + T_1, \tag{65}$$
$$[L^{(k)}_{10}, L^{(n)\dagger}_{21}] = K^{(k,n)} - T_1, \quad [L^{(k)\dagger}_{10}, L^{(n)}_{21}] = -K^{(k,n)} - T_1,$$

where

$$K^{(k,n)}_{\pm} = r \cos \theta [g'_n(u) \pm \frac{1}{r^2} f'_k(\theta)].$$

By virtue of (28) we have

$$K^{(k,n)}_+ = \frac{1}{2}r \cos \theta [H_2^{(k)} - H_0^{(n)}], \quad K^{(k,n)}_- = \frac{1}{2}r \cos \theta [H_0^{(k)} + H_2^{(n)} - 2H_1]. \tag{66}$$

It will be convenient to start with the symmetry generators of $H_1$

$$X^{(k)}_1 = L^{(k)}_{10} L^{(k)\dagger}_{10} = H_{PT} - \lambda_{1,k}, \tag{67}$$
$$Y^{(n)}_1 = L^{(n)\dagger}_{21} L^{(n)}_{21} = H_{SO} - \lambda_{2,n}, \tag{68}$$

where we have made use of $T_2 g_n(u) = g'_n(u)$. $H_{PT}$ and $H_{SO}$ are defined by (49) and (50). Second order symmetry generators of $H_0^{(k)}$ and $H_2^{(n)}$ can also be written as follows

$$X^{(k)}_0 = L^{(k)\dagger}_{10} L^{(k)}_{10} = V^{(k)} - J^2 = \tilde{H}^{(k)}_{PT} - \lambda_{1,k}, \tag{69}$$
$$X^{(n)}_2 = L^{(n)}_{21} L^{(n)\dagger}_{21} = g'^2_n + g'_n - T_2^2 = \tilde{H}^{(n)}_{SO} - \lambda_{2,n}, \tag{70}$$

where

$$\tilde{H}^{(k)}_{PT} = -\frac{d^2}{d\theta^2} + V_{PT} - 2\partial^2_\theta \ln \psi_k(\theta),$$
$$\tilde{H}^{(n)}_{SO} = -\frac{d^2}{du^2} + V_{SO} - 2\partial^2_u \ln \Psi_n(u). \tag{71}$$

These are the so called super partners of $H_{PT}$ and $H_{SO}$. As a result, the Hamiltonians of 1D auxiliary problems are, up to some constants, the second order symmetry generators of $H_1$ and their super partners are the second order symmetry generators of $H_0^{(k)}$ and $H_2^{(n)}$.

The simplest forms of remaining fourth order symmetry generators seem to be their factorized forms given by (5). Making use of (65) and (67-70) these can be expressed in a variety of ways, some of which are as follows;
\[ Y_0^{(k,n)} = L_{10}^{(k)} Y_1^{(n)} L_{10}^{(k)} = L_{10}^{(k)} H_{SO} L_{10}^{(k)} - \lambda_{2,n} X_0^{(k)}, \]
\[ = Y_1^{(n)} X_0^{(k)} - [(K^{(k,n)}_+ - T_1) L_{21}^{(n)} + L_{21}^{(n)} (K^{(k,n)}_+ + T_1)] L_{10}^{(k)}, \]
\[ = X_0^{(n)} Y_1^{(k)} - L_{10}^{(k)} [(K^{(k,n)}_+ - T_1) L_{21}^{(n)} + L_{21}^{(n)} (K^{(k,n)}_+ + T_1)], \]
\[ Y_2^{(k,n)} = L_{21}^{(n)} X_1^{(k)} L_{21}^{(n)} = L_{21}^{(n)} H_{PT} L_{21}^{(n)} - \lambda_{1,k} X_2^{(n)}, \]
\[ = X_1^{(k)} X_2^{(n)} - [(K^{(k,n)}_- + T_1) L_{10}^{(k)} T - L_{10}^{(k)} (K^{(k,n)}_+ + T_1)] L_{21}^{(n)}, \]
\[ = X_2^{(n)} X_1^{(k)} - L_{21}^{(n)} [L_{10}^{(k)} (K^{(k,n)}_- + T_1) - (K^{(k,n)}_+ - T_1) L_{10}^{(k)}]. \]

At this point we have to emphasize the followings. The existence of \( \lambda_{1,k} \) and \( \lambda_{2,n} \) as additive constants in \( X_1^{(k)} \), \( Y_1^{(n)} \) and \( X_0^{(k)} \), \( X_2^{(n)} \) seem to be redundant in regard of superintegrability of \( H_1 \). In particular, our labelling of the fourth order generators with two indices may seem as if we have more symmetries than is needed for superintegrability. But an inspection of the first lines of Eqs. (73) and (74) immediately shows that, for a given, say, \( k \) and all \( n \) the set \( \{ Y_0^{(k,n)}, X_0^{(k)} \} \) spans only a 2D vector space. As fourth order symmetries labelled with one index one may take

\[ Y_0^{(k)} \equiv Y_0^{(k,n)} + \lambda_{2,n} X_0^{(k)} = L_{10}^{(k)} H_{SO} L_{10}^{(k)}, \]
\[ Y_2^{(n)} \equiv Y_2^{(k,n)} + \lambda_{1,k} X_2^{(n)} = L_{21}^{(n)} H_{PT} L_{21}^{(n)} - \lambda_{1,k} X_2^{(n)} . \]

However, for overall consistency of the hierarchy such as intertwining of \( H_1 \) with \( H_0^{(k)} \), \( H_2^{(n)} \) and, as we will show in the next section, in determining the spectra of \( H_0^{(k)} \) and \( H_2^{(n)} \) these seemingly redundant constants and labels play an essential role.

One of virtues of our approach is that the commutativity of the symmetry generators with the corresponding Hamiltonian is guaranteed by construction from the outset. For justification we first note that the relations

\[ [H_1, X_1^{(k)}] = 0 = [H_1, Y_1^{(k)}], \quad k = 0, 1, \ldots, \]  

immediately follow from the fact that \( X_1^{(k)} \) and \( Y_1^{(k)} \) emerge from the separation of \( H_1 \) in different coordinate systems. Secondly at a glimpse of Eqs. (67-72) we observe that

\[ X_0^{(k)} = X_1^{(k)} - 2 \partial^2_\theta \ln \psi_k(\theta), \quad X_2^{(n)} = Y_1^{(n)} - 2 \partial^2_\theta \ln \Psi_n(u). \]

That is, the second order symmetry generators of \( H_0^{(k)} \) and \( H_2^{(n)} \) are Darboux transforms of symmetry generators of \( H_1 \) as are, apart from the factor \( r^{-2} \), \( H_0^{(k)} \) and \( H_2^{(n)} \) Darboux transforms of \( H_1 \) along different legs of diagram (3). In view of this fact the relations

\[ [H_0^{(k)}, X_0^{(k)}] = 0 = [H_2^{(n)}, X_2^{(n)}], \quad n, k = 0, 1, \ldots, \]  

follow from, or, in a sense, are Darboux transforms of (75). Only the explicit check of

\[ [H_0^{(k)}, Y_0^{(k,n)}] = 0 = [H_2^{(n)}, Y_2^{(k,n)}], \quad n, k = 0, 1, \ldots, \]  

takes tediously a lot of time. This shows an advantage of our method compared with the conventional approach in which much effort is devoted to verify the commutativity for
specified forms of generators. There it is known that for symmetries higher than second order, equations resulting from commutativity are almost intractable.

It is not so hard to check that \([X_j^{(i)}, Y_j^{(i)}] \neq 0, j = 0, 1, 2\), since the highest order derivatives with constant coefficients will appear at the right hand side. For example,

\[
[X_0^{(k)}, Y_0^{(k,n)}]_{\text{hot}} = [J^2, JT_2 J] = 4T_1 T_2 J^3 + 2(2T_1^2 - 2T_2^2 + T_1 T_2)J^2 - 8T_1 T_2 J,
\]

\[
[X_2^{(n)}, Y_2^{(k,n)}]_{\text{hot}} = [T_2^2, T_2 J T_2] = 2T_1^4 - 6T_2^2 T_2 - 4T_1 T_2^3 J,
\]

where \([X_0^{(k,n)}, Y_0^{(k,n)}]_{\text{hot}}\) represents only the highest order terms resulting from \([X_0^{(k)}, Y_0^{(k,n)}]\).

Therefore, the symmetry generators of each potential do not close in a finite dimensional Lie algebra structure. Note that by the Jacobi identity \(Z_j = [X_j, Y_j], j = 0, 1, 2\), is also a symmetry generator, but it is algebraically dependent to \(X_j\) and \(Y_j\).

There is an elegant way of expressing the symmetry algebra of \(H_1\). For this purpose we introduce the generators

\[
X_\pm = \frac{1}{4\beta^2}(-\partial_x^2 \pm \frac{1}{2} \alpha x^2 + \frac{b}{x^2}), \quad D_1 = \frac{1}{4}(1 + x \partial_x), \quad (78)
\]

\[
Y_\pm = \frac{1}{4\beta^2}(-\partial_y^2 \pm \frac{1}{2} \alpha y^2 + \frac{c}{y^2}), \quad D_2 = \frac{1}{4}(1 + y \partial_y), \quad (79)
\]

which obey the Lie algebras

\[
[X_\pm, D_1] = X_\mp, \quad [X_+, X_-] = D_1, \quad (80)
\]

\[
[Y_\pm, D_2] = Y_\mp, \quad [Y_+, Y_-] = D_2, \quad (81)
\]

with the Casimir invariants

\[
X_+^2 - X_-^2 + D_1^2 = \frac{4b - 3}{16}, \quad Y_+^2 - Y_-^2 + D_2^2 = \frac{4c - 3}{16}. \quad (82)
\]

It is straightforward to show that in terms of (78) and (79) we have

\[
H_1 = 4\beta^2(X_+ + Y_+),
\]

\[
X_1^{(k)} = 8(X_+ Y_+ - X_- Y_- + D_1 D_2) + K, \quad (83)
\]

\[
Y_1^{(n)} = 4\beta^2 Y_+ - \lambda_{2,n},
\]

where \(K = b + c - \lambda_{1,k} - (1/2)\). Eqs. (80-82) are defining relations of two commuting copies of a \(su(1, 1)\) algebra which can be written as a direct sum \(su(1, 1) \oplus su(1, 1)\). The basis given by (78-79) is connected with that mentioned in Sec. VIII by linear transformations, for instance, by comparing (57) and (79) we have \(Y_+ = J_0, Y_- = D_2 + J_+\). Eqs. (83) show that the symmetries of \(H_1\) are quadratic in the generators of centrally extended (because of the constant \(K\)) \(su(1, 1) \oplus su(1, 1)\) algebra.

By defining

\[
W^{(k,n)} \equiv \frac{1}{8}[X_1^{(k)}, Y_1^{(n)}] = 4\beta^2(X_- D_2 - Y_- D_1), \quad (84)
\]

one can easily show that
\[ [X^{(k)}_1, W^{(k,n)}] = \{X^{(k)}_1, Y^{(n)}_1\} + X^{(k)}_1(2\lambda_{2,n} - H_1) + (2Y^{(n)}_1 + 2\lambda_{2,n} - H_1)(\lambda_{1,k} - 1) + H_1(b - c), \tag{85} \]

\[ [Y^{(n)}_1, W^{(k,n)}] = (Y^{(n)}_1 + \lambda_{2,n})(H_1 - Y^{(n)}_1 - \lambda_{2,n}) - 2\beta^4(X^{(k)}_1 - K), \tag{86} \]

where \{,\} represents the anti commutator. These explicitly show that the extended symmetry algebra of \(H_1\) spanned by \(\{H_1, X^{(k)}_1, Y^{(n)}_1, W^{(k,n)}\}\), with the inclusion of \(W^{(k,n)}\), closes in a quadratic associative algebra for all values of \(k, n\). We also observe that this algebra is a cubic associative algebra in the enveloping algebra of the centrally extended \(su(1, 1) \oplus su(1, 1)\). Recently such finitely generated associative algebras have attracted a great deal of interest. The structure we have obtained coincides, up to some additive constants, with that presented in Ref. [32] for the Winternitz potential \(V_1\). In Ref. [30] this structure is constructed as a cubic associative algebra in which counterparts of \(X_1, Y_1\) are taken to be purely quadratic in the generators of \(su(1, 1) \oplus su(1, 1)\). We end this section by emphasizing that exploring similar algebraic structures for \(H_0^{(k)}\) and \(H_2^{(n)}\) and connection between them seems to be an important problem which deserves to be taken up in another study.

\section*{X. Bound States of \(H_0^{(K)}, H_2^{(N)}\) and Their Degeneracies}

Representing \(\Psi_{n_1}^\epsilon, \Psi_{n_2}^\epsilon\) and \(\Psi_{\ell}^{(1)\bar{\epsilon}\epsilon}\) given by (52) and (56), in the Dirac notation, respectively by the kets \(|n_1\bar{\epsilon}>, |n_2\epsilon>\) and \(|1; \ell\bar{\epsilon}\epsilon>\), we write (56) as follows

\[ |1; \ell\bar{\epsilon}\epsilon> = |n_1\bar{\epsilon}> |n_2\epsilon>, \quad \ell = n_1 + n_2. \tag{87} \]

In this notation, the corresponding isospectral states of \(H_2^{(n)}\) are

\[ |2n; \ell\bar{\epsilon}\epsilon> = L^{(n)}_{21}|1; \ell\bar{\epsilon}\epsilon>. \tag{88} \]

From (53) (or, 55), (56), (68) and (87) one can easily show that

\[ <2n; \ell\bar{\epsilon}\epsilon|2n; \ell\bar{\epsilon}\epsilon> = <1; \ell\bar{\epsilon}\epsilon|Y^{(n)}_1|1; \ell\bar{\epsilon}\epsilon>, \]

\[ = <n_2\epsilon|H_{SO}|n_2\epsilon> - \lambda_{2,n}, \tag{89} \]

\[ = 2\hbar\omega(n_2 - n), \]

where \(<|>\) represents the usual inner product of \(\mathcal{H} = L^2(R^2)\) and in the third line we have included \(2m/\hbar^2\) into the notation. Since \(\ell = n_1 + n_2\), this implies that as physically acceptable states only those with \(\ell > n\) will survive in the spectrum of \(H_2^{(n)}\). Moreover, the degeneracies of the survived states will be shifted to \(\ell - n\) since the states corresponding to \(n_2 \leq n\) can not be normalized. As a result, the normalized states of \(H_2^{(n)}\) are as follows

\[ |2n; \ell\bar{\epsilon}\epsilon> = [2\hbar\omega(n_2 - n)]^{-1/2} L^{(n)}_{21}|1; \ell\bar{\epsilon}\epsilon>, \tag{90} \]

provided that \(\ell = n_1 + n_2\) and \(n_2 > n\).

In a similar way, if we represent \(\psi_{k_2}\) and \(R_{k_1}\)'s given by (59) and (61), respectively by the kets \(|k_2\epsilon\bar{\epsilon}>\) and \(|k_1\epsilon\bar{\epsilon}>,\) the states given by (63) can be expressed as
\[ |1; \ell \bar{\varepsilon} \varepsilon> = |k_1 \bar{\varepsilon} \varepsilon> |k_2 \bar{\varepsilon} \varepsilon>, \quad \ell = k_1 + k_2. \]  

In that case the corresponding isospectral states of \( H_0^{(k)} \) are \( |0k; \ell \bar{\varepsilon} \varepsilon> = \mathcal{L}_1^{(k)\dagger} |1; \ell \bar{\varepsilon} \varepsilon> \) and by virtue of (62), (67) and (91) we have

\[
< 0k; \ell \bar{\varepsilon} \varepsilon |0k; \ell \bar{\varepsilon} \varepsilon> = < 1; \ell \bar{\varepsilon} \varepsilon |X_1^{(k)} |1; \ell \bar{\varepsilon} \varepsilon>, \]
\[
= < k_2 \bar{\varepsilon} \varepsilon |H_{PT} |k_2 \bar{\varepsilon} \varepsilon> - \lambda_{1,k}, \]
\[
= \frac{2\hbar^2}{m} (k_2 - k)(k + k_2 + \bar{\varepsilon}\nu + \varepsilon\nu + 1). \]

Hence, the normalized states of \( H_0^{(k)} \) are

\[
|0k; \ell \bar{\varepsilon} \varepsilon> = \left[ \frac{2\hbar^2}{m} (k_2 - k)(k + k_2 + \bar{\varepsilon}\nu + \varepsilon\nu + 1) \right]^{-1/2} \mathcal{L}_1^{(k)\dagger} |1; \ell \bar{\varepsilon} \varepsilon>, \]

provided that \( \ell = k_1 + k_2 \) and \( k_2 > k \). In this case the degeneracy of the state \( |0k; \ell \bar{\varepsilon} \varepsilon> \) is \( \ell - k \). Explicit functional realizations of the states (91) and (93) can easily be obtained by applying \( \mathcal{L}_2^{(n)} \), \( \mathcal{L}_1^{(k)\dagger} \) to the wave functions given by (56) and (63).

**XI. CONCLUDING REMARKS**

The method of intertwining is a unified approach widely used in various fields of physics and mathematics such as in investigating particle propagation on a curved space \([16,24,33]\), in constructing matrix-Hamiltonian to realize higher dimensional superalgebras \([17,34]\), in solving both ordinary and partial differential equations \([16]\), in generating exact solutions of non-stationary Schrödinger equation \([17,35]\), and in constructing isospectral potentials in an arbitrary space dimension \([25]\). The method we have introduced increases the power and enlarges the range of applicability of the intertwining operator idea. It allows us to perform Darboux transformations in higher dimensions in such a manner that, in addition to their isospectral deformation property they acquire integrability and superintegrability preserving property. In particular, as we have shown the realization of this method for 2D systems generates two infinite families of isospectral and superintegrable quantum systems intertwined to a 2D Winternitz system. Work on 3D realization of the method is in progress.

The space of purely second order operators quadratic in the generators of \( e(2) \) has, under the adjoint action of \( E(2) \), only four orbits whose representatives can be taken to be; \( T_1^2, J^2, J^2 + a_0 T_1^2 \) and \( T_1 J + JT_1 \), where \( a_0 \) is a constant. Existence of only four types Winternitz potentials is closely connected with this orbit structure since each corresponds to a different 2D orthogonal coordinate system \([3,24]\). \( T_1^2, J^2 \) constitute the differential parts of the symmetry generators of \( V_1 \) and account for its separation in the Cartesian and polar (hence in elliptic) coordinates. Therefore, the appearance of the Winternitz potential \( V_1 \) as the common member of two families is of no surprise; it is a direct result of our orbit prescription in constructing the intertwinings in Sec. V. We also observe that since only \( T_1^2 \) and \( J^2 \) can be factorized as \( \mathcal{L} \mathcal{L}^\dagger \) (or, as \( \mathcal{L}^\dagger \mathcal{L} \)) the other three Winternitz potentials can not be utilized as \( V_1 \) in the context of this paper. In this regard, a combination of our method and the conventional approach may be used for similar purposes. Finally we point out that what made it possible to implement Darboux transformations in our approach is that when
the eigenvalue equation of $V_1$ is separated in the Cartesian and polar coordinates, at least one of the separated equation is of the Schrödinger type.

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