Nonequilibrium Quantum Fields and the Classical Field Theory Limit

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We calculate the far-from-equilibrium dynamics and thermalization both for the quantum and the classical $O(N)$–model. The early and late-time behavior can be described from the $2PI$–loop expansion for weak couplings or the nonperturbative $2PI–1/N$ expansion of the effective action beyond leading order. A comparison with exact simulations in 1+1 dimensions in the classical limit shows that the $2PI–1/N$ expansion at next-to-leading order gives quantitatively precise results already for moderate values of $N$. We derive a criterion for the validity of the classical approximation and verify it by comparing far-from-equilibrium quantum and classical dynamics. At late times one observes the expected deviations due to the difference between classical and quantum thermal equilibrium.

In recent years we have witnessed an enormous increase of interest in the dynamics of quantum fields out of equilibrium. Strong motivation comes from a wide range of applications including current and upcoming relativistic heavy-ion collision experiments, phase transitions in the early universe or the dynamics of Bose-Einstein condensation. Directly simulating quantum fields in real time, such as solving the Schrödinger equation for the wave functional is prohibitively difficult and one has to find suitable approximations. The nonequilibrium time evolution is inherently nonperturbative in the sense that approximations based on a finite order in standard perturbation theory break down at sufficiently late times. Practicable approximations for nonequilibrium dynamics may be based on the two-particle irreducible ($2PI$) generating functional for Green’s functions [1,2]. Recently, the $2PI$ effective action has been solved for a 1+1 dimensional scalar quantum field theory at next-to-leading order in the $2PI$–loop expansion [3] and in the $2PI–1/N$ expansion [4]. Both the far-from-equilibrium early-time behavior and the late-time physics of thermalization were successfully described. For a recent review on the use of the $2PI$ effective action in nonequilibrium field theory see Ref. [5].

A unique possibility to calculate the exact time evolution, which includes all orders in loops or $1/N$, is provided by the classical statistical field theory limit. The exact evolution (up to statistical errors) of correlation functions can be constructed by numerical integration and sampling of initial conditions from a given probability distribution function. On the level of correlation functions classical and quantum evolution equations are remarkably similar\(^1\) and the same approximation schemes and initial conditions can be applied. This aspect of classical field theory has been stressed in Refs. [7–10]. In Ref. [9] this is applied to the next-to-leading order classical $\phi^4$-model. In Ref. [10] it is shown that

\(^1\)Of course, in contrast to the quantum theory the classical limit suffers from Rayleigh-Jeans divergences and has to be regulated. In 1+1 dimensions such divergences are absent in $\phi\phi$-correlation functions [6].
the 1/N expansion at next-to-leading order converges to the exact result by increasing
N already for moderate values of N. Apart from benchmarking approximation schemes
employed in quantum field theory, the classical field limit is of great practical importance
and often applied for the approximate description of nonequilibrium quantum fields. In
this note I want to elaborate on Ref. [10], done together with G. Aarts, and study the
conditions under which far-from-equilibrium quantum dynamics can be reliably described
by classical fields.

We consider a real N-component scalar quantum field theory with a λ(φ_a φ_a)/(4N)
interaction in the symmetric phase (a = 1, . . . , N). There are two linearly independent
two-point functions which can be related to the anti-commutator and commutator of two
field operators [11,4]

\[ F_{ab}(x, y) = \langle [\phi_a(x), \phi_b(y)]_+ \rangle / 2, \quad ρ_{ab}(x, y) = i \langle [\phi_a(x), \phi_b(y)]_- \rangle \]  

(1)

Here F is the “symmetric” propagator and ρ denotes the spectral function. The classical
equivalent of the spectral function is obtained by replacing the commutator by the Poisson
bracket. For the analytic presentation we consider here the three-loop expansion of the
2PI effective action for N = 1, which has been employed in Ref. [3] to study late-time
thermalization in a quantum field theory. Numerical results from the 1/N expansion of
the 2PI effective action at next-to-leading order for N > 1 are shown below. For spatially
homogeneous fields the dynamics of the Fourier transformed F and ρ is described by [11,4]

\[ \left[ \partial_{t_x}^2 + M^2(t_x) \right] F(t_x, t_y; p) = -\int_0^{t_x} dt_z \Sigma_ρ(t_x, t_x; p) F(t_z, t_y; p) \]

\[ + \int_0^{t_y} dt_z \Sigma_F(t_x, t_z; p) ρ(t_z, t_y; p), \]  

(2)

\[ \left[ \partial_{t_x}^2 + M^2(t_x) \right] ρ(t_x, t_y; p) = -\int_0^{t_x} dt_z \Sigma_ρ(t_x, t_x; p) ρ(t_z, t_y; p). \]  

(3)

which are exact for known Σ_F, Σ_ρ. From the three-loop 2PI effective action the effective
mass term is \( M^2(t_x) = m^2 + (λ/2) \int \frac{d^d k}{(2π)^d} F(t_x, t_x; q) \) and the self energies are

\[ \Sigma_F(t_x, t_y; p) = -\frac{λ^2}{6} \int \frac{d^d q}{(2π)^d} \frac{d^d k}{(2π)^d} F(t_x, t_y; p - q - k) \]

\[ \left[ F(t_x, t_y; q) F(t_x, t_y; k) - \frac{3}{4} \rho(t_x, t_y; q) ρ(t_x, t_y; k) \right], \]  

\[ \Sigma_ρ(t_x, t_y; p) = -\frac{λ^2}{2} \int \frac{d^d q}{(2π)^d} \frac{d^d k}{(2π)^d} \rho(t_x, t_y; p - q - k) \]

\[ \left[ F(t_x, t_y; q) F(t_x, t_y; k) - \frac{1}{12} \rho(t_x, t_y; q) ρ(t_x, t_y; k) \right]. \]

The classical statistical field theory limit of a scalar quantum field theory has been studied
extensively in the literature. An analysis along the lines of Refs. [12,13,9,10] shows that
all equations (2)–(3) remain the same in the classical limit except for differing expressions
for the self energy:

\[ Σ_F(t_x, t_y; p) \text{ classical limit} \Rightarrow -\frac{λ^2}{6} \int \frac{d^d q}{(2π)^d} \frac{d^d k}{(2π)^d} F(t_x, t_y; p - q - k) F(t_x, t_y; q) F(t_x, t_y; k), \]  

(4)
Figure 1. Nonequilibrium classical time evolution from the three-loop approximation of the 2PI effective action for \( \lambda/m^2 = 1 \). Shown is the mode temperature \( T'(t; q) \), as defined in the text, with \( T'(t = 0; q)/m = 5 \) [14]. The equivalent approximation for the corresponding quantum field theory has been used in Ref. [3] to demonstrate the late-time approach to quantum thermal equilibrium. In contrast, the classical field theory approaches classical thermal equilibrium as one clearly observes from the approach \( T'(t; q) \rightarrow T_{\text{cl}} \) which corresponds to classical equipartition.

One observes that the classical self energies are obtained from the expressions in the quantum theory by dropping terms with two spectral (\( \rho \)) components compared to two statistical (\( F \)) functions. In particular, it becomes obvious that the leading order equations (similarly for leading-order large \( N \), Hartree or mean field) are identical for the quantum and the classical theory, and the inclusion of direct scattering effects is crucial.

The solution of the above classical evolution equations is shown in Fig. 1 for Gaussian initial correlations with temperature \( T'(t = 0; q)/m = 5 \) and \( \lambda/m^2 = 1 \). (For the numerical implementation see Ref. [4].) The “mode temperature” is defined by [7]

\[
T'(t_x; t_y) = \partial_{t_x} \partial_{t_y} F(t_x, t_y; p)|_{t_x = t_y}. \tag{6}
\]

One observes that the system relaxes to a final temperature \( T_{\text{cl}}/m = 5.5 \). The thermalization time turns out to be very large and we find an exponential late-time relaxation to thermal equilibrium with rate \( \gamma^{(\text{therm})} \simeq 2 \times 10^{-4} \) for \( T(t; q = 0) \) [14]. We emphasize that the exponential behavior with similarly long thermalization times are found as well from exact simulations as in Ref. [7].

\[\Sigma_{\rho}(t_x, t_y; p) \xrightarrow{\text{classical limit}} -\frac{\lambda^2}{2} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{d\mathbf{k}}{(2\pi)^d} \rho(t_x, t_y; p - \mathbf{q} - \mathbf{k}) F(t_x, t_y; \mathbf{q}) F(t_x, t_y; \mathbf{k}). \tag{5}\]

For the classical theory we employ a lattice regularization with spatial lattice spacing \( ma_s = 0.4 \) corresponding to a fixed momentum cutoff \( \Lambda = \pi/a_s \). We observe that at sufficiently late times the contributions from early times to the dynamics are effectively suppressed. This fact is shown in detail in Ref. [14] and has been employed in Fig. 1 to reach the very late times.

Note that these authors employ a fixed lattice cutoff \( \pi/a_s = 4\pi m \).
The classical field approximation is expected to become a reliable description for the quantum theory if the number of field quanta in each mode is sufficiently high. Comparing Eqs. (3) and (4) one observes the sufficient condition for classical evolution:

$$F(t_x, t_y; q) F(t_x, t_y; k) \gg \frac{3}{4} \rho(t_x, t_y; q) \rho(t_x, t_y; k).$$

To obtain an estimate on the occupation numbers we define a time-dependent effective particle number $n(t; p)$ and mode energy $\epsilon_p(t)$ [11]

$$n(t_x; p) + \frac{1}{2} \equiv \left( F(t_x, t_y; p) \frac{\partial_{t_x} \partial_{t_y} F(t_x, t_y; p)}{F(t_x, t_y; p)} \right)^{1/2} |_{t_x = t_y}, \quad \epsilon_p(t_x) \equiv \left( \frac{\partial_{t_x} \partial_{t_y} F(t_x, t_y; p)}{F(t_x, t_y; p)} \right)^{1/2} |_{t_x = t_y}.$$  

With these definitions $F(t_x, t_x; p) \equiv (n(t_x; p) + 1/2)/\epsilon_p(t_x)$. Note that $\rho \equiv 0$ for $t_x = t_y$ and at unequal times we find $|\rho(t_x, t_y; p)| \lesssim \max[(\epsilon_p(t))^2 |_{t_x = t_y}]$. The latter relation has indeed to be valid for the free theory or if the weakly coupled quasiparticle picture applies, which is underlying the above particle number and mode energy definitions. Time averaged over an oscillation period $2\pi/\epsilon_p(t)$ one can translate (7) into an estimate on the lower bound for the effective particle number:

$$\left[ n(t; p) + \frac{1}{2} \right]^2 \gg \frac{3}{4} \quad \text{or} \quad n(t; p) \gg 0.37.$$
This limit agrees rather well with what is found in thermal equilibrium. For a Bose-Einstein distributed particle number \( n_T = (e^{\varepsilon/T} - 1)^{-1} \) with temperature \( T \) one finds \( n_T(\varepsilon = T) = 0.58 \), below which deviations from the classical thermal distribution become sizeable.

A similar estimate can also be obtained beyond the weak coupling regime from the \( 1/N \) expansion of the \( 2PI \) effective action at next-to-leading order. In Fig. 2 we consider the full next-to-leading order time evolution in the \( (2PI-) 1/N \) expansion \([4,10]\). The far-from-equilibrium initial particle number, \( n_0(p) \), is described by a Gaussian \( n_0(p) = \mathcal{A} \exp(-\frac{1}{2\sigma^2}(|p| - p_{ts})^2) \) peaked around \( p_{ts} = 2.5m_R \) with \( \sigma = 0.25m_R \) and \( \mathcal{A} = 4 \). We add to \( n_0 \) a thermal “background” \( n_T \) with initial temperature \( T = 4m_R \). Here \( m_R \) is the one-loop renormalized mass in vacuum \((n \equiv 0)\).

For the current initial distribution the classicality condition (6) is approximately valid for momenta \( p \lesssim 2p_{ts} \) with \( n(t = 0; p = 2p_{ts}) \approx 0.35 \) \((n(t = 0; p = p_{ts}) \approx 4.5)\) and a slightly larger final density at this momentum of about \( n(p = 2p_{ts}) \approx 0.5 \). We observe a good agreement of quantum and classical evolution for this momentum range as shown in Fig. 2. The right figure shows the time dependent inverse slope parameter \([4]\)

\[
T(t, p) \equiv -n(t, \epsilon_p)[n(t, \epsilon_p) + 1](dn/d\epsilon)^{-1}. \tag{9}
\]

This parameter is constant for a Bose-Einstein distributed particle number and remains momentum dependent for classical thermal equilibrium. In Fig. 2 we plot the function \( T(t, p) \) for \( p_{low} \approx 0 \) and \( p_{high} \approx 2p_{ts} \). Initially one observes a very different behavior of \( T(t, p) \) for the low and high momentum modes, indicating that the system is far from equilibrium. The quantum evolution approaches quantum thermal equilibrium with a momentum independent inverse slope \( T_{qm} = 4.7m_R \) to high accuracy. In contrast, in the classical limit this slope parameter remains momentum dependent and the system relaxes towards classical thermal equilibrium, as exemplified in Fig. 1.

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REFERENCES