monopole condensation in $\Omega^{1/2}(S^4)$
There have been many attempts to prove the monopole condensation in QCD [7, 8]. Unfortunately the effective action of QCD obtained from these earlier attempts has failed to establish the desired magnetic condensation, because the magnetic condensation was unstable. This instability of the magnetic condensation has been widely accepted and never been convincingly revoked. In retrospect there are many reasons why the earlier attempts have not been so successful. First the attempts to calculate the effective action of QCD were gauge dependent. In fact the separation of the magnetic background from the quantum fields were not gauge independent. So there is no way of knowing whether the desired magnetic condensation is indeed a gauge independent phenomenon. Moreover the origin of the magnetic background in the earlier attempts was completely obscure, and could not be associated to the non-Abelian monopoles. Consequently the magnetic condensation could not be interpreted as the monopole condensation. But the most serious defect was the appearance of an imaginary part in the effective action, which was due to the improper infra-red regularization. This improper infra-red regularization was the critical defect which really destroyed the magnetic condensation in the earlier attempts [7, 8]. In this paper we start from the gauge independent separation of the monopole background from the quantum fields in our calculation of the effective action. More importantly we make a proper infra-red regularization which respects the causality, and show that the causality makes our monopole condensation stable.

Recently Faddeev and Niemi have discovered the knot-like topological solitons in the Skyrme-type non-linear sigma model, and made an interesting conjecture that the Skyrme-Faddeev action could be interpreted as an effective action for QCD in the low energy limit [9, 10]. With the effective action at hand we discuss the possible connection between Skyrme-Faddeev theory and QCD. We show that indeed the two theories are closely related, and demonstrate that we can derive a generalized Skyrme-Faddeev action from the effective action of QCD.

The paper is organized as follows. In Section II we review the Abelian projection and the gauge independent decomposition of the non-Abelian potential into the restricted potential and the valence potential. In Section III we derive the integral expression of the one loop effective action of SU(2) QCD in the presence of pure monopole background, using the background field method. In Section IV we derive the integral expression of the effective action for an arbitrary constant (color) electromagnetic background, which we need to establish the stability of the monopole condensation. In Section V we obtain the effective action for the pure monopole background, and demonstrate the existence of the monopole condensation which generates a dynamical symmetry breaking in QCD. In Section VI we obtain the effective action for pure electric background, and show that the electric background generates the pair annihilation of the valence gluons. In Section VII we demonstrate the stability of the monopole condensation. We provide three independent arguments (the causality, the duality, and the perturbative expansion) which support the stability of the vacuum condensation. In Section VIII we establish a deep connection between the Skyrme-Faddeev theory and QCD, and derive a generalized Skyrme-Faddeev action from our effective action as an effective action of QCD in the infra-red limit. Finally in Section IX we discuss the physical implications of our results.

II. ABELIAN PROJECTION AND VALENCE GLUON: A REVIEW

Consider SU(2) QCD for simplicity. A natural way to identify the monopole potential is to introduce an isorotet unit vector field $n$ which selects the “Abelian” direction (i.e., the color charge direction) at each space-time point, and to decompose the connection into the restricted potential (called the Abelian projection) $\tilde{A}_\mu$ which leaves $n$ invariant and the valence potential $\tilde{X}_\mu$ which forms a covariant vector field [2, 3],

$$\tilde{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \tilde{X}_\mu = A_\mu + \tilde{X}_\mu,$$

$$\hat{n} \times \hat{n} = 0,$$  \hspace{1cm} (1)

where $\hat{n} = \hat{n} \cdot \tilde{X}_\mu$ is the “electric” potential. Notice that the restricted potential is precisely the connection which leaves $n$ invariant under the parallel transport,

$$D_\mu \hat{n} = \partial_\mu \hat{n} + g A_\mu \times \hat{n} = 0.$$  \hspace{1cm} (2)

Under the infinitesimal gauge transformation

$$\delta n = -\tilde{\alpha} \times n, \quad \delta \tilde{A}_\mu = \frac{1}{g} D_\mu \tilde{\alpha},$$  \hspace{1cm} (3)

one has

$$\delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \tilde{\alpha}, \quad \delta A_\mu = \frac{1}{g} D_\mu \tilde{\alpha},$$

$$\delta \tilde{X}_\mu = -\tilde{\alpha} \times \tilde{X}_\mu.$$  \hspace{1cm} (4)

This shows that $\tilde{A}_\mu$ by itself describes an SU(2) connection which enjoys the full SU(2) gauge degrees of freedom. Furthermore $\tilde{X}_\mu$ transforms covariantly under the gauge transformation. Most importantly, the decomposition is gauge-independent. Once the color direction $n$ is selected, the decomposition is fixed independent of the choice of a gauge. Our decomposition, which has recently become known as the Cho decomposition [10] or Cho-Faddeev-Niemi decomposition [11], was first introduced long time ago in an attempt to demonstrate the monopole condensation in QCD [2, 3]. But only recently the importance of
the decomposition in clarifying the non-Abelian dynamics has been appreciated by many authors [10, 11]. Indeed it is this decomposition which has played a crucial role to establish the Abelian dominance in Wilson loops in QCD [12], and the possible connection between the Skyrme-Faddeev action and the effective action of QCD [13, 14].

To understand the physical meaning of our decomposition notice that the restricted potential $A_\mu$ actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as

\[ F_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu}) n, \]

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

\[ H_{\mu\nu} = -\frac{1}{g} n \cdot (\partial_\mu n \times \partial_\nu n) = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu, \]

where $\tilde{C}_\mu$ is the “magnetic” potential [2, 3]. Notice that we can always introduce the magnetic potential (at least locally section-wise), because $H_{\mu\nu}$ is closed

\[ \partial_\mu H_{\mu\nu} = 0 \quad (H_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} H_{\rho\sigma}). \]

This allows us to identify the non-Abelian magnetic potential by

\[ \tilde{C}_\mu = -\frac{1}{g} \tilde{n} \times \partial_\mu \tilde{n}, \]

in terms of which the magnetic field is expressed as

\[ \tilde{H}_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu + g \tilde{C}_\mu \times \tilde{C}_\nu \]

\[ = -g \tilde{C}_\mu \times \tilde{C}_\nu - \frac{1}{g} \partial_\mu \tilde{n} \times \partial_\nu \tilde{n} \]

\[ = H_{\mu\nu} n. \]

Another important feature of $A_\mu$ is that, as an $SU(2)$ potential, it retains all the essential topological characteristics of the original non-Abelian potential. This is because the topological field $\tilde{n}$ can naturally describe the non-Abelian topology $\pi_2(S^3)$ and $\pi_2(S^3) \simeq \pi_2(S^3)$. Clearly the isolated singularities of $\tilde{n}$ defines $\pi_2(S^3)$ which describes the non-Abelian monopoles. Indeed $A_\mu$ with $A_\mu = 0$ and $\tilde{n} = \tilde{r}$ (or equivalently, $\tilde{C}_\mu$ with $\tilde{n} = \tilde{r}$) describes precisely the Wu-Yang monopole [15, 16]. Besides, with the $S^3$ compactification of $R^3$, $\tilde{n}$ characterizes the Hopf invariant $\pi_2(S^3) \simeq \pi_2(S^3)$ which describes the topologically distinct vacua [17, 18]. This tells that the restricted gauge theory made of $A_\mu$ could describe the dual dynamics which should play an essential role in $SU(2)$ QCD [2, 12, 19].

With (1) we have

\[ \tilde{F}_{\mu\nu} = F_{\mu\nu} + D_\mu \tilde{X}_\nu - D_\nu \tilde{X}_\mu + g \tilde{X}_\mu \times \tilde{X}_\mu, \]

so that the Yang-Mills Lagrangian is expressed as

\[ \mathcal{L} = -\frac{1}{4} \tilde{F}_{\mu\nu}^2 \]

\[ = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} (D_\mu \tilde{X}_\nu - D_\nu \tilde{X}_\mu)^2 \]

\[ -\frac{g}{2} F_{\mu\nu} \cdot (\tilde{X}_\mu \times \tilde{X}_\nu) - \frac{g}{4} (\tilde{X}_\mu \times \tilde{X}_\nu)^2 \]

\[ + \lambda n^2 n \cdot \tilde{X}_m n, \]

where $\lambda$ and $\lambda_\mu$ are the Lagrangian multipliers. From the Lagrangian we have

\[ \partial_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) = -g n \cdot (\tilde{X}_\mu \times \tilde{X}_\nu), \]

\[ D_\mu (D_\mu \tilde{X}_\nu - D_\nu \tilde{X}_\mu) = g (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) n \times \tilde{X}_\mu. \]

Notice that here $n$ has no equation of motion even though the Lagrangian contains it explicitly. This implies that it is not a local degrees of freedom, but a topological degrees of freedom [19]. From this we conclude that the non-Abelian gauge theory can be viewed as a restricted gauge theory made of the restricted potential, which has an additional colored source made of the valence gluon.

Obviously the Lagrangian (10) is invariant under the active gauge transformation (3). But notice that the decomposition introduces a new gauge symmetry that we call the passive gauge transformation [13, 19],

\[ \delta n = 0, \quad \delta \tilde{A}_\mu = \frac{1}{g} D_\mu \tilde{a}, \]

under which we have

\[ \delta \tilde{A}_\mu = \frac{1}{g} (\tilde{n} \times D_\mu \tilde{a}) n, \quad \tilde{A}_\mu = \frac{1}{g} (\tilde{n} \times D_\mu \tilde{a}) n, \]

\[ \delta \tilde{X}_\mu = \frac{1}{g} D_\mu \tilde{a} - (\tilde{n} \times D_\mu \tilde{a}) n. \]
This is because, for a given $\tilde{A}_\mu$, one can have infinitely many different decomposition of (1), with different $A_\mu$ and $\tilde{X}_\mu$ by choosing different $n$. Equivalently, for a fixed $n$, one can have infinitely many different $\tilde{A}_\mu$ which are gauge-equivalent to each other. So it must be clear that with our decomposition we automatically have another type of gauge invariance which comes from different choices of decomposition. This extra gauge invariance plays the crucial role in quantizing the theory [19].

Another advantage of the decomposition (1) is that it can actually “Abelianize” (or more precisely “dualize”) the non-Abelian dynamics [2, 12, 19]. To see this let $(n_1, n_2, n)$ be a right-handed orthonormal basis and let

$$\tilde{X}_\mu = X^{1}_\mu n_1 + X^{2}_\mu n_2,$$

$$(X^{1}_\mu = n_1 \cdot \tilde{X}_\mu, \quad X^{2}_\mu = n_2 \cdot \tilde{X}_\mu)$$

and find

$$D_\mu X_\nu = \left[ \partial_\mu X_\nu + g(A_\mu + \tilde{C}_\mu)X_\nu \right] n_1$$

$$+ \left[ \partial_\nu X_\mu + g(A_\mu + \tilde{C}_\mu)X_\mu \right] n_2.$$  \hspace{1cm} (15)

So with

$$B_\mu = A_\mu + \tilde{C}_\mu,$$


$$X_\mu = \frac{1}{\sqrt{2}} (X^1_\mu + iX^2_\mu),$$  \hspace{1cm} (16)

one could express the Lagrangian explicitly in terms of the dual potential $B_\mu$ and the complex vector field $X_\mu$,

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu} + H_{\mu\nu})^2 - \frac{1}{2} \left| D_\mu X_\nu - D_\nu X_\mu \right|^2$$

$$+ ig (F_{\mu\nu} + H_{\mu\nu}) X^*_\mu X_\nu$$

$$\frac{1}{2} \left| (X^*_\mu X_\mu)^2 - (X^*_\mu)^2 (X_\mu)^2 \right|, \hspace{1cm} (17)$$

where now

$$D_\mu X_\nu = (\partial_\mu + ig B_\mu)X_\nu.$$  \hspace{1cm} 

Clearly this describes an Abelian gauge theory coupled to the charged vector field $X_\mu$. But the important point here is that the Abelian potential $B_\mu$ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. In this form the equations of motion (11) is re-expressed as

$$\partial_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) = ig X^*_\mu (D_\mu X_\nu - D_\nu X_\mu) - ig X_\mu (D_\mu X_\nu - D_\nu X_\mu)^*,$$

$$D_\mu (D_\mu X_\nu - D_\nu X_\mu) = ig X_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}).$$ \hspace{1cm} (18)

where now

$$X_{\mu\nu} = -ig(X^*_\mu X_\nu - X^*_\nu X_\mu).$$

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. The remarkable change in this “Abelian” formulation is that here the topological field $n$ is replaced by the magnetic potential $C_\mu$.

But notice that here we have never fixed the gauge to obtain this Abelian formalism, and one might ask how the non-Abelian gauge symmetry is realized in this “Abelian” theory. To discuss this let

$$\tilde{a} = a_1 n_1 + a_2 n_2 + \theta n,$$

$$a = \frac{1}{\sqrt{2}} (a_1 + i a_2),$$

$$\tilde{C}_\mu = -\frac{1}{g} n \times \partial_\mu n = -C^1_\mu n_1 - C^2_\mu n_2,$$

$$C_\mu = \frac{1}{\sqrt{2}} (C^1_\mu + i C^2_\mu).$$ \hspace{1cm} (19)

Then the Lagrangian (17) is invariant not only under the active gauge transformation (3) described by

$$\delta A_\mu = \frac{1}{g} \partial_\mu \theta - i (C^*_\mu \alpha - C_\mu \alpha^*), \quad \delta C_\mu = -\delta A_\mu,$$

$$\delta X_\mu = 0,$$ \hspace{1cm} (20)

but also under the passive gauge transformation (13) described by

$$\delta A_\mu = \frac{1}{g} \partial_\mu \theta - i (X^*_\mu \alpha - X_\mu \alpha^*), \quad \delta C_\mu = 0,$$

$$\delta X_\mu = \frac{1}{g} D_\mu \alpha - i \theta X_\mu.$$ \hspace{1cm} (21)

This tells that the “Abelian” theory not only retains the original gauge symmetry, but actually has an enlarged (both the active and passive) gauge symmetries. But we emphasize that this is not the “naive” Abelianization of the SU(2) gauge theory which one obtains by fixing the gauge. Our Abelianization is a gauge independent Abelianization. Besides, here the Abelian gauge group
III. MONOPOLE BACKGROUND

With this preparation we will now derive the integral expression of the one-loop effective action in the presence of the pure monopole background $\tilde{C}_\mu$. To do this we resort to the background field method [20, 21]. So we first divide the gauge potential $\tilde{A}_\mu$ into two parts, the slow-varying classical part $\tilde{A}_\mu^{(c)}$ and the fluctuating quantum part $\tilde{A}_\mu^{(q)}$, and identify the magnetic potential $\tilde{C}_\mu$ as the classical background [13, 19],

$$\tilde{A}_\mu = \tilde{A}_\mu^{(c)} + \tilde{A}_\mu^{(q)},$$

$$\tilde{A}_\mu^{(c)} = \tilde{C}_\mu, \quad \tilde{A}_\mu^{(q)} = A_\mu \bar{n} + \bar{X}_\mu. \quad (22)$$

With this we introduce two types of gauge transformations, the background gauge transformation and the physical gauge transformation. Naturally we identify the background gauge transformation as

$$\delta \tilde{C}_\mu = \frac{1}{g} \tilde{D}_\mu \tilde{\alpha},$$

$$\delta (A_\mu \bar{n} + \bar{X}_\mu) = -\tilde{\alpha} \times (A_\mu \bar{n} + \bar{X}_\mu), \quad (23)$$

where now $\tilde{D}_\mu$ is defined with only the background potential $\tilde{C}_\mu$

$$\tilde{D}_\mu = (\partial_\mu + g \tilde{C}_\mu \times). \quad (24)$$

As for the physical gauge transformation which leaves the background potential invariant, we must have

$$\delta \tilde{C}_\mu = 0, \quad \delta (A_\mu \bar{n} + \bar{X}_\mu) = \frac{1}{g} D_\mu \tilde{\alpha}. \quad (25)$$

Notice that both (23) and (25) respect the original gauge transformation,

$$\delta \tilde{A}_\mu = \frac{1}{g} D_\mu \tilde{\alpha}. \quad (26)$$

Now, we fix the gauge by imposing the following gauge condition to the quantum fields,

$$\tilde{F}_\mu = D_\mu (A_\mu \bar{n} + \bar{X}_\mu) = 0,$$

$$\mathcal{E}_{gf} = \frac{1}{2\xi} \left[ (\partial_\mu A_\nu)^2 + (D_\mu \bar{X}_\nu)^2 \right]. \quad (27)$$

The corresponding Faddeev-Popov determinant is given by

$$M_{FP}^{ab} = \frac{\delta \tilde{F}_{\mu} \delta F_{\mu}}{\delta \tilde{\alpha} \delta \alpha} = (\tilde{D}_\mu D_\mu)^{ab}. \quad (28)$$

With this gauge fixing the effective action takes the following form,

$$\exp \left[ i S_{eff}(\tilde{C}_\mu) \right] = \int D A_\mu D \bar{X}_\mu D \bar{\alpha} D \bar{\alpha}^* \exp \left\{ i \int \left[ \frac{g^2}{4} F_{\mu\nu} (\bar{X}_\mu \times \bar{X}_\nu)^2 - \frac{g^2}{4} (\bar{X}_\mu \times \bar{X}_\nu)^2 + \bar{\alpha} \times \bar{\alpha}^* \tilde{D}_\mu D_\mu \bar{\alpha}^* - \frac{1}{2\xi} (\partial_\mu A_\nu)^2 - \frac{1}{2\xi} (D_\mu \bar{X}_\nu)^2 \right] d^4 x \right\}, \quad (29)$$

where $\bar{\alpha}$ and $\bar{\alpha}^*$ are the ghost fields. Notice that the effective action (29) is explicitly invariant under the background gauge transformation (3), if we add the following gauge transformation of the ghost fields to (3),

$$\delta \bar{\alpha} = -\alpha \times \bar{\alpha}, \quad \delta \bar{\alpha}^* = -\alpha \times \bar{\alpha}^*. \quad (30)$$

This guarantees that the resulting effective action we obtain after the functional integral should be invariant under the remaining background gauge transformation which involves only $\tilde{C}_\mu$. This, of course, is the advantage of the background field method which greatly simplifies the calculation of the effective action [20, 21].

Now, we can perform the functional integral in (29). Remember that in one loop approximation only the terms quadratic in quantum fields become relevant in the functional integral. So the $A_\mu$ integration becomes trivial, and the $\bar{X}_\mu$ and ghost integrations result in the following functional determinants (with $\xi = 1$),

$$\det \tilde{F}_{\mu\nu}^{ab} \simeq \det \left[ -g_{\mu\nu} (\tilde{D}\tilde{D})^{ab} - 2g H_{\mu\nu\rho} \epsilon^{abc} \tilde{c}_\rho \right],$$

$$\det M_{FP}^{ab} \simeq \det (-\tilde{D}\tilde{D})^{ab}. \quad (31)$$
One can simplify the determinant $K$ [13, 22]

\[
\ln \det \frac{1}{\sqrt[3]{2}} K = -\frac{1}{2} \ln \det \left[ (-\bar{D} D)^{ab} + i\sqrt{2}g H e^{ibc \eta^c} \right] - \frac{1}{2} \ln \det \left[ (-\bar{D} D)^{ab} - i\sqrt{2}g H e^{ibc \eta^c} \right] - \ln \det (-\bar{D} D)^{ab},
\]

where

\[
H = \sqrt{\bar{H}^2_{\mu \nu}}.
\]

With this the one loop contribution of the functional determinants to the effective action can be written as

\[
\Delta S = i \ln \det (-\bar{D}^2 + \sqrt{2}g H) + i \ln \det (-\bar{D}^2 - \sqrt{2}g H),
\]

where now $\bar{D}_\mu$ acquires the following Abelian form

\[
\bar{D}_\mu = \partial_\mu + igC_\mu.
\]

Remarkably the functional determinants (33) acquire the Abelian form. This, of course, is precisely due to the fact that our decomposition (1) Abelinizes QCD. But we emphasize again that this Abelianization is gauge independent.

One can evaluate the functional determinants in (33) with the Fock-Schwinger proper time method, and for a constant background $H$ we find

\[
\Delta \mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty dt \frac{g H / \sqrt{2} \mu^2}{\sinh(g H t / \sqrt{2} \mu^2)} \left[ \exp(\sqrt{2} g H t / \mu^2) + \exp(\sqrt{2} g H t / \mu^2) \right],
\]

where $\mu$ is a dimensional parameter.

IV. ARBITRARY BACKGROUND

Before we evaluate the above integral and establish the monopole condensation we now derive the integral expression of the one-loop effective action in the presence of arbitrary background $A_\mu$, which we need to establish the stability of the monopole condensation. So we repeat the above procedure, but now replacing the monopole background $\bar{C}_\mu$ by the restricted potential $\bar{A}_\mu$. So we first divide the gauge potential $\bar{A}_\mu$ into two parts, and now identify the restricted potential $\bar{A}_\mu$ as the classical background,

\[
\bar{A}_\mu = \bar{A}^{(c)}_\mu + \bar{A}^{(q)}_\mu,
\]

\[
\bar{A}^{(c)}_\mu = \bar{A}_\mu, \quad \bar{A}^{(q)}_\mu = \bar{X}_\mu.
\]

With this we introduce two types of gauge transformations, the background gauge transformation and the physical gauge transformation. Naturally we identify the gauge transformation (3) as the background gauge transformation. As for the physical gauge transformation which leaves the background potential invariant, we must have

\[
\delta \bar{A}_\mu = 0, \quad \delta \bar{X}_\mu = \frac{1}{g} D_\mu \bar{\alpha}.
\]

Again notice that both (3) and (37) respect the original gauge transformation (26). Now, we fix the gauge by imposing the following gauge condition to the quantum field [19, 22],

\[
\bar{F} = D_\mu \bar{X}_\mu = 0,
\]

\[
\mathcal{L}_{gf} = -\frac{1}{2g^2} (D_\mu \bar{X}_\mu)^2.
\]

The corresponding Faddeev-Popov determinant is given by

\[
\mathcal{M}^{ab}_{FP} = \frac{\delta F^a}{\delta \bar{A}^b} = (D_\mu D_\mu)^{ab}.
\]

With this gauge fixing the effective action takes the following form,

\[
\exp \left[ i S_{eff}(\bar{A}_\mu) \right] = \int \mathcal{D} \bar{X}_\mu \mathcal{D} \bar{\alpha} \mathcal{D} \bar{\alpha}^* \exp \left\{ i \int \left[ -\frac{1}{4} F_{\mu \nu}^2 - \frac{1}{4} (D_\mu \bar{X}_\nu - D_\nu \bar{X}_\mu)^2 \right] \right\}
\]

\[
-\frac{g}{2} F_{\mu \nu} \cdot (\bar{X}_\mu \times \bar{X}_\nu) - \frac{g^2}{4} (\bar{X}_\mu \times \bar{X}_\nu)^2 + \bar{\alpha}^* D_\mu D_\mu \bar{\alpha} - \frac{1}{2g^2} (D_\mu \bar{X}_\mu)^2 \right\} d^4 \bar{x}.
\]

Notice again that, with (30), the effective action (40) is explicitly invariant under the background gauge transformation (3) which involves only $A_\mu$. 
Now, we can perform the functional integral. To do this let the background field $F_{\mu\nu}$ be

$$F_{\mu\nu} = G_{\mu\nu}\hat{n},$$
$$G_{\mu\nu} = F_{\mu\nu} + H_{\mu\nu}.$$

Since, in one loop approximation only the terms which are quadratic in the quantum fields are relevant to the functional integral, we find that the $\bar{X}_\mu$ and ghost integrations result in the following functional determinants (with $\xi = 1$),

$$\text{Det}^{-\frac{1}{2}} K_{\mu\nu}^{ab} \simeq \text{Det}^{-\frac{1}{2}} \left[ -g_{\mu\nu}(DD)^{ab} - 2gG_{\mu\nu}\epsilon^{abc}\eta^{c}\right],$$

$$\text{Det}M_{FP} = \text{Det} \left[ - (DD)^{ab} \right]. \quad (41)$$

where now $D_\mu$ is defined with an arbitrary background field $A_\mu$. Using the relation

$$G_{\mu\alpha}G_{\nu\beta}G_{\alpha\beta} = \frac{1}{2}G^2G_{\mu\nu} + \frac{1}{2}(GG\bar{G})G_{\mu\nu}$$
$$\left(\bar{G}_{\mu\nu} = \frac{1}{2}G_{\mu\nu}\eta^{\rho\sigma}G_{\rho\sigma}\right), \quad (42)$$

one can simplify the functional determinants of the valence gluon and the ghost loops to the following Abelian form,

$$\ln \text{Det}^{-\frac{1}{2}} \left[ -g_{\mu\nu}(DD)^{ab} - 2gG_{\mu\nu}\epsilon^{abc}\eta^{c}\right]$$

$$= \ln \text{Det} \left[ (-\bar{D}^2 + 2a)(-\bar{D}^2 - 2a) \right]$$
$$(-\bar{D}^2 - 2ib)(-\bar{D}^2 + 2ib),$$

$$\ln \text{Det}M_{FP} = 2\ln \text{Det}(-\bar{D}^2), \quad (43)$$

where now $\bar{D}_\mu$ is defined with an arbitrary background $A_\mu + \bar{C}_\mu$,

$$\bar{D}_\mu = \partial - ig(A_\mu + \bar{C}_\mu), \quad (44)$$

and

$$a = \frac{g}{2}\sqrt{G^2 + (G\bar{G})^2 + G^2},$$

$$b = \frac{g}{2}\sqrt{G^2 + (G\bar{G})^2 - G^2}. \quad (45)$$

Notice that in the Lorentz frame where the electric field becomes parallel to the magnetic field, $a$ becomes purely magnetic and $b$ becomes purely electric.

From this we have

$$\Delta S = i\ln \text{Det} \left[ (-\bar{D}^2 + 2a)(-\bar{D}^2 - 2a) \right]$$
$$+ i\ln \text{Det} \left[ (-\bar{D}^2 - 2ib)(-\bar{D}^2 + 2ib) \right]$$
$$- 2i\ln \text{Det}(-\bar{D}^2). \quad (46)$$

We can evaluate the functional determinant, and for a general background with arbitrary $a$ and $b$, the contribution of the gluon and ghost loops is given by [22]

$$\Delta \mathcal{L} = \frac{1}{16\pi^3} \int_0^\infty dt \frac{a\beta t^2}{\beta^2 - \sin^2(\alpha t/\mu^2)}$$

$$\left[ \exp(-2at/\mu^2) + \exp(2at/\mu^2) \right]$$

$$+ \exp(2ibt/\mu^2) + \exp(-2ibt/\mu^2) - 2. \quad (47)$$

The integral expression (47) of the effective action has been known for some time [8], but the actual integration of it is not easy to perform. Indeed, as far as we understand, the integration has never been evaluated correctly. This is because the integral contains (not only the usual ultra-violet divergence around $t \simeq 0$) a severe infra-red divergence around $t \simeq \infty$, which has to be regularized correctly. In the following we will perform the integral for pure magnetic and pure electric backgrounds separately.

V. MONOPOLE CONDENSATION

For the pure monopole background the integral (35) reduces to

$$\Delta \mathcal{L} = \Delta \mathcal{L}_+ + \Delta \mathcal{L}_-,$$

$$\Delta \mathcal{L}_+ = \frac{1}{16\pi^3} \int_0^\infty dt \frac{a/\mu^2}{\beta^2 - \sin^2(\alpha t/\mu^2)} \exp(-2at/\mu^2)$$

$$\Delta \mathcal{L}_- = \frac{1}{16\pi^3} \int_0^\infty dt \frac{a/\mu^2}{\beta^2 - \sin^2(\alpha t/\mu^2)} \exp(2at/\mu^2), \quad (48)$$

where

$$a = \frac{gB}{\sqrt{2}}.$$

Notice that this is precisely the same integral that we obtain from (47) for the pure magnetic background (i.e.,
for \( b = 0 \). This tells that the evaluation of the effective action for an arbitrary magnetic background becomes mathematically identical to that for the pure monopole background.

As we have remarked both integrals have the usual ultra-violet divergence at the origin, but the second integral has a severe infra-red divergence at the infinity. To find the correct infra-red regularization, one must understand the origin of the divergence. The infra-red divergence can be traced back to the magnetic moment interaction of the gluons that we have in (10), which is also well-known to be responsible for the asymptotic freedom [2,3]. This magnetic interaction generates negative eigenvalues in \( \text{Det} \, K \) in the long distance region, which cause the infra-red divergence. More precisely when the momentum \( k \) of the gluon parallel to the background magnetic field becomes smaller than the background field strength (i.e., when \( k^2 < a \)), the lowest Landau level gluon eigenfunction whose spin is parallel to the magnetic field acquires an imaginary energy and thus becomes tachyonic. It is these unphysical tachyonic states which cause the infra-red divergence. So one must exclude these tachyonic modes in the calculation of the effective action, when one makes a proper infra-red regularization. Including the tachyons in the physical spectrum will surely destabilize QCD and make it ill-defined.

The correct infra-red regularization is dictated by the causality. To implement the causality in (48) we first go to the Minkowski time with the Wick rotation, and find

\[
\Delta \mathcal{L}_+ = -\frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^2} \, \frac{a/\mu^2}{\sinh(at/\mu^2)} \exp(-2iat/\mu^2),
\]
\[
\Delta \mathcal{L}_- = \frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^2} \, \frac{a/\mu^2}{\sinh(at/\mu^2)} \exp(+2iat/\mu^2). \tag{49}
\]

In this form the infra-red divergence has disappeared, but now we face the ambiguity in choosing the correct contours of the integrals in (49). Fortunately this ambiguity can be resolved by the causality. To see this notice that the two integrals \( \Delta \mathcal{L}_+ \) and \( \Delta \mathcal{L}_- \) originate from the two determinants in (33), and the standard causality argument requires us to identify \( 2a \) in the first determinant as \( 2a - i\epsilon \) but in the second determinant as \( 2a + i\epsilon \). This tells that the poles in the first integral in (49) should lie above the real axis, but the poles in the second integral should lie below the real axis. From this we conclude that the contour in \( \Delta \mathcal{L}_+ \) should pass below the real axis, but the contour in \( \Delta \mathcal{L}_- \) should pass above the real axis. With this causality requirement the two integrals become complex conjugate to each other, which guarantees that \( \Delta \mathcal{L} \) is explicitly real, without any imaginary part. This removes the infra-red divergence. We emphasize that this causality for the infra-red regularization is precisely the same causality that determines the Feynman propagators in field theory. With this observation we finally have

\[
\Delta \mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^2} \, \frac{a/\mu^2}{\sinh(at/\mu^2)} \left[ \exp(-2iat/\mu^2) + \exp(-2iat/\mu^2) \right], \tag{50}
\]

where now \( \epsilon \) is the ultra-violet cutoff which we have introduced to regularize the ultra-violet divergence.

Now we can perform the integral, and obtain

\[
\Delta \mathcal{L} = \frac{11a^2}{48\pi^2} \left( \frac{1}{\epsilon - \gamma} - \frac{1}{\epsilon - \gamma} \ln \frac{a}{\mu^2} - c \right),
\]

where

\[
\epsilon = 1 - \ln 2 - \frac{24\zeta(-1, \frac{3}{2})}{11\pi^2} = 0.94556..., \tag{51}
\]

with \( \zeta(x, y) \) is the generalized Hurwitz zeta function. So with the ultra-violet regularization by modified minimal subtraction we finally obtain the following effective Lagrangian [13, 22]

\[
\mathcal{L}_{\text{eff}} = \frac{1}{16\pi^2} a^2 - \frac{11a^2}{48\pi^2} \left( \ln \frac{a}{\mu^2} - \epsilon \right). \tag{52}
\]

This completes our derivation of the one-loop effective Lagrangian of \( SU(2) \) QCD in the presence of the monopole background. Notice that, as expected, the effective Lagrangian is explicitly invariant under the background gauge transformation (23) which involves only \( \bar{\psi} \).

As we have indicated, there is another way to obtain the effective action which is more physical. Remember that the two integrals in (48) come from the two determinants in (33), and the infra-red divergence in the second integral comes from the tachyonic modes contained in the second determinant in (33). So one can calculate the effective action by calculating the determinant correctly. Now, to evaluate the determinant one is supposed to use a complete set of eigenfunctions which is made of the physical states. But obviously the tachyonic modes can not be regarded as physical, because they violate the causality. A remarkable point is that by calculating the determinants with the physical states one can show that the second determinant become identical to the first one. This means that, by calculating the functional determi-
nants correctly, one can obtain exactly the same effective action that we have obtained with the infra-red regularization by causality. This provides another justification of our effective action (52).

Now we are ready to establish the monopole condensation. To do this we renormalize the effective action first. For this notice that the effective action provides the following non-trivial effective potential

\[ V = \frac{1}{4} H^2 \left[ 1 + \frac{11g^2}{24\pi^2} \left( \ln \frac{gH}{\mu^2} - c_1 \right) \right], \tag{53} \]

where

\[ c_1 = 1 - \frac{1}{2} \ln 2 - \frac{24}{11} \zeta(-1, \frac{3}{2}) = 1.29241\ldots. \]

So we can define the running coupling \( \tilde{g} \) by [7]

\[ \frac{\partial^2 V}{\partial H^2}_{H=\tilde{g}^2} = \frac{1}{2} g^2 \tilde{g}^2. \tag{54} \]

With the definition we obtain

\[ \frac{1}{\tilde{g}^2} = \frac{1}{g^2} + \frac{11}{24\pi^2} \left( \ln \frac{g\tilde{g}^2}{\mu^2} - c_1 + \frac{3}{2} \right), \tag{55} \]

from which we obtain the following \( \beta \)-function,

\[ \beta(\tilde{g}) = -\frac{11}{24\pi^2} \tilde{g}^3. \tag{56} \]

This is exactly the same \( \beta \)-function that one obtained from the perturbative QCD to prove the asymptotic freedom [23]. This confirms that our effective action is consistent with the asymptotic freedom.

The fact that the \( \beta \)-function obtained from the effective action becomes identical to the one obtained by the perturbative calculation is really remarkable, because this is not always the case. In fact in QED it has been demonstrated that the running coupling and \( \beta \)-function obtained from the effective action is different from those obtained from the perturbative method [24, 25].

In terms of the running coupling the renormalized potential is given by

\[ V_{\text{ren}} = \frac{1}{4} H^2 \left[ 1 + \frac{11}{24\pi^2} \tilde{g}^3 \left( \ln \frac{H}{\tilde{g}^2} - \frac{3}{2} \right) \right], \tag{57} \]

which generates a non-trivial local minimum at

\[ < H > = \tilde{g}^2 \exp \left( -\frac{24\pi^2}{11\tilde{g}^2} + 1 \right). \tag{58} \]

Notice that with \( \tilde{\alpha}_s = 1 \) we have

\[ < H > = 0.13819\ldots. \tag{59} \]

This is nothing but the desired magnetic condensation. This proves that the one loop effective action of QCD in the presence of the constant magnetic background does generate a dynamical symmetry breaking through the monopole condensation [13, 22].

The corresponding effective potential is plotted in Fig.1, where we have assumed \( \tilde{\alpha}_s = 1, \tilde{g} = 1 \). The effective potential clearly shows that there is indeed a dynamical symmetry breaking in QCD.

The renormalization group invariance of the effective action is guaranteed by the Callan-Symanzik equation

\[ \left( \tilde{g} \frac{\partial}{\partial \tilde{g}} + \beta(\tilde{g}) \frac{\partial}{\partial \tilde{g}} - \gamma(\tilde{g}) \tilde{g} \frac{\partial}{\partial \tilde{g}} \right) V_{\text{ren}} = 0, \tag{60} \]

where \( \gamma(\tilde{g}) \) is the anomalous dimension for \( \tilde{g} \),

\[ \gamma(\tilde{g}) = -\frac{11}{48\pi^2} \tilde{g}^2 + O(\tilde{g}^4). \tag{61} \]

This should be compared with that of the gluon field in perturbative QCD, \( \gamma(A) = -\frac{5g^2}{24\pi^2} \) for SU(2), in the absence of the quarks.
VI. ELECTRIC BACKGROUND

To make sure that our infra-red regularization is indeed the correct one it is necessary to have an independent confirmation of the above result. To do this it is instructive to calculate the effective action with a pure electric background first.

From (46) and (47) we have for a pure electric background (i.e., for \( \alpha = 0 \))

\[
\Delta S = i \ln \text{Det}(-\vec{B}^2 - 2i\beta) + i \ln \text{Det}(-\vec{B}^2 + 2i\beta),
\]

and

\[
\Delta \mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty dt \frac{b/m^2}{t^2 - c \sinh(bt/m^2)} \left[ \exp(2ibt/m^2) + \exp(-2ibt/m^2) \right].
\]  

There are different ways to evaluate the integral, but a simple and nice way of doing this follows from the observation that in the imaginary time (i.e., in the Minkowski time) the role of the electric and magnetic fields are reversed. So with the Wick rotation the above integral acquires the same form as (48). Indeed with the Wick rotation (63) becomes

\[
\Delta \mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty dt \frac{b}{t^2 - c \sinh(bt)} \left[ \exp(-2bt) + \exp(2bt) \right].
\]  

Now, adopting the same infra-red regularization as in the pure magnetic background, we obtain

\[
\Delta \mathcal{L} = -\frac{11b^2}{48\pi^2}(1 - \gamma) + \frac{11b^2}{48\pi^2}(\ln b - c) - i\frac{11b^2}{96\pi}
\]

So with the modified minimal subtraction we have (with the pure electric background)

\[
\mathcal{L}_{\text{eff}} = \frac{k^2}{2g^2} + \frac{11b^2}{48\pi^2}(\ln b - c) - i\frac{11b^2}{96\pi}.
\]

We emphasize that in evaluating the above integral the same infra-red regularization is applied as in the pure magnetic background. With the pure electric background the eigenfunctions of the second determinant in (62) becomes anti-causal and thus unphysical in the long distance region (i.e., for \( k^2 < b \)), just like the eigenfunctions under the pure magnetic background become tachyonic and unphysical in the infra-red region (i.e., for \( k^2 < a \)). So we must again exclude these unphysical modes to evaluate the above integral.

Another way to perform the integral (63) is by choosing the proper contour. Notice that (unlike the pure magnetic background) the integrand here has poles on the real axis, so that we must specify the contour of the integral. To find the proper contour, first notice that the eigenvalues of the two determinants in (62) are complex conjugate to each other. This means that the contour of the two integrals in (63) should also be complex conjugate to each other. Secondly, one can make the first integral finite by choosing the contour to pass above the real axis and rotating it to the positive imaginary axis (i.e., by replacing \( t \) with \( -it \)). This is justifiable, because the first integral is free of the controversial acausal states. With this the contour of the second integral is fixed by complex conjugating the first contour. This means that the second contour must pass below the real axis, which one can rotate to the negative imaginary axis (by replacing \( t \) with \( -it \)). This makes the second integral finite. Finally the causality requires us to replace \( b \) with \( b + \epsilon \) in the first determinant but \( b - \epsilon \) in the second determinant in (62). This means that the first contour should start from \( 0 + \epsilon \), but the second one from \( 0 - \epsilon \) in (63). From this we conclude that the half of the residue at the origin should contribute to the integral. This recipe reproduces (66), and justifies the result.

Notice that it is the causality that produces the imaginary part in (65). This is remarkable, because it was the same causality which has made (51) explicitly real. So in both pure magnetic and pure electric backgrounds the causality determines the imaginary part of the effective action.

The contrast between the effective actions (52) and (66) is remarkable. First, the effective potential derived from (66) has no local minimum. This implies that the electric background does not generate a condensation. Secondly, (66) has an imaginary part

\[
\text{Im} \mathcal{L} = -\frac{11b^2}{96\pi}.
\]

This implies that the electric background is unstable. But perhaps a more important point here is that the imaginary part is negative. This means that the electric background generates the pair annihilation, rather than the pair creation, of the gluons. This is because the negative imaginary part can be interpreted as the negative probability of the pair creation. This implies that the gluons in QCD, unlike the electrons in QED, tend to annihilate among themselves in the color electric field. This might sound strange, but actually is not difficult to understand. Indeed this is precisely what the asymptotic freedom dictates. To understand this remember that the gluon loop contributes positively, but the quark loop contributes negatively, to the asymptotic freedom [23]. Exactly for the same reason the gluon and quark loops contribute oppositely to the imaginary part of the effective action. But the quark loop in QCD, just like the electron loop in QED, generates a positive imaginary part [22]. This tells that the gluon loop should generate a negative imaginary part in the effective action. This means that
the asymptotic freedom, the anti-screening, and the pair annihilation all originate from the same physics. This is really remarkable.

VII. STABILITY OF MONOPOLE CONDENSATION

There have been many attempts to construct the effective action of QCD in the literature and in the appearance our vacuum (58) looks very much like the old Savvidy-Nielsen-Olesen (SNO) vacuum [7, 8]. The major difference is that the effective action in the earlier approaches contained an imaginary part, which made the magnetic condensation unstable. In contrast our effective action is explicitly real, which guarantees the stability of our monopole condensation. Indeed it has been asserted that the SNO vacuum should be unstable, because the effective action which defines the vacuum develops an imaginary part [7, 8],

\begin{equation}
Im \mathcal{L}|_{SNO} = \frac{1}{8\pi} a^2.
\end{equation}

This destabilizes the vacuum through the pair creation of gluons. This assertion of the instability of the SNO vacuum, which comes from improper infra-red regularizations, has been widely accepted and never been convincingly revoked. As a consequence it has been generally believed that the one-loop effective action cannot establish the monopole condensation in QCD. Our analysis tells that this misleading belief has no foundation.

But since the absence of the absorptive part in our effective action is such a crucial point which distinguishes our effective action from the SNO action, one might like to have an independent proof that our infra-red regularization is indeed the correct one. Fortunately there are various ways to make an independent confirmation of our effective actions (52) and (66). To see this first notice that the imaginary part (68) of the SNO action as well as ours are quadratic in the background fields. This, with the definition (45), tells that the imaginary part of the one loop effective action is second order in the coupling constant \( g \). So one can find the correct imaginary part of the effective action perturbatively, just by calculating the effective action up to the second order in the coupling constant in the perturbative expansion. There are different ways of doing this. In fact one can just calculate the relevant Feynman diagrams of the perturbative expansion [26], or can adopt the Schwinger's method used in QED to obtain the imaginary part [27]. Now, a remarkable point is that these perturbative calculations do reproduce the result which is identical to ours [28],

\begin{equation}
Im \Delta \mathcal{L} = \begin{cases} 
0 & b = 0 \\
-\frac{11\alpha^2}{96\pi} & a = 0.
\end{cases}
\end{equation}

This confirms that our infra-red regularization is indeed correct. More importantly this confirms that we do have the desired dynamical symmetry breaking and the magnetic condensation in QCD. It must be pointed out that the possibility that one could calculate the imaginary part of the effective action by the perturbative method, and that the SNO action could probably be incorrect, was first raised by Schubacker [26]. Unfortunately this remarkable work has been completely neglected so far, probably because this work is also plagued by the defect that it is not gauge independent.

We emphasize that this perturbative calculation of the imaginary part in QCD is justified precisely because the imaginary part of the effective action is second order in \( g \). This is remarkable, because in general the one loop effective action does not allow a perturbative expansion. For example in QED, the perturbative expansion of the imaginary (as well as the real) part of the effective action is divergent and does not make sense, because the point \( e = 0 \) is singular [24, 25]. This means that in QED the perturbative calculation does not reproduce the result of one loop effective action.

To reinforce our assertion we now provide a third independent argument which supports our results. An important point to observe here is that the effective actions (52) and (66) are actually the mirror image of each other. To see this notice that we can obtain (66) from (52) simply by replacing \( a \) with \(-ib\), and similarly (52) from (66) by replacing \( b \) with \( ia \). This is the first indication that there exists a fundamental symmetry which we call the duality in the effective action of QCD. The duality states that the effective action must be invariant under the replacement

\begin{equation}
\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} ib \\ a \end{pmatrix}, \quad b \rightarrow ia.
\end{equation}

This type of duality was first established in the effective action of QED [24, 25]. But we emphasize that exactly the same duality should also hold in our effective action of QCD, because we have already Abelianized it. An important point of the duality is that the duality provides a very useful tool to check the consistency of the effective action. In the present case the duality indeed confirm the consistency of our effective actions (52) and (66). Obviously this endorses that our calculation of the imaginary parts (69) is probably correct, or at least consistent with the duality. This tells that the causality, the perturbative expansion, and the duality all strongly endorse the stability of our monopole condensation.

It must be emphasized that there are fundamental differences between the earlier attempts and the present approach. The earlier attempts had three problems. First the separation between the classical background and the quantum field was not gauge independent, which made it difficult to establish the gauge invariance of the one loop effective action. Secondly the origin of the magnetic background has never been clarified. As a consequence the magnetic condensation could not be associated with the monopole background. These defects were serious
enough, but perhaps the most serious problem was that 
the infra-red divergence was not properly regularized in 
the earlier attempts. Because of this the SNO effective 
action contained an imaginary part. This destabilizes the 
vacuum through the pair creation of gluons.

In contrast in our approach the separation of the 
monopole background from the quantum fluctuation is 
clearly gauge independent. Moreover our infra-red regu-
larization generates no imaginary part in the effective 
action. Because of these we obtain a stable vacuum mode 
of monopole condensation which is both gauge and Lorentz 
manifest. Notice that the infra-red regularization in (50) 
is not just to remove the infra-red divergence (there are 
ininitely many ways to do this). The infra-red diver-
gence that we face here in QCD is also different from 
those one encounters in the effective action of the mass-
less QED [24, 25]. The infra-red divergence in the mass-
less QED comes from the zero modes. But these zero 

modes are physical modes, which should not be excluded 
in the calculation of the effective action. On the other 

hand the infra-red divergence that we have here comes 
from the unphysical modes, so that one must exclude 
these unphysical modes from the physical spectrum with 
a proper infra-red regularization. Our analysis has pro-
vided ample reason why this has to be so (Notice that 
in the earlier attempts these tachyonic modes are incorrect-
ly identified as the “unstable” modes, but we emphasis 
that they are not just unstable but unphysical). And it 
is precisely these unphysical modes that generate the 
controversial imaginary part in the SNO action. So with 
the exclusion of the unphysical modes the instability of 
the vacuum disappears completely. As importantly in 
our approach we can really claim that the magnetic cond-
ensation is a gauge independent phenomenon. Furthermore 
here we have demonstrated that it is precisely the Wu-
Yang monopole that is responsible for the condensation.

VIII. QCD versus SKYRME-FADDEEV 
ACTION

Recently Faddeev and Niemi have discovered the 
knot-like topological solitons in the Skyrme-type non-
linear sigma model [9],

$$\mathcal{L}_{SF} = \frac{\mu^2}{2} (\partial_{\mu} n)^2 - \frac{1}{4} (\partial_{\mu} n \times \partial_{\nu} n)^2,$$  \hspace{1cm} (70)

and made an interesting conjecture that the Skyrme-
Faddeev action could be interpreted as an effective action 
for QCD in the low energy limit [10]. But we emphasize 
that from our decomposition (1) it should have been evi-
dent that the above action is closely related to QCD. 
Indeed from the decomposition we have [14]

$$\mathcal{L}_{SF} = \frac{1}{4} \tilde{H}_{\mu\nu \rho} - \frac{\mu^2}{2} \tilde{C}_\mu.$$ \hspace{1cm} (71)

This tells that the Skyrme-Faddeev theory can be inter-
preted as a massive Yang-Mills theory where the gauge 
potential has the special form (7). Furthermore we can 
claim that it is a theory of monopoles and at the same 
time a theory of confinement, where the monopole-anti-
monopole pairs are confined to form the knots [14, 19].

But now with the effective action of QCD at hand we 
can discuss the connection between QCD and Skyrme-
Faddeev theory in more detail.

Evidently the effective action (52) is invariant un-
der both gauge and Lorentz transformations. On the 
other hand we can express the effective action explicitly 
in terms of the monopole field strength $\tilde{H}_{\mu\nu}$. This, of 
course, is not accidental. The background field method 
guarantees that the effective action should be expressed 
by the gauge invariant form, invariant under the back-
ground gauge transformation (23). What is remarkable 
here is that, with (7), the background magnetic field $\tilde{H}_{\mu\nu}$ 
can be expressed completely by the magnetic potential $\tilde{C}_\mu$:

$$\tilde{H}_{\mu\nu} = -g \tilde{C}_\mu \times \tilde{C}_\nu,$$

so that the effective potential (53) can actually be written 
completely in terms of $\tilde{C}_\mu$,

$$V = \frac{g^2}{4} (\tilde{C}_\mu \times \tilde{C}_\nu)^2 \left[ 1 + \frac{11g^2}{24\pi^2} \ln \left( \frac{g(\tilde{C}_\mu \times \tilde{C}_\nu)^2}{\mu^2} \right)^{1/2} - c_1 \right].$$ \hspace{1cm} (72)

Now, just for a heuristic reason, suppose we choose a 
particular Lorentz frame and express the vacuum (58) 
by the vacuum expectation value of $\tilde{C}_\mu$. In this case 
the above effective potential generates the following mass 
matrix for $\tilde{C}_\mu$:

$$M^{\mu \nu}_{\tilde{C}} = \left( \frac{\partial^2 V}{\partial \tilde{C}_\mu \partial \tilde{C}_\nu} \right) = m^2 \delta^{\mu \nu} - n^2 \tilde{g}_{\mu\nu},$$ \hspace{1cm} (73)

where

$$m^2 = \frac{11g^2}{96\pi^2} \left( \frac{(\tilde{C}_\mu \times \tilde{H}_{\mu\nu})^2}{H^2} \right).$$ \hspace{1cm} (74)

can be interpreted as the “effective mass” for $\tilde{C}_\mu$. This 
demonstrates that the magnetic condensation indeed 
generates the mass gap necessary for the dual Meissner 
effect and the confinement.

With the above understanding we can now study the 
possible connection between the Skyrme-Faddeev action 
and the effective actin of QCD. To do this we first expand 
the effective potential in terms of the monopole potent-
ial around the vacuum and make the following Taylor ex-
ansion,

$$V = \frac{g^2}{4} (\tilde{C}_\mu \times \tilde{C}_\nu)^2 \left[ 1 + \frac{11g^2}{24\pi^2} \ln \left( \frac{g(\tilde{C}_\mu \times \tilde{C}_\nu)^2}{\mu^2} \right)^{1/2} - c_1 \right].$$
\[ V = V_0 + \frac{1}{2!} \left( \frac{\delta^2 V}{\delta C_\mu \delta C_\mu} \right) C_\mu^i C_\mu^j + \frac{1}{3!} \left( \frac{\delta^3 V}{\delta C_\mu \delta C_\nu \delta C_\rho} \right) C_\mu^i C_\nu^j C_\rho^k + \frac{1}{4!} \left( \frac{\delta^4 V}{\delta C_\mu \delta C_\nu \delta C_\rho \delta C_\sigma} \right) C_\mu^i C_\nu^j C_\rho^k C_\sigma^l + \ldots \]  

where \( C_\mu^i = C_\mu^i - < C_\mu^i > \). Now, near the vacuum we could neglect the higher order terms and keep only the quartic polynomial in \( \bar{C}_\mu \) for simplicity. In this approximation the corresponding effective Lagrangian will acquire the form

\[ \mathcal{L}_{eff} = -\frac{1}{2} m^2 (\bar{C}_\mu)^2 - \frac{\alpha}{4} (\bar{C}_\mu \times \bar{C}_\nu)^2 - \frac{\beta}{4} (\bar{C}_\mu \cdot \bar{C}_\nu)^2 - \frac{\gamma}{4} (\bar{C}_\mu)^4 + \ldots \]

where \( \alpha, \beta, \) and \( \gamma \) are numerical parameters which can be fixed from (75). This is nothing but a generalized Skyrme-Faddeev Lagrangian [13, 14]. This shows that one can indeed derive a generalized Skyrme-Faddeev action from QCD by expanding the effective potential around the vacuum. This, together with (71), establishes a firm connection between the Skyrme-Faddeev theory and QCD. In fact we can go further, and establish a deep connection between QCD and the Skyrme theory itself [14, 19].

An important feature in our analysis is that the Skyrme-Faddeev action is intimately connected to the monopole condensation in QCD. In particular our analysis makes it clear that the mass scale in the Skyrme-Faddeev action is directly related to the mass of the monopole potential, which determines the confinement scale in QCD. This is not surprising. Indeed any attempt to relate the Skyrme-Faddeev action to QCD must produce the mass scale that the Skyrme-Faddeev action contains, and the only way to interpret this mass scale in QCD is through the confinement.

But it must be emphasized that our approximation (76) is by no means exact. There are two points that should be kept in mind here. First, we have kept only the quadratic part and neglected all the higher order terms in (76). More seriously, in deriving the effective action we have neglected the derivatives of \( \tilde{H}_{\mu\nu} \) and thus the derivatives of \( \bar{C}_\mu \), assuming that \( H \) is constant. Secondly, we had to choose a particular Lorentz frame to justify the expansion (75) of the effective action around the vacuum. So our derivation appears to have compromised the Lorentz invariance, although the generalized Skyrme-Faddeev action is obviously Lorentz invariant. Consequently our analysis establishes a possible connection between a “generalized” non-linear sigma model of Skyrme-Faddeev type and QCD only in a limited sense. In particular it does not assert that the simple-minded Skyrme-Faddeev action describes QCD in the infra-red limit. In spite of these drawbacks our analysis strongly endorses the fact that the Skyrme-Faddeev action has something in common with QCD, which is really remarkable.

**IX. DISCUSSION**

In this paper we have established the monopole condensation, which describes a stable vacuum of QCD. Furthermore we have demonstrated the existence of a genuine dynamical symmetry breaking in QCD triggered by the monopole condensation. We were able to do this by calculating the one-loop effective action of \( SU(2) \) QCD in the presence of pure monopole background. There have been earlier attempts to calculate the effective action, but our result differs from the earlier results. The main difference with the earlier attempts was the controversial imaginary part in the effective action in the earlier attempts. This has made the SNO vacuum unstable. In contrast, with a proper infra-red regularization, we have shown that the QCD vacuum made of the monopole condensation is stable. We have provided three independent arguments to support our conclusion.
It is truly remarkable that the principles of the quantum field theory allow us to demonstrate the monopole condensation within the framework of the conventional quantum field theory. The assertion of the instability of the SNO vacuum has created a wrong impression that one can not demonstrate the monopole-condensation with the one-loop effective action. Our analysis tells that in truth one can demonstrate the monopole-condensation with the effective action. Notice, however, that this does not prove that the monopole condensation is the true vacuum of QCD. To prove this we have to calculate the effective action in an arbitrary color electromagnetic background, and show that indeed the monopole condensation is the true minimum of the effective potential. This is not an easy task. Even for the “simple”QCD the calculation of the one loop effective action in an arbitrary background has been completed only recently [24, 25], fifty years after the Schwinger’s seminal work [27]. In the subsequent paper we obtain the one loop effective action of QCD for an arbitrary background and demonstrate that indeed the monopole condensation is the true vacuum of QCD, at least at one loop level [22, 29].

We conclude with the following remarks:

1) It should be emphasized that the gauge independent decomposition (1) of the non-Abelian gauge potential plays the crucial role in our analysis. The decomposition has been known for more than twenty years [2, 3], but its physical significance appears to have been appreciated very little till recently. Now we emphasize that it is this decomposition which has made the gauge independent separation of the classical background from the quantum field, and allows us to obtain the effective action of QCD without ambiguity. In particular, it is this decomposition which shows that the vacuum condensation is indeed made of the monopole condensation. Many of the earlier approaches had the critical defect that the decomposition of the non-Abelian gauge potential to the U(1) potential and the charged vector field was not gauge independent, which has made these approaches controversial. In particular, in these approaches one can not make sure that the effective action (and the resulting magnetic condensation) obtained with the Abelian background really has a gauge independent meaning.

2) There have been two competing proposals for the correct mechanism of the confinement in QCD, the one emphasizing the role of the instantons and the other emphasizing that of the monopoles. Our analysis strongly favors the monopoles as the physical source for the confinement.

It provides a natural dynamical symmetry breaking, and generates the mass gap necessary for the confinement in QCD. Notice that the multiple vacua, even though it is an important characteristic of the non-Abelian gauge theory, did not play any crucial role in our calculation of the effective action. Moreover our result shows that it is the monopole condensate, not the q-vacuum, which describes the physical vacuum of QCD.

3) We have established a firm connection between the Skyrme-Faddeev action and QCD. On the other hand the Skyrme-Faddeev theory (and the Skyrme theory itself) contains the topological knot states. If so, QCD could also likely to admit such states, which might naturally be interpreted as the “glueballs”. But these knots are not the ordinary glueballs made of the valence quarks. They are made of the magnetic, not electric, flux. In this sense they should be called the “magnetic” glueballs [14]. The existence of such magnetic glueballs has been predicted long time ago [2, 3]. Once the monopole condensation sets in, one should expect the fluctuation of the condensed vacuum. But obviously the fluctuation modes have to be magnetic, which could be identified as the magnetic glueballs (A new feature here is that they have a topological stability. But this could be an artifact of the effective theory, not a genuine feature of QCD). We can even predict that the mass of these glueballs starts from around 1.4 GeV [14]. If so, the remaining task is to look for a convincing experimental evidence of the magnetic glueball states in hadron spectrum [2, 3].

Although we have concentrated to SU(2) QCD in this paper, it must be clear from our analysis that the magnetic condensation is a generic feature of the non-Abelian gauge theory. A more detailed discussion which contains the calculation of the effective action in the presence of an arbitrary color electromagnetic background will be presented in an accompanying paper [29].

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