Large N Limit of SO(N) Gauge Theory of Fermions and Bosons

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Abstract

In this paper we study the large $N_c$ limit of $SO(N_c)$ gauge theory coupled to a Majorana field and a real scalar field in 1 + 1 dimensions extending ideas of Rajeev\cite{1}. We show that the phase space of the resulting classical theory of bilinears, which are the mesonic operators of this theory, is $\text{OSp}(\mathcal{H}|\mathcal{H})/U(\mathcal{H}_+|\mathcal{H}_+)$, where $\mathcal{H}|\mathcal{H}$ refers to the underlying complex graded space of combined one-particle states of fermions and bosons and $\mathcal{H}_+|\mathcal{H}_+$ corresponds to the positive frequency subspace. In the beginning to simplify our presentation we discuss in detail the case with Majorana fermions only (the purely bosonic case is treated in [2]). In the Majorana fermion case the phase space is given by $O_1(\mathcal{H})/U(\mathcal{H}_+)$, where $\mathcal{H}$ refers to the complex one-particle states and $\mathcal{H}_+$ to its positive frequency subspace. The meson spectrum in the linear approximation again obeys a variant of the ’t Hooft equation. The linear approximation to the boson/fermion coupled case brings an additional bound state equation for mesons, which consists of one fermion and one boson, again of the same form as the well-known ’t Hooft equation.

1 Introduction

Gauge theories play a fundamental role in our description of nature. Nevertheless our understanding of confining phase of gauge theories is not so complete. In principle we should be able to calculate the hadronic spectrum starting from Quantum Chromodynamics(QCD),
which is a gauge theory, yet this has not been possible up to now. It is believed that
the hadrons are colorless excitations of the underlying gauge theory and we never see the
constituent quarks as free particles. This suggests that in this case we should have an in-
dependent formulation of gauge theories in terms of color singlet operators of the original
gauge theory. In general this is a very hard task.

Gauge theories in $1 + 1$ dimensions provide a great testing ground for many ideas about
realistic theories. This is a great simplification, various difficult problems of higher dimen-
sional theories will not be there, yet there are still interesting aspects of these theories which
make them worth studying in depth. In [1] Rajeev has constructed a theory of mesons in
two dimensions in the limit $N_c$, the number of colors in $SU(N_c)$, goes to infinity using only
the color invariant variables (which correspond to the meson operators). The idea that QCD
should simplify while keeping all its essential features in this limit goes back to ’t Hooft[3, 4]
and that this limit should be a kind of classical mechanics to Migdal and Witten [5]. This
is a very promising step in simplifying gauge theories, but the large-$N_c$ theory is also quite
complicated and it is not possible as yet to understand it in four dimensions.

Originally ’t Hooft studied two dimensional QCD in the large-$N_c$ limit to understand
the meson spectrum and obtained his bound state equation in his seminal paper[4]. Soon after
the scalar two dimensional QCD was worked out by Shei and Tsao in [6] following ’t Hooft,
and later by Tomaras using Hamiltonian methods in [7]. These works obtained the analog of
the ’t Hooft equation for this case. A natural extension of these would be to look at combined
(fermionic) QCD and scalar QCD, this is done in a paper of Aoki[8] where it is shown that
three types of mesons are possible and they all obey a certain type of ’t Hooft equation(see
also [9]). Cavicchi [10] using a path integral approach with bilocal fields, developed in [11],
studied coupled fermions and bosons as well as some other models in two dimensions and he
obtained some generalized versions of the ’t Hooft equation.

To understand gauge theories better, we study the problem of bosons and fermions coupled
to $SO(N_c)$ gauge fields in $1 + 1$ dimensions. We will apply the methods developed by Rajeev
to this toy model. We recommend his lectures for a more detailed exposition of the underlying
ideas and various other directions [12]. In [1] it was shown that the phase space of the two
dimensional QCD is an infinite dimensional Grassmannian[13]. Using the same methods
scalar version of QCD is worked out in [14],the phase space of the theory comes out to be
an infinite dimensional disc. Recently Konechny and the second author obtained the large-$N_c$
phase space of bosons and fermions coupled to $SU(N_c)$ gauge theory; a certain kind of
super-Grassmannian [15]. The linearized equations agree with the ones found in [8]. The
correct equations are nonlinear and various approximation schemes are also discussed in [15].
There are some ideas in the literature which suggest that gauge theories in two dimensions
all behave in a very similar way[16], therefore it will be interesting to see how much of this
holds for $SO(N_c)$ gauge theory.

The organization of our work is as follows, since we did not want to go into technical
details of super-geometry immediately, we first study the purely fermionic case. The essential
calculations are very similar to the ones in Rajeev’s lectures[12] and for the geometry basic
ideas are already in [17, 13], we also recommend the article [18] for a good discussion. We
show that one can formulate the large-$N_c$ limit in terms of bilinears along the lines in [1].
We obtain a variant of the ’t Hooft equation in the linear approximation. We explain the
geometry of the phase space and show that it is a homogeneous manifold, \( O_1(\mathcal{H})/U(\mathcal{H}_+)(\text{see explanations in section IV}), \) and the symplectic form is the natural one. In the second part, we study the combined system of bosons and fermions, this part is very brief, we state mostly the results. We obtain a super-Poisson structure of the bilinears in the large-\( N_c \) limit and the resulting Hamiltonian. The equations of motion in the linear approximation agree with the purely bosonic and purely fermionic ones with an additional one for the mesons made up of one fermion and one boson. This is again a variant of the well-known ‘t Hooft equation. The discussion on the geometry of the resulting infinite dimensional supersymplectic space requires some new ideas. This part is technically complicated, we use essentially Berezin’s ideas [19], but we do not claim that all the technicalities of the infinite dimensional case is understood. We show that the underlying phase space should be the super-homogeneous manifold \( \mathbb{OSp}(1|1) \), and the supersymplectic form is the natural one on this space. We plan to come to the more mathematical aspects of this problem in a future publication.

2 The \( SO(N_c) \) Majorana Fermions in the Light-cone

Since the basic philosophy was explained in [1] we can be brief and only state our conventions and define our theory. We will use the light cone coordinates \( x^+ = \frac{1}{\sqrt{2}}(t+x), \ x^- = \frac{1}{\sqrt{2}}(t-x), \) (we recommend [20] for an introduction to light-cone quantization, and [21] for a more comprehensive review), the action functional is

\[
S = \int \left\{ \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi}_M \gamma^\mu (\partial_\mu + g A_\mu) \Psi_M - m \bar{\Psi}_M \Psi_M \right\},
\]

(1)

where we have an \( SO(N_c) \) gauge theory for which the matter fields are in the fundamental representation and \( \mathrm{Tr} \) denotes an invariant inner product in the Lie algebra. The Lie algebra condition for \( SO(N_c) \) implies that \( A^T_\mu = -A_\mu \). To compute the variations of the action we need the independent degrees of freedom, we can expand \( A_\mu = A^a_\mu T^a \) where \( T^a \) are the generators of the Lie algebra of \( SO(N_c) \), chosen such that \( \mathrm{Tr} T^a T^b = -\frac{1}{2} \delta^{ab} \). Our conventions for the Majorana fermions are as follows: we choose the Majorana representation in which the fermions are real, i.e. \( \Psi_M^T = \Psi_M \) (transpose here also includes the color indices to simplify the notation). The gamma matrices now are given by,

\[
\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(2)

Note that \( \gamma^5 \) happens to be diagonal in 1 + 1 dimensions and we set \( \bar{\Psi}_M = \Psi_M^T \gamma^0 \). We now rewrite the action in the light-cone coordinates and eliminate all nondynamical degrees of freedom. We write \( \Psi_M = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \), and use \( \gamma^+ = \frac{1}{\sqrt{2}} (\gamma^0 + \gamma^1), \gamma^- = \frac{1}{\sqrt{2}} (\gamma^0 - \gamma^1) \). We further set \( A_- = 0 \) and choose \( x^+ \) as the evolution variable, which we call “time”.

\[
S = \int dx^+ dx^- \left\{ \frac{1}{2} (\partial^- A^a_+)^2 + i \sqrt{2} \psi_1^T \partial^- \psi_1 + i \sqrt{2} \psi_2^T \partial^- \psi_2 + i 2 m \psi_1^T \psi_2 + i \sqrt{2} g \psi_2^T A^a_+ T^a \psi_2 \right\},
\]

(3)
(from now on $T$ only means tranpose in the color space). We note also that we have a real two component fermion, they are Grassmann valued obeying $\psi_1 \psi_2 = -\psi_2 \psi_1$. We can check that the action is real if we use the following complex conjugation convention for spinors, $(\psi \xi)^* = \xi^* \psi^*$. We see that $\psi_1^\alpha$ is non-dynamical, and hence can be eliminated using its equation of motion,

$$\psi_1^\alpha = -\frac{m}{\sqrt{2}\partial_-} \psi_2^\alpha. \tag{4}$$

Similarly we solve for the nondynamical $A_+^a$, and get

$$A_+^a = \frac{i\sqrt{2}g}{\partial_-} \psi_2^T T^a \psi_2. \tag{5}$$

A remark is in order here to clarify what we mean by ‘real fermions’ while the action and the constraint equation for $A_+^a$ have explicit factors of $i$. The resolution of this seeming paradox is that it is the equations of motion which are actually real. To see this note first that the symplectic form has a factor of $i$ in it, and the $\psi_1^\alpha$ constraint only has real operators. The $A_+^a$ constraint is also real if we choose complex conjugation of fermions to be $(\psi \xi)^* = \xi^* \psi^*$. This convention implies that the product of two real fermions is imaginary, and this is the reason for the extra factors of $i$ in the action. The equation of motion for $\psi_2^\alpha$ reads,

$$\partial_+ \psi_2^\alpha = \frac{m}{\sqrt{2}} \psi_1^\alpha, \tag{6}$$

which is manifestly real. This shows that the “time” evolution preserves the real valuedness condition imposed on the fermions.

If we insert the above constraints into the action we arrive at

$$S = \int dx^+ dx^- [i\sqrt{2} \partial_+ \psi_2^\alpha - \sqrt{2} m^2 \frac{1}{2} \partial_- \psi_2^\alpha - g^2 \psi_2^T T^a \psi_2^\alpha - \frac{1}{\partial_-^2} \psi_2^T T^a \psi_2^\alpha]. \tag{7}$$

This defines our theory at the classical level with the redundant degrees of freedom eliminated. Since it is written entirely in terms of $\psi_2$ we will refer to this field as $\psi$ from now on. The real fermions have a super-poisson bracket, which can be read off from the action, given by

$$\{\psi_1^\alpha(x^-, x^+), \psi_2^\beta(y^-, x^+)\} = \frac{-i}{2\sqrt{2}} \delta^{\alpha\beta} \delta(x^--y^-). \tag{8}$$

Since our real fermions are Grassmann valued we use a symplectic structure which is $i$ times a real symmetric operator, and the Hamiltonian is actually $i$ times an antisymmetric one, as we will see in more detail in the next section. There is an ambiguity in the quantization, we follow Rajeev’s original approach[12, 1], we will remove the nondynamical fields after quantizing the dynamical field, $\psi^\alpha$. Using the Dirac rule we get an anticommutator for $\psi$,

$$[\psi_1^\alpha(x^-), \psi_2^\beta(y^-)] = i\hbar \frac{-i}{2\sqrt{2}} \delta^{\alpha\beta} \delta(x^- - y^-) = \frac{\hbar}{2\sqrt{2}} \delta^{\alpha\beta} \delta(x^- - y^-). \tag{9}$$

(Note that for the orthogonal group, the distinction of upper and lower indices is irrelevant, since the metric tensor is unity). The reason for our convention of complex conjugation is to
arrive at this more familiar form of the Clifford algebra. We could have chosen a convention in which the product of real fermions is real, then we would arrive at a Clifford algebra with a factor of $i$ as in [17], but this usual form is preferable (it leads to a positive inner product, or the hermitian conjugation is compatible with the inner product in the fermionic Fock space). Let us introduce the Fourier decomposition, which is done in a complex Hilbert space (to simplify notation we drop the subscript in $\psi_\alpha$)

\[
\psi^\alpha(x^-) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \chi^\alpha(p) e^{-ipx^-}.
\]

(To be precise, in the above expansion, we should assume a cut-off $\epsilon_0$ around zero momentum, to be taken to zero at the end of our calculations). We see that $\chi^\alpha(p)$ satisfies the basic anticommutator,

\[
[\chi^\alpha(p), \chi^\beta(q)]_+ = \delta^{\alpha\beta} \delta[p + q],
\]

where we write $\delta[p - q] = 2\pi\delta(p - q)$. Real valuedness of the original field implies that $\chi^\alpha(p) = \chi^\alpha(-p)$. In standard physics notation the expansion would be written as

\[
\psi^\alpha(x^-) = \int_{0}^{\infty} \frac{dp}{2\pi} \chi^\alpha(p)e^{-ipx^-} + \frac{1}{2}\delta^{\alpha\beta}\delta[p + q][1 + \text{sgn}(p)].
\]

As well known in the physics literature, to make the Hamiltonian bounded from below, we should choose a vacuum to be used to construct a fermion Fock space, and further impose a normal ordering prescription. This is done by simply requiring that $\chi^\alpha(p)|0> = 0$ for $p > 0$, and defining

\[
: \chi^\alpha(p)\chi^\beta(q) := \begin{cases} -\chi^\beta(q)\chi^\alpha(p) & \text{if} \quad p > 0, q < 0, \\ \chi^\alpha(p)\chi^\beta(q) & \text{otherwise} \end{cases}.
\]

This can be stated in one formula as,

\[
: \chi^\alpha(p)\chi^\beta(q) := \chi^\alpha(p)\chi^\beta(q) - \frac{1}{2}\delta^{\alpha\beta}\delta[p + q][1 + \text{sgn}(p)].
\]

For most of our calculations we need only the bilinears and the above expressions. (For the Hamiltonian we actually need the normal ordering of product of four such operators, and it is defined as usual all the annihilation operators are to be taken to the right of creation operators, it will be briefly explained later on). We can reduce our Hamiltonian after this quantization process, and we get,

\[
H = \int dx^- \left( \frac{m^2\sqrt{2}}{2} : \psi^\alpha \frac{1}{i\partial^-} \psi^\alpha : -\frac{g^2}{2} : \psi^\alpha(x^-)\psi^\beta(x^-) : |x^- - y^-| : \psi^\beta(y^-)\psi^\alpha(y^-) : \right).
\]

Above we used $(T^a)^{\alpha\beta}(T^a)^{\lambda\sigma} = -\frac{1}{2}(\delta^{\alpha\lambda}\delta^{\beta\sigma} - \delta^{\alpha\sigma}\delta^{\beta\lambda})$ and the Green function $\frac{1}{2}\delta(x^- - y^-) = \partial^- 2\delta(x^- - y^-)$. The last normal orderings can be rearranged to act only on the color invariant combinations in the large-$N_c$ limit, we will discuss this in the next section.

Next we introduce the algebra of color invariant bilinears and study the resulting system in the large-$N_c$ limit following [1, 12].
3 Classical mechanics of color invariant operators

We define color invariant bilinears as in [1, 12, 15] to be our dynamical variables and find the large-$N_c$ limit by postulating Poisson algebra of these bilinears and defining the phase space to be a manifold where these Poisson brackets make sense. Since the theory is super-renormalizable we expect this to be related to the Hilbert-Schmidt ideal condition which is well-known in the literature on the Fock spaces[13, 22, 23, 24]. We will see these aspects in more detail in the next section when we talk about the geometry of the phase space. We define our basic dynamical variables, bilinears,

\[ \hat{R}(p, q) = \frac{2}{N_c} \sum_{\alpha} \chi^\alpha(p) \chi^\alpha(q), \]

which are color invariant combinations of the fermion operators. We find it useful to define a related operator, \( \hat{F}(p, q) = \hat{R}(-p, q) \), we will see that this is the correct variable for the geometry of the phase space. We assume that there are proper large-$N_c$ limits of our operators, then they become classical variables when they are restricted to color invariant sector of the full Fock space. Following [1], we postulate the following Poisson brackets(we choose the quantization parameter to be \( \frac{1}{N_c} \)),

\[ \{ R(p, q), R(s, t) \} = -2i \left( \hat{R}(p, t) \delta[q + s] - R(q, t) \delta[p + s] + R(s, p) \delta[q + t] - R(s, q) \delta[p + t] + (\delta[q + t] \delta[p + s] - \delta[p + t] \delta[q + s]) (\text{sgn}(t) + \text{sgn}(s)) \right). \]

Our dynamical system is not defined completely yet, since there is still a left over global color invariance, generated by

\[ \hat{Q}^{\alpha\beta} = \int [dp] : \chi^{\alpha\dag}(p) \chi^\beta(p) : = \int_0^\infty [dp] \chi^{\alpha\dag}(p) \chi^\beta(p) - \int_0^\infty [dp] \chi^{\beta\dag}(p) \chi^\alpha(p). \]

The commutators of these generators satisfy the Lie algebra of \( SO(N_c) \).

If we restrict ourselves to the color invariant states, we find a constraint equation satisfied in the large-$N_c$ limit, which can be best expressed in terms of \( F(p, q) = R(-p, q) \),

\[ \int [dq] F(p, q) F(q, s) - \text{sgn}(p) F(p, s) - F(p, s) \text{sgn}(s) = 0. \]

We define \( \epsilon(p, q) = -\text{sgn}(p) \delta[p - q] \), then we can rewrite this constraint as a simple quadratic operator equation,

\[ (F + \epsilon)^2 = 1, \]

(we interpret \( F, \epsilon \) as integral kernels acting on \( L_2 \) space of initial data). In the next section we will analyze the geometric meaning of these constraints. The Hamiltonian and the above Poisson brackets determine the evolution of our classical system; the Poisson brackets are consistent with the constraint equation.

The large-$N_c$ Hamiltonian is obtained by dividing the original Hamiltonian by \( N_c \) and rewriting it in terms of our large-$N_c$ variables. After certain manipulations which are sketched below, we obtain the following Hamiltonian,

\[ H = \mathcal{P} \int \frac{1}{8} (m^2 - \frac{g^2}{2\pi} \frac{[dp]}{p} R(-p, p) - \frac{g^2}{64} \mathcal{F} \mathcal{P} \int \frac{[dpdqdsdt]}{(p + s)^2} R(p, q) R(s, t) \delta[p + q + s + t] \],

(20)
where $\mathcal{P}$ and $\mathcal{FP}$ refer to the principal value and finite part prescriptions, respectively. In the following we will often write $f$ short for $\mathcal{P}f$ and $\mathcal{FP}f$, but one should keep in mind that these regularization prescriptions are used to define the singular integrals. The main steps of the derivation of the above Hamiltonian are very similar to the one in [12], although there are some small differences. Here we supply the basic ingredients to help the reader: for simplicity in many places we write $x, y$ instead of $x^-, y^-$, we define $\epsilon(z) = \mathcal{P} \int_{-\infty}^{\infty} \text{sgn}(p) e^{+ipz}$, note the sign of the exponent. We have the vacuum expectation value of our field product,

$$
<0|\psi^\alpha(x^-)\psi^\beta(y^-)|0> = \frac{1}{4\sqrt{2}}[\delta(x^- - y^-) - \epsilon(x^- - y^-)].
$$

(21)

An important formula for the reduction is given in [12]: If $f(x, y) = \int [dpdq] e^{ipx + ipy} \tilde{f}(p, q)$,

$$
\int \epsilon(x - y)|x - y|f(x, y) = -\frac{1}{\pi} \mathcal{P} \int \frac{[dp]}{p} \tilde{f}(-p, p).
$$

(22)

We also have $|x - y| = \mathcal{FP} \int \frac{|dp|}{p} e^{ip(x-y)}$. We use a form of Wick’s theorem for normal ordered products,

$$
:\psi^\alpha(x)\psi^\beta(y) :: \psi^\beta(y)\psi^\alpha(x) :: =: \psi^\alpha(x)\psi^\beta(x)\psi^\beta(y)\psi^\alpha(y) + <0|\psi^\alpha(x)\psi^\alpha(y)|0>: \psi^\beta(x)\psi^\beta(y) :
$$

$$
+ <0|\psi^\beta(x)\psi^\beta(y)|0>: \psi^\alpha(x)\psi^\alpha(y) :- <0|\psi^\alpha(x)\psi^\beta(y)|0>: \psi^\beta(x)\psi^\alpha(y) :
$$

$$
- <0|\psi^\beta(x)\psi^\alpha(y)|0>: \psi^\alpha(x)\psi^\beta(y) + <0|\psi^\alpha(x)\psi^\alpha(y)|0>: <0|\psi^\beta(x)\psi^\beta(y)|0>.
$$

Note that when we take the large-$N_c$ limit we can expand the full normal ordering in the leading order to get $\psi^\alpha(x)\psi^\alpha(y) :: \psi^\beta(x)\psi^\beta(y) ::$. In the above equality the fourth and fifth terms on the right are of smaller order in the large-$N_c$ limit as well as the last term in the equality. The sixth term is an infinite vacuum expectation value, but that is a constant term which will not contribute to the equations of motion hence we can drop it. As a result,

$$
:\psi^\alpha(x)\psi^\beta(x) :: \psi^\beta(y)\psi^\alpha(y) :: =: \psi^\alpha(x)\psi^\beta(x)\psi^\beta(y)\psi^\alpha(y) + <0|\psi^\alpha(x)\psi^\alpha(y)|0>: \psi^\beta(x)\psi^\beta(y) :
$$

$$
+ <0|\psi^\beta(x)\psi^\beta(y)|0>: \psi^\alpha(x)\psi^\alpha(y) :- <0|\psi^\alpha(x)\psi^\beta(y)|0>: \psi^\beta(x)\psi^\alpha(y) :
$$

$$
- <0|\psi^\beta(x)\psi^\alpha(y)|0>: \psi^\alpha(x)\psi^\beta(y) + <0|\psi^\alpha(x)\psi^\alpha(y)|0>: <0|\psi^\beta(x)\psi^\beta(y)|0>.
$$

Using the above formulae we get a finite renormalization of the mass term.

Let us compute the equations of motion at the linear approximation. What we mean by this is to linearize the constraint as well as the equations of motion. The linearization of the constraint simply says that $R(u, v) = 0$ if $u, v$ have different signs. We thus restrict ourselves to $u, v > 0$ and compute

$$
\frac{\partial R(u, v; x^+)}{\partial x^+} = \{R(u, v; x^+); H\},
$$

(23)

We also put $P = u + v, x = u/P$ and make the ansatz $R(u, v; x^+) = \zeta_R(x)e^{-iP_+x^+}$. For further details we refer to the previous works [1, 12] where similar calculations are done in more detail with the same type of ansatz; this yields an eigenvalue equation,

$$
\mu^2 \zeta_R(x) = (m^2 - \frac{g^2}{2\pi})\frac{1}{x} + \frac{1}{1 - x}\zeta_R(x) - \frac{g^2}{2\pi} \int_0^1 dy \frac{\zeta_R(y) - \zeta_R(1 - y)}{(y - x)^2},
$$

(24)
where $\mu^2 = 2P_x P$ is the invariant mass of the excitation. By looking at the behaviour of this equation under $x \mapsto 1 - x$, and $y \mapsto 1 - y$, we see that we can choose our wave functions to be antisymmetric under $y \mapsto 1 - y$, thus $\zeta(1 - y) = -\zeta(y)$. This gives us,
\[
\mu^2 \zeta_R(x) = (m^2 - \frac{g^2}{2\pi}) \left[ \frac{1}{x} + \frac{1}{1-x} \right] \zeta_R(x) - \frac{g^2}{4\pi} \int_0^1 dy \frac{\zeta_R(y)}{(y-x)^2}.
\]
(25)
This equation is one of our main results and it is a variant of the well-known 't Hooft equation. Apart from the numerical factors this equation is the same as the original one, and this result fits to the ideas in [16]. Its properties are well known, the most important one is that there are only bound state solutions.

An interesting question is the existence of “baryon” like excitations. These should correspond to operators of the form
\[
\frac{1}{Z} \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_{N_c}} \chi^{\alpha_1 \dagger}(p_1) \chi^{\alpha_2 \dagger}(p_2) \ldots \chi^{\alpha_{N_c} \dagger}(p_{N_c}),
\]
(26)
but the meaning of these operators as $N_c \to \infty$ is not so obvious. Yet we can think about normalized states of this form when all the momenta are positive acting on the Fock vacuum, they should correspond such baryon like states. Perhaps our large-$N_c$ theory can detect their presence. Indeed, one can check that the operator,
\[
\hat{B} = \frac{1}{N_c} \int_0^\infty [dp] \chi^{\alpha \dagger}(p) \chi^\alpha(p),
\]
(27)
measures the number of such excitations. This operator can be given a meaning in our theory: in the large-$N_c$ limit therefore it is natural to expect that the operator, $B = \frac{1}{2} \int_0^\infty [dp] F(p, p)$ gives us this number and as we will see it is well-defined. In our classical limit we can ask if this number makes sense for our system, that is if it is a conserved quantity. The answer, not surprisingly, is no: the above baryon number is not conserved by our equations of motion. Thus there are really no baryons in this theory.

4 Geometry of the Phase space

To understand the geometry behind the classical system that we introduce in the previous section, we must take a look at the finite dimensional orthogonal group. Our approach will be similar to one in [2] where we discussed the bosonic version of this theory. The basic ideas of the quantization of free Weyl fermions and the underlying geometry is discussed in the paper of Bowick and Rajeev [17], but we would like to expand on it and there are some differences in our conventions.

We recall that the real orthogonal group can be defined as the set of linear transformations which leave a quadratic form invariant.
\[
Q(Au, Av) = Q(u, v),
\]
(28)
(here $Q(u, v) = u^T Q v$ represents this quadratic form, and superscript $T$ denotes the ordinary transpose). In our case the quadratic form is diagonal, so it is the standard inner product.
We work with the complexification of the original real Hilbert space, and if our Hilbert space is even dimensional, in this complex space we can use a different quadratic form, simply by using an invertible transformation $S$, $Q_2 = S^T Q S$. Assume now that we have a complex structure $J$ acting on our original real Hilbert space, that is, a real antisymmetric matrix with respect to this form, which is also orthogonal, implying $J^2 = -1$. If the quadratic form is the identity, we may think of such a matrix as $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in an appropriate basis of the *real Hilbert space*. Let us split our Hilbert space into two isomorphic pieces with respect to the above decomposition of the complex structure, $W \oplus \tilde{W}$, and complexify the real Hilbert space, naturally we have $W \otimes \mathbb{C} \oplus \tilde{W} \otimes \mathbb{C}$. Choose with respect to this decomposition, $S = \begin{pmatrix} i1 & -i1 \\ 1 & 1 \end{pmatrix}$. (29)

This is the transform which we can use to diagonalize our complex structure. Of course our original quadratic form now changes as we described above: we get, $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (30)

(In our problem we actually transform the inverse of this form, but one can see that as matrices these two forms are identical). The complex orthogonal group is the set of transformations which leaves the form $Q$ invariant. Thus a general complex matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orthogonal if,

$$
a^T c = -c^T a \quad a^T d + c^T b = 1 \quad b^T d = -d^T b. \quad (31)
$$

In finite dimensions the quadratic form is $Q(z, z) = z_1 z_{m+1} + z_2 z_{m+2} + \ldots + z_m z_{2m}$. We see then that the original *real orthogonal group* is embeded into the complex orthogonal group defined by this quadratic form as a set of matrices $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, (32)

with now $a, b$ satisfying, $a^T \bar{a} + b^T \bar{b} = 1$ and $a^T \bar{b} = -b^T a$ (where we decomposed the matrix in the obvious way). This explicitly shows that the complex structure, which is a real orthogonal matrix, becomes diagonal, $J = \begin{pmatrix} -i1 & 0 \\ 0 & i1 \end{pmatrix}$. In our physical example these diagonalizations will be accomplished by the Fourier transform.

An immediate consequence of this way of looking at the real orthogonal group is that the real orthogonal group actually carries a copy of the unitary group in it, corresponding to the elements, $\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$. The quadratic form implies $a^T \bar{a} = 1$, as well as $aa^\dagger = 1$, this implies $aa^\dagger = a^\dagger a = 1$. It is the unitary group of $\mathcal{H}_+$, where $\mathcal{H}_+$ refers to the subspace on which $J$ acts as $i$.

For our purposes we should extend these discussions to the infinite dimensional case. In the infinite dimensional one we should not use the full orthogonal group but the one with a convergence condition[17]. This condition is the well-known Hilbert-Schmidt condition.
in the quasi-free representations of canonical anticommutation algebra. We will comment further on the convergence conditions when we make contact with our system. We define the restricted orthogonal group on the complexified Hilbert space as follows,

$$O_1(\mathcal{H}) = \{g^T Q g = Q | g = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \, \, b \in \mathcal{I}_2\},$$

(33)

where $\mathcal{I}_2$ is the ideal of Hilbert-Schmidt operators [25]. We can state the convergence condition more economically as $[\epsilon, g] \in \mathcal{I}_2$, where $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to the above decomposition. This is basically the complex structure we had, except that a factor of $i$ has been removed. The Lie algebra of this group can be found from an infinitesimal group element,

$$g = 1 + i \Delta u = 1 + i \Delta \begin{pmatrix} S & R \\ -R & -S \end{pmatrix},$$

(34)

with $R^T = -R$ and $S^\dagger = S$ and $\Delta$ represents an infinitesimal parameter. The reader can verify that $u^T Q + Q u = 0$. We would like to define a classical phase space using this infinite dimensional orthogonal group. This will be our phase space for the large-$N_c$ theory, but for the moment let us define it as a mathematical system. We introduce a variable $\Phi$,

$$\Phi = g \epsilon g^{-1} \, \, g \in O_1(\mathcal{H}).$$

(35)

The orbit of $\epsilon$ under the restricted orthogonal group is parametrized by this operator. It is easy to see that the orbit is diffeomorphic to

$$O_1(\mathcal{H})/U(\mathcal{H}_+).$$

(36)

The operator $\Phi$ satisfies,

$$\Phi^2 = 1 \quad \Phi = -Q^{-1} \Phi^T Q \quad \Phi - \epsilon = \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \end{pmatrix} \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \end{pmatrix}$$

(37)

where $\mathcal{I}_1$ denotes the trace class operators in the appropriate space of operators( here $Q^{-1}$ is identical to $Q$ as a matrix, but transforms differently). The second condition really says that $\Phi$ is in the Lie algebra of this group (it is possible to think of this space as a real subset of the restricted Grassmanian, and there is an analogous construction of a line-bundle on this space, see [26]). The tangent space of this orbit is given by the infinitesimal action of the group at any point, and in fact it is a copy of the Lie algebra of this group at every point. The action of a vector field on the basic variable $\Phi$ becomes $V_u(\Phi) = i[u(\Phi), \Phi]$, for a Lie algebra element $u(\Phi)$, which changes differentiably over the orbit. So a vector field at a point $\Phi = g \epsilon g^{-1}$ comes from a Lie algebra element $g^{-1} u(\Phi) g$.

It is well-known [27] that such orbits in finite dimensions typically carry a symplectic structure. If we formally define a two form,

$$\Omega = \frac{i}{4} \text{Tr} \Phi d\Phi \wedge d\Phi,$$

(38)
following the methods in [1] we can check that it is closed and non-degenerate. The form evaluated at two vector fields $V_u, V_v$ is given by

$$\Omega(V_u, V_v) = \frac{i}{8} \text{Tr}[\epsilon, g^{-1}u(\Phi)g][\epsilon, g^{-1}v(\Phi)g]$$

(39)

which shows that it is well-defined, due to the Hilbert-Schmidt conditions, non-degenerate, homogeneous and Kähler. The group action on this phase space $\Phi \mapsto g^{-1}\Phi g$ is actually Hamiltonian, that is there are moment maps which generate this action, given by a conditional trace, $F_u = -\frac{1}{2} \text{Tr}(\Phi - \epsilon) u$, with $\text{Tr}(A) = \frac{1}{2} \text{Tr}(A + \epsilon A \epsilon)$. Just for completeness we record that

$$\{F_u, F_v\} = F - i[u, v] - 2\Re \text{Tr}(R_1 R_2^\dagger),$$

(40)

if we decompose $u, v$ as above. The last term represents a central part and cannot be removed in this classical theory.

How does this tie up with our system? Recall that we had a symplectic form which was $i$ times a quadratic form $Q$, and a Hamiltonian for the free theory which is the mass part, $i$ times an antisymmetric form $\omega$, the combination of the two provides a natural operator: $\tilde{\omega} = Q^{-1}\omega$ is a type $(1, 1)$ tensor hence a proper linear transformation. Its polar decomposition will have all the basic pieces we need. Of course we have also $\omega^{-1}Q$, so which one we choose is determined by the equations of motion. If we look at this general system in the Hamiltonian formalism,

$$S_0 = \int dt \frac{1}{2} i\psi Q \partial_t \psi - \int dt H = \int dt \frac{1}{2} i\psi Q \partial_t \psi - \int dt \frac{1}{2} i\psi \omega \psi,$$

(41)

the equations of motion will give us,

$$\partial_t \psi = Q^{-1}\omega \psi.$$

(42)

Hence the operator $Q^{-1}\omega$ is the one we should use. We find the polar decomposition of this operator, $\tilde{\omega} = KJ$, where $K$ is positive symmetric and $J^TJ = 1$, orthogonal (we should be using the natural inner product defined by $Q$ to define the transpose, and in the infinite dimensional case to define underlying real Hilbert space of initial data). However $\tilde{\omega}$ is antisymmetric with respect to our quadratic form, this means that $J^2 = -1$ and orthogonal, thus a complex structure (the complex structure coming from the other choice differs from this by a minus sign). In our example we see that the quadratic form is $2\sqrt{2}\delta(x^1 - y^1)\delta_{\alpha\beta}$ (thus all the calculations can be done with the usual matrix transpose), and the antisymmetric form is $-\sqrt{2}m^2 \partial^{-1}$, so we get from the polar decomposition, $K = \frac{m^2}{2}[-\partial^2]^{-1/2}, J = -[-\partial^2]^{1/2}\partial^{-1}$ (we omit the identity in the color space). When we use a basis which diagonalizes $\tilde{\omega}$ we get solutions which oscillate in time with a frequency given by the eigenvalues of $K$. In our example, if we decompose the field $\psi^\alpha$ using a Fourier mode decomposition,

$$\psi^\alpha(x^-) = \int_{-\infty}^{\infty} \frac{dp}{2^{3/4}} w^\alpha(p)e^{-ipx^-},$$

(43)

we have

$$w^\alpha(p, x^+) = w^\alpha(p, 0)e^{-i\frac{m^2}{2}x^+} \text{ for } p > 0 \quad w^\alpha(p, x^+) = w^\alpha(p, 0)e^{+i\frac{m^2}{2}x^+} \text{ for } p < 0.$$
(Note that the above combinations on the exponents are relativistically invariant if we recall
the mass-shell condition \( p_+ = \frac{m^2}{2p} \)). This suggests that the \( i \) subspace of \( J \) goes to creation
operators, and \(-i\) subspace goes to the annihilation operators, it is better therefore to
represent our Fourier coefficients as \( w^\alpha(p) = \xi^\alpha(p) \) and \( w^\alpha(-p) = \bar{\xi}^\alpha(p) \) for \( p > 0 \). If we act
with \( J \) on our field variables,

\[
(J \psi)^\alpha(x^-) = \int_0^\infty \frac{dp}{2^{3/4}} (-i \xi^\alpha(p) e^{-ipx^-} + i \bar{\xi}^\alpha(p) e^{ipx^-}).
\]  

(45)

We see now that this Fourier transform diagonalizes our complex structure. If we look at the
inverse of the quadratic form it transforms as

\[
\int dx^- dy^- 2^{3/4} e^{ipx^-} (2\sqrt{2})^{-1} \delta(x^- - y^-) 2^{3/4} e^{iqy^-},
\]

which gives us \( \delta[p + q] \). This is the form of \( Q \) that we wanted to obtain.

From the Fourier decomposition, creation and annihilation operators therefore are as-
signed according to \( \text{sgn}(p) \), \( \xi(p) \mapsto \chi^\dagger \alpha(p) \) and \( \bar{\xi}(p) \mapsto \chi^\alpha(p) \). The ultimate reason for the
choice of Fock vacuum is to make the Hamiltonian bounded from below, if we write our
Hamiltonian in the Fourier space,

\[
H_0 = \mathcal{P} \int \frac{[dp]}{2^{3/2}} \frac{\sqrt{2m^2}}{|p|} \text{sgn}(p) : \chi^\alpha(-p) \chi^\alpha(p) := \int_{0^+}^\infty \frac{dp}{2^{3/2}} \frac{\sqrt{2m^2}}{|p|} \left[ : \chi^{\dagger \alpha}(p) \chi^\alpha(p) : - : \chi^\alpha(-p) \chi^{\dagger \alpha}(-p) : \right].
\]

Notice that \( \text{sgn}(p) \) appears in the Hamiltonian, which is basically the complex structure
we have, and the normal ordering (according to our choice of creation and annihilation
operators) now makes the Hamiltonian bounded from below:

\[
H_0 = \int_{0^+}^\infty \frac{m^2}{|p|} \chi^{\dagger \alpha}(p) \chi^\alpha(p).
\]  

(46)

We could question the effect of the interactions since we have been describing everything in
terms of the free part of the Hamiltonian. Here we see a clear advantage of our light-cone
point of view, the complex structure we start with using the free Hamiltonian is indepen-
dent of any of the parameters of the theory, thus the choice of quasi-free representation of
the canonical anticommutation relations is not affected by the change of parameters due to
interactions. In our case we explicitly keep the change of mass due to the interactions with
the gauge fields, so we are not taking advantage of this property. In more general case this
property may be helpful, in fact for the scalar theory it is essential. We thus conclude our
discussion on the choice of Fock space and its relation to the natural complex structure in
our system.

Next we show that \( \Phi - \epsilon \) really represents our basic bilinears: let us decompose the
complexification of our one-particle Hilbert space as \( \mathcal{H}_+ \otimes \mathcal{H}_- \) according to \(-\text{sgn}(p)\), we can write a general bilinear as an operator acting on the one-particle space and decomposed
according to this direct sum, one checks that

\[
F = \begin{pmatrix}
  S & R \\
  -R & -S
\end{pmatrix},
\]  

(47)

with exactly \( S^\dagger = S \) and \( R^T = -R \). We also know that \( (F + \epsilon)^2 = 1 \). But these are
exactly the properties satisfied by \( \Phi \). Our physical system has a one-particle Hilbert space
given by the initial data on the light-cone $x^+ = 0$, we complexify this space and use Fourier transform to put our operators into the desired form. Then $\mathcal{H}_-$ corresponds to the negative frequency components in the physics language. The Poisson bracket relations can be meaningfully extended to the Hilbert-Schmidt type $R$, so we need the convergence conditions. The convergence conditions are also a natural consequence of the super-renormalizability of this system. The time evolution of the finite $N_c$ system should keep us in the same free Fock space, and in the large-$N_c$ limit this should be expressible as an operator like $\Phi$. In fact the smeared out Poisson brackets are given by the Poisson bracket relations of the moment maps. Thus the symplectic structure we have on this homogeneous manifold is the one we have found for our bilinears.

It is useful to look at the same issue from the point of view of generalized coherent states: assume that we have a Lie group which is representable on a Hilbert space by unitary operators through a highest weight vector. If we look at the orbit of this vector under the action of the group, this orbit has a natural symplectic structure, and all the vectors on the orbit correspond to the generalized coherent states [28, 29, 5]. In our case the group of Bogoliubov automorphisms, which do not act on the color part of our fermions are represented on the fermion Fock space by the color invariant bilinears. The highest weight vector is the vacuum and its orbit under this group therefore carries a natural symplectic structure. The corresponding group is the restricted orthogonal group $O_1(\mathcal{H})$ and the orbit is our phase space. (In fact physically we should be using the projective Fock space, since the phase does not change the physical content of a state. The bilinears provide a unitary representation of the central extension $\hat{O}_1(\mathcal{H})$ of the group $O_1(\mathcal{H})$, when we use the projective Fock space, the central part disappears and we descend to the restricted orthogonal group). The convergence conditions are now a result of the implementability of these automorphism in the Fock space, which is defined by our choice of the vacuum [23, 22, 13]. The large-$N_c$ limit allows us to restrict to the bilinears and the super-normalizability keeps us in the restricted class of implementable automorphisms. Thus taking the large-$N_c$ limit provides a classical limit in this sense.

This shows that our large-$N_c$ limit has a well-defined classical phase space with a natural symplectic structure. This opens up various possibilities, such as studying large fluctuations of the field in this limit. There are various delicate questions, such as the domain of the Hamiltonian, existence of finite time evolution, completeness of the trajectories which we plan to come back in the future.

5 Bosons and Fermions

This is the beginning of the second part of our paper. The second part has two themes again: the construction of the phase space via the large-$N_c$ limits of the bilinears and the geometry of the ensuing phase space. Since the bosonic theory is developed in [2] and the fermionic version is explained in detail in the previous sections the construction of the phase space and finding the Hamiltonian will be very brief. We recommend the reader to look at [2] and we use the results of the previous sections freely. The geometry part, which is in the next section, will require new methods and in some sense it is not as complete. It may be helpful
if the reader also consults to [15] where the $SU(N_c)$ version is discussed. We will develop these aspects as much as we can and in some cases we indicate what the idea should be.

We start our first theme: we use the same conventions as in the previous sections and our previous paper. The action functional of the combined system of bosons and fermions can be written as,

$$S = \int \left[ \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi}_M \gamma^{\mu} D_\mu \Psi_M - m_F \bar{\Psi}_M \Psi_M + \frac{1}{2} (D^\mu \phi)^T (D_\mu \phi) - \frac{1}{2} m_B^2 \phi^T \phi \right].$$

(48)

where we use the same conventions as in section IV for the Majorana fermions. The transpose refers to the color indices for the scalar field. Again the covariant derivative is $D_\mu = \partial_\mu + g A_\mu$, where $A_\mu$ has values in the Lie algebra of $SO(N_c)$. We choose $x^+$ as time and set $A_- = 0$ as our gauge fixing condition. Then the action in the light-cone formalism reads,

$$S = \int dx^+ dx^- \left[ i \sqrt{2} \bar{\psi}_1^\dagger \partial_- \psi_1 + i \sqrt{2} \bar{\psi}_2^\dagger \partial_+ \psi_2 + 2 m_F \bar{\psi}_1 \psi_2 + \frac{1}{2} \phi^T (\partial_- \phi) + \frac{1}{2} (\phi^T T^a \phi - \phi^T T^a \partial_- \phi) \right].$$

The advantage of the light-cone formalism is again clear, we are already in the Hamiltonian picture. We can read off the Poisson brackets satisfied by the dynamical fields. We also see that $\psi_1$ is not dynamical, as well as $A_\alpha^a$, therefore they can be eliminated through their equations of motion. The dynamical fermion field $\psi_2^\dagger$ will be called $\psi^a$ for simplicity as in the previous sections. We will assume that the field $A_\alpha^a$ is eliminated after the dynamical fields are quantized, this will give us the quantized Hamiltonian of the system,

$$H = \int dx^- \left( \frac{1}{2} m_B^2 \phi^T \phi : + \frac{1}{2} \sqrt{2} m_F^2 : \psi^T \frac{1}{i \partial_-} \psi : - \frac{g^2}{2} : J^a : \frac{1}{\partial_-^2} : J^a : \right),$$

(49)

where

$$J^a = \left[ i \sqrt{2} \bar{\psi}^T T^a \psi + \frac{1}{2} (\partial_- \phi^T T^a \phi - \phi^T T^a \partial_- \phi) \right].$$

(50)

The quantization process is defined for the Fermionic sector in section II and for bosons in the reference [2]. We expand fermions and bosons into Fourier modes in a complex space,

$$\psi^a(x^-) = \int_{-\infty}^{\infty} \frac{dp}{2 \pi} \chi^a(p) e^{-ipx^-} \phi^a(x^-) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2 |p|}} a^\dagger(p) e^{-ipx^-},$$

(51)

with now $\chi^{\alpha(p)} = \chi^{\alpha}(-p)$ and $a^{\alpha\dagger}(p) = a^\dagger(-p)$. (We should again assume that there is an infinitesimal cut-off around the zero momentum to be taken to zero at a later stage). The Poisson bracket relations go to

$$[\chi^\alpha(p), \chi^\beta(q)]_+ = \delta[p+q] \delta^{\alpha\beta}, \quad [a^\alpha(p), a^\beta(q)] = \text{sgn}(p) \delta[p+q] \delta^{\alpha\beta}. $$

(52)

These are exactly the same as before, there is one more commutator now,

$$[\chi^\alpha(p), a^\beta(q)] = 0.$$

(53)

As we will see in the next section the definition of the Fock vacuum brings new features—a larger symmetry algebra appears. We introduce the vacuum state $|0>_s$, characterized by
Then we express our Hamiltonian in the large-

\[ \chi^\alpha(p)|0>_s = 0, a^\alpha(p)|0>_s = 0 \] for \( p > 0 \), where we put a subscript \( _s \) to emphasize that the vacuum is for the full algebra of the boson/fermion system. We repeat for the convenience of the reader the normal ordering rules of the bilinears (rewritten to fit to our needs),

\[ : \chi^\alpha(p)\chi^\beta(q) : = \chi^\alpha(p)\chi^\beta(q) - \frac{1}{2}\delta^{\alpha\beta}(1 + \text{sgn}(p))\delta[p + q] \]

\[ : a^\alpha(p)a^\beta(q) : = a^\alpha(p)a^\beta(q) - \frac{1}{2}\delta^{\alpha\beta}(1 + \text{sgn}(p))\delta[p + q]. \]

There is an obvious extension of the general definition of normal ordering to the product of more than two operators, which one needs for the reduction of the Hamiltonian: set all the annihilation operators to the right of creation operators in a recursive way.

We first introduce our bilinears for the large-\( N_c \) limit and work out their Poisson brackets. Then we express our Hamiltonian in the large-\( N_c \) limit in terms of these bilinears. We can see that the basic color invariant observables are:

\[ \hat{F}(p, q) = \frac{2}{N_c} : \chi^\alpha(p)\chi^\alpha(q) : \]

\[ \hat{B}(p, q) = \frac{2}{N_c} : a^\alpha(p)a^\alpha(q) : \]

\[ \hat{C}(p, q) = \frac{2}{N_c} : a^{\alpha\dagger}(p)a^\alpha(q) : \]

\[ \hat{\bar{C}}(p, q) = \frac{2}{N_c} : a^\alpha(p)\chi^\alpha(q) : \]

note that we have no need for normal ordering in the last two operators since they consist of commuting operators. In the large-\( N_c \) limit \( \hat{C} \) and \( \hat{\bar{C}} \) are related, \( \hat{C} = \hat{\bar{C}}^\dagger \), and there are similar conditions on \( F, B \) (when we represent the resulting classical observables as integral kernels and think of them as now abstract operators). For our calculational purposes it is better to introduce the following variables as in section III and the reference [2],

\[ \hat{T}(p, q) = \frac{2}{N_c} : a^\alpha(p)a^\alpha(q) : = \hat{B}(-p, q) \]

\[ \hat{R}(p, q) = \frac{2}{N_c} : \chi^\alpha(p)\chi^\alpha(q) : = \hat{F}(-p, q), \quad (54) \]

and also the variable,

\[ \hat{S}(p, q) = \frac{2}{N_c} : a^\alpha(p)a^\alpha(q) : = \hat{C}(-p, q). \quad (55) \]

These variables in the large-\( N_c \) limit satisfy the following (super)Poisson brackets,

\[ \{T(p, q), T(s, t)\} = -2i\left( \text{sgn}(p)\delta[p + s]T(q, t) + \text{sgn}(q)\delta[q + s]T(p, t) + \text{sgn}(p)\delta[p + t]T(s, q) \right. \]

\[ + \left. \text{sgn}(q)\delta[q + t]T(s, p) + (\text{sgn}(p) + \text{sgn}(q))(\delta[p + s]\delta[q + t] + \delta[p + t]\delta[s + q]) \right) \]

\[ \{R(p, q), R(s, t)\} = -2i\left( R(p, t)\delta[q + s] - R(q, t)\delta[p + s] + R(s, p)\delta[q + t] - R(s, q)\delta[p + t] \right. \]

\[ + \left. (\delta[q + t]\delta[p + s] - \delta[p + t]\delta[q + s])(\text{sgn}(t) + \text{sgn}(s)) \right) \]

\[ \{T(p, q), S(s, t)\} = -2i\left( S(s, q)\text{sgn}(p)\delta[p + t] + S(s, p)\text{sgn}(q)\delta[q + t] \right) \]

\[ \{R(p, q), S(s, t)\} = -2i\left( S(s, t)\delta[q + s] - S(q, t)\delta[p + s] \right) \]

\[ \{S(p, q), S(s, t)\}_+ = -2i\left( T(q, t)\delta[p + s] - R(s, p)\text{sgn}(q)\delta[q + t] + \delta[p + s]\delta[q + t](1 + \text{sgn}(p)\text{sgn}(q)) \right) \]

We note that the last one is symmetric in the variables and the third and forth ones show that \( S \) behaves as a module of the algebras defined by the Poisson brackets of \( T, R \), thus it
carries a representation of these two algebras. This is the general form of a super-algebra structure. We will denote the full set of these brackets as a super-Poisson bracket \( \{, \} \).

The conversion of the normal ordered products of non-color invariant combinations appearing in the above Hamiltonian to the full normal ordering in the large-\( N_c \) limit can be achieved as before resulting with the same changes in the masses \( m_F^2 \mapsto m_F^2 - g^2/2\pi \) and \( m_B^2 \mapsto m_B^2 - g^2/2\pi \), where \( m_B^2 = m_B^2 - g^2/4\pi \ln(\Lambda_U/\Lambda_I) \) denotes the renormalized mass of the boson. We skip the details of this reduction, since they are the extensions of the details in [12] and we have given some essential steps in section III. The resulting Hamiltonian of our system in the large-\( N_c \) can be expressed as a free part and an interacting part:

\[
H_0 = \frac{1}{8} (m_F^2 - \frac{g^2}{2\pi}) \mathcal{P} \int \frac{[dp]}{|p|} T(-p, p) + \frac{1}{8} (m_F^2 - \frac{g^2}{2\pi}) \mathcal{P} \int \frac{[dp]}{p} R(-p, p).
\]

The interaction part is written as

\[
H_I = \mathcal{F}\mathcal{P} \int [dqdqdsdt] \left( G_1(p, q; s, t) T(p, q) T(s, t) + G_2(p, q; s, t) R(p, q) R(s, t) + G_3(p, q; s, t) S(p, q) S(s, t) \right),
\]

where the kernels are given by

\[
G_1(p, q; s, t) = \frac{g^2 \delta[p + q + s + t]}{64 \sqrt{|pqst|}} \frac{sq - st + pt - pq}{(p + s)^2}
\]

\[
G_2(p, q; s, t) = \frac{g^2 \delta[p + q + s + t]}{64 (p + s)^2}
\]

\[
G_3(p, q; s, t) = \frac{g^2 \delta[p + q + s + t]}{64 \sqrt{|tq|(p + s)^2}}.
\]

We have not completed the definition of our large-\( N_c \) limit yet, there is a constraint. Recall that we still have a left over global color invariance, which is generated by the operator,

\[
\hat{Q}^{\alpha\beta} = \int [dp] (: \chi^{\alpha\dagger}(p) \chi^{\beta}(p) : + \text{sgn}(p) : a^{\alpha\dagger}(p) a^{\beta}(p) :).
\]

When we restrict our color invariant bilinears to the color invariant sector of the full Fock space, we find that

\[
(F + \epsilon)^2 + C\epsilon C = 1 \\
C\epsilon B + B\epsilon + FC + C = 0 \\
\epsilon B C = C + \epsilon C^\dagger \epsilon + \epsilon C^\dagger F = 0 \\
(\epsilon B + \epsilon)^2 + C = 1,
\]

where we define as in section III, \( \epsilon(p, q) = -\text{sgn}(p) \delta[p - q] \) (here the minus sign is crucial, in our previous works that was not important, but in the super case there is a preferred choice) and we also employ the product convention as before for example \( (FC)(p, s) = \int [dq] F(p, q) C(q, s) \). We warn the reader that above the two epsilons have the same matrix.
elements but they are acting on different spaces. The meaning of this constraint could best be understood if we introduce a super operator,

$$\Phi = \begin{pmatrix} \epsilon B + \epsilon & \epsilon C^\dagger \\ C & F + \epsilon \end{pmatrix}. \quad (58)$$

The above constraint is simply given by

$$\Phi^2 = 1. \quad (59)$$

It also satisfies a Lie algebra condition, it is better to write it in the following form: use a decomposition of our super-space into $\mathcal{H}_+|\mathcal{H}_+ \oplus \mathcal{H}_-|\mathcal{H}_-$, according to the sign of $\epsilon$ in even and odd parts respectively. Then we have $\dot{\epsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and we introduce with respect to this decomposition $\hat{\omega}_s = \begin{pmatrix} 0 & -\bar{\epsilon} \\ 1 & 0 \end{pmatrix}$, then:

$$\hat{\omega}_s \Phi^\tau + \Phi \hat{\omega}_s = 0, \quad (60)$$

we invite the reader to verify this.

There are also convergence conditions, which come from the super-renormalizability of this system again. The time evolution should leave this system in the same Fock space. Another way to see this is to think about the smeared out operators, and see that the central terms make sense only for the restricted set of operators, for which the off-diagonal blocks are in the Hilbert-Schmidt class. We can write down these convergence conditions in an economical way as

$$[\hat{\epsilon}, \Phi] \in \mathcal{I}_2, \quad (61)$$

where $\mathcal{I}_2$ refers to the ideal of Hilbert-Schmidt operators in this super-space. We have proposed elsewhere [30] a method of introducing such operators in the super-context, and we assume this definition is used. Since these technical matters are not completely settled we are brief at this point, see also the next section on the geometry. This completes the construction of our large-$N_c$ limit: we postulate the above Hamiltonian, the super-Poisson brackets with the constraint and this defines a classical system. The “time” evolution is given by the basic rule: for any observable $O_s$ of the theory

$$\frac{\partial O_s}{\partial x^+} = \{O_s, H\}_s, \quad (62)$$

where the Hamiltonian is in general an even function of our bilinears– which we should consider as the coordinates of this phase space.

It is possible to carry out the analysis given in [15], but we will be content with describing only the linear approximation. We plan to report on these in a separate publication (they will appear in the PhD thesis of the first author).

We start with the linearization of the constraint $\Phi^2 = 1$, which gives us

$$F \epsilon + \epsilon F = 0 \quad \epsilon B \epsilon + B = 0 \quad \epsilon C + C \epsilon = 0. \quad (63)$$

The first two are exactly the conditions we have found before, for mesons made up of only bosons in [2], the first one is in section III, and the last one is the new condition on our
odd variable. In terms of $S$ that means we have $S(u,v) = 0$ unless $u,v > 0$ or $u,v < 0$. If we assume $u,v > 0$ and evaluate the equations of motion in the linear approximation for $S(u,v)\partial_x S(u,v; x^+) = \{S(u,v; x^+), H\}$ and furthermore we make the same type of ansatz as in [12, 2, 15] $S(u,v; x^+) = \zeta_S(x)e^{-iP_p x^+}$, with $P = u + v, x = \frac{u}{P}$, 

$$
\mu^2_S \zeta_S(x) = \left[ \frac{m_F^2 - g^2/2\pi}{x} + \frac{m_R^2 - g^2/2\pi}{1-x} \right] \zeta_S(x) - \frac{g^2}{8\pi} \int_0^1 dy \frac{x + y}{\sqrt{xy}(x-y)^2} \zeta_S(1-y).
$$

(64)

The other linearized equations are the same as before (see section III and [2]).

There are baryonic states that we can measure by the operator 

$$
\hat{B} = \frac{1}{N_c} \int_0^\infty [dp] (\chi^{\alpha_1^+}(p)\chi^{\alpha}(p) + a^{\alpha_1^+}(p)a^{\alpha}(p)),
$$

(65)

in the large-$N_c$ limit this operator should go to $B = \frac{1}{2} \int_0^\infty [dp] (F(p,p) + B(p,p))$. The baryonic states for finite $N_c$ correspond to states of the form

$$
\frac{1}{Z} \epsilon_{\alpha_1\alpha_2...\alpha_{N_c}} \chi^{\alpha_1^+}(p_1)...\chi^{\alpha_{s+1}^+}(p_s)a^{\alpha_{s+1}^+}(p_{s+1})...a^{\alpha_{N_c}^+}(p_{N_c}),
$$

(66)

where $p_1...p_{N_c} > 0$ and products of them acting on $|0>_s$.

Not surprisingly the above baryon number is not a conserved quantity, so it does not have the physical importance as it has in the case of Dirac fermions where it is a conserved number, in fact a topological number(see [12] for the discussion of this in the large-$N_c$ limit and its extension in [15]).

6 The Geometry of the Phase Space

Let us define a super space $\mathcal{H}|\mathcal{H}$, where we use a splitting to even and odd according to the grading $+, -$ (we are using a $\mathbb{Z}_2$ graded real Hilbert space). We recall some of the conventions, following Berezin [19]: we work with the Grassmann envelop of this graded vector space (thus we acquire a $\mathbb{Z}$ grading). Its mathematical theory is delicate and we will comment on it later (some good examples of homogeneous super-symplectic manifolds are worked out in [31], this is a good reference to learn by examples). We decompose every linear transformation or tensor according to this grading, the standard matrix form of a linear transformation is

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}: \mathcal{H}|\mathcal{H} \to \mathcal{H}|\mathcal{H},
$$

(67)

where $A, D$ are even and $B, C$ are odd. This means that $A = A_B + A_S, D = D_B + D_S$ where subscript $B$ refers to the body that is the ordinary numbers, subscript $S$ refers to the soul, that is only the Grassmann part. $B, C$ have no body they are purely Grassmann valued.

We have the usual hermitian conjugation of such block matrices, but the transpose has to be carefully defined. We introduce a super-transpose, $\tau$,

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^\tau = \begin{pmatrix}
A^T & C^T \\
-B^T & D^T
\end{pmatrix}
$$

(68)
where $T$ denotes the ordinary matrix transpose. One can verify that this form satisfies $(AB)^T = B^T A^T$. It will be useful to record the following properties, $\text{Str}(A^T) = \text{Str} A$, if we decompose our graded space into a direct sum, for example in our case into $\mathcal{H}_+ \oplus \mathcal{H}_- | \mathcal{H}_-$, the operators can also be decomposed into super-operators, say into $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix}.$$  \hfill (69)

Realness is related to an involution in the Grassmann algebra, $\xi \mapsto \xi^*$ and we assume that this involution obeys $(\xi^i \xi^j)^* = (\xi^j)^* (\xi^i)^*$ and $(a \xi)^* = \bar{a} \xi^*$, where $a$ is a complex number and bar denotes the ordinary complex conjugation. The real Grassmann algebra is the part which is invariant under this involution. This means that there will be factors of $i$ to make things invariant. This implies that the real graded Hilbert space is defined residing inside a complex graded Hilbert space.

On the space of linear transformations there is a complex conjugation operator, according to Berezin conventions it should be given by the following: write a linear transformation in its standard form, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & -b^* \\ -c^* & d^* \end{pmatrix}. \hfill (70)$$

we note that $A^{**} = A$, and $(A^\tau)^* = \begin{pmatrix} a^\dagger & -c^\dagger \\ -b^\dagger & d^\dagger \end{pmatrix} = \bar{E} A^\dagger \bar{E}$, here $\bar{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, whereas $(A^\tau)^\tau = A^\dagger$.

We have the set of real linear transformations, this set is invariant under the above conjugation, $M^* = M$, it remains so under the product of super-matrices, thanks to $(A_1 A_2)^* = A_1 A_2$. The set of real linear operators thus is an algebra.

Let us assume that the even part has a symplectic form $\omega$ and the odd part has a standard quadratic form $1$. On the complexification of this space we introduce a super-symplectic form, $\omega_s = \begin{pmatrix} \omega & 0 \\ 0 & i1 \end{pmatrix}$, $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. \hfill (71)

Note that multiplying the last part with an $i$ does not really change anything as far as only the even transformations are concerned but for the full case we need this factor. We look at the space of real transformations which will leave this form invariant,

$$g^T \omega_s g = g \omega_s g^T = \omega_s,$$ \hfill (72)

this is a super-group, and it is denoted by $OSp(\mathcal{H}|\mathcal{H})$. Its even part has body isomorphic to $Sp(\mathcal{H}) \oplus O(\mathcal{H})$, the odd parts are modules over the Grassmann envelops of these groups. If we write down the group conditions for an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -b^* \\ c^* & d^* \end{pmatrix}$, where $a, d$ are real even, $c$ real odd and $b$ imaginary odd operators,

$$a^T \omega a + ic^T c = \omega \quad a^T \omega b + ic^T d = 0 \quad -b^T \omega a + id^T c = 0 \quad -b^T \omega b + id^T d = i1.$$ \hfill (73)

Since we have the complex conjugation convention $(\psi \xi)^* = -\psi^* \xi^*$, the complex conjugate of a product of odd operators become imaginary, this is why we have $ic^T c$, then it becomes a real even element of the Grassmann envelop.
Decompose our spaces according to the matrix representation of $\omega_s$, $W \oplus \tilde{W} | W \oplus \tilde{W}$. Let us assume that we also have a super-complex structure, which is a type $(1,1)$ tensor,

$$J_s = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (74)$$

Assume that we extend everything to the complexification of our original Hilbert space. The we can perform a transformation $S$ that will put the above complex structure into diagonal form in this complexified space. To accomplish this it is better to represent it in a slightly different way, use a decomposition $W | W \oplus \tilde{W} | \tilde{W}$, then

$$J_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \quad (75)$$

Now compute $S^{-1} J_s S$ and see that we get $\hat{J}_s = i \hat{\epsilon} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, which defines $\hat{\epsilon}$ in this decomposition. We use a decomposition according to the sign of $i$ and the resulting graded Hilbert space becomes $\mathcal{H}_+ | \mathcal{H}_+ \oplus \mathcal{H}_- | \mathcal{H}_-$. If we compute the transformation of $\omega_s$, it goes into $S^r \omega_s S$ since it is a two form, and we get

$$\hat{\omega}_s = \begin{pmatrix} 0 & -\hat{\epsilon} \\ 1 & 0 \end{pmatrix}, \quad \hat{\epsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (76)$$

with respect to the above decomposition. Obviously our real group also transformed by the same rule as $J_s$, so a typical group element becomes according to the above decomposition,

$$g = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}. \quad (77)$$

Note that each of the blocks are super-operators with standard decompositions, and for each one we are using Berezin definition of the complex conjugate $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & -b^* \\ c^* & d^* \end{pmatrix}$. We have a full complex group which leaves invariant the above transformed version of the two form, this is the complex $OSp$ group,

$$g^T \hat{\omega}_s g = \hat{\omega}_s, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (78)$$

and the real group now sits inside this complex group. Thus a complex transformation satisfies

$$A^T \hat{\epsilon} C - C^T A = 0 \quad A^T \hat{\epsilon} D - C^T B = \hat{\epsilon} - B^T \hat{\epsilon} C + D^T A = 1 \quad B^T \hat{\epsilon} D - D^T B = 0. \quad (79)$$

The reader may question the consistency of these equations. We should remember that $(M^T)^* = \hat{\epsilon} M \hat{\epsilon}$, then we can see that they are consistent. There is an interesting subgroup, given by elements of the form $g = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$ and $A$ satisfies

$$A^T \hat{\epsilon} A^* = \hat{\epsilon} \quad A^{*T} A = 1, \quad (80)$$
recall that $A^{\tau} = A^\dagger$ and $(A^*)^\tau = \bar{\epsilon} A^\dagger \bar{\epsilon}$, this is the same as before except that we express it in the subspace, so we should use $\bar{\epsilon}$ instead of $\bar{E}$, we get $A^\dagger A = AA^\dagger = 1$. Let us see what it means when we expand $A = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$,

$$a^\dagger a = 1 + \gamma^\dagger \gamma \quad a^\dagger \beta + \gamma^\dagger d = 0 \quad \beta^\dagger a + d^\dagger \gamma = 0 \quad d^\dagger d = 1 + \beta^\dagger \beta,$$

(81)

we see that the body parts satisfy $a^\dagger B a = 1, d^\dagger B d = 1$, these are the ordinary unitary groups inside. Therefore we have shown that this group’s even part has body $\mathcal{U} (\mathcal{H} + \mathcal{H}^*)$. This group is denoted by $\mathcal{U} (\mathcal{H}+|\mathcal{H}^+)$ and it is the super-unitary group of $\mathcal{H}_+$.

Let us define the orbit of $\hat{\bar{\epsilon}}$, this is really the complex structure if we remove the factors of $i$, under the real group $OSp$: 

$$\Phi = g^{-1} \hat{\bar{\epsilon}} g.$$

(82)

It is immediate that $\Phi^2 = 1$. We will now show that we also have

$$\hat{\omega}_s \Phi^\tau + \Phi \hat{\omega}_s = 0,$$

(83)

so it is an element of the Lie algebra of $OSp$. Define $\hat{\bar{E}} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \bar{\epsilon} \end{pmatrix}$, this is really our $\bar{E}$ written in this splitting of the Hilbert space, and note $(\hat{\omega}_s^\tau)^{-1} = \hat{\omega}_s, \hat{\omega}_s^\tau = \hat{\omega}_s \hat{\bar{E}}, \hat{\omega}_s^2 = \hat{\bar{E}}$ and $\hat{\omega}_s = -\hat{\omega}_s \hat{\bar{\epsilon}}$, then,

$$\Phi^\tau \omega_s = (\hat{\omega}_s^{-1} g^\tau \hat{\omega}_s \hat{\bar{\epsilon}} g)^\tau \omega_s = g^\tau \hat{\omega}_s \hat{\bar{E}} g^\tau \hat{\bar{E}} \hat{\omega}_s g = g^\tau \hat{\omega}_s g = -g^\tau \hat{\omega}_s g = -\hat{\omega}_s g^{-1} \hat{\bar{\epsilon}} g = -\hat{\omega}_s \Phi,$$

where we used $\hat{E} g^\tau \hat{E} = g$. Let us look at the stability subgroup of $\hat{\bar{\epsilon}}$, that is given by operators of the form $\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$, and we have seen that this can be identified with the unitary operators on $\mathcal{H}_+|\mathcal{H}_+, U(\mathcal{H}_+|\mathcal{H}_+)$. Hence we conclude that our variable $\Phi$ is actually parametrizing the space $OSp(\mathcal{H}|\mathcal{H})/U(\mathcal{H}_+|\mathcal{H}_+)$.

(84)

What is the advantage of this parametrization? The above super manifold is actually a symplectic manifold with a super-symplectic structure most naturally written in terms of the variable $\Phi$:

$$\Omega_s = \frac{i}{4} \text{Str} \Phi d\Phi \wedge d\Phi.$$

(85)

This is formally defined, but we use the rules of super analysis to define our differential forms. Clearly it is closed, use

$$d\text{Str} \Phi d\Phi \wedge d\Phi = \text{Str} d\Phi \wedge d\Phi \wedge d\Phi = \text{Str} \Phi^2 d\Phi \wedge d\Phi \wedge d\Phi = \text{Str} d\Phi \wedge d\Phi \wedge d\Phi = -\text{Str} \Phi^2 d\Phi \wedge d\Phi \wedge d\Phi,$$

where we used $\text{Str} AB = \text{Str} BA$. It is also clear that this form is homogeneous. Its nondegeneracy can be proved at $\hat{\bar{\epsilon}}$, and homogeneity proves it everywhere.

Upto now we have really used a finite dimensional approach, but to identify the large-$N_c$ phase space of the previous section, we need to extend these notions to the infinite
The extension is formally simple, we assume that we have super-Hilbert spaces, that is even and odd spaces each one are coming from a separable Hilbert space and we use a proper extension of the Grassmann envelop to this case (this is not so obvious and we assume our proposal in [30], this may not be the only possibility see [32, 33]). In this infinite dimensional setting we introduce a Hilbert-Schmidt condition, the group that we use should be the restricted real \( OSp \) group,

\[
OSp_1(\mathcal{H}|\mathcal{H}) = \{ g = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} | g^{-1} \text{ exists} , g^* \omega_s g = 1 \ [\hat{\epsilon}, g] \in \mathcal{I}_2 \}. \tag{86}
\]

The variable \( \Phi \) now satisfies some convergence conditions, indeed one can check that

\[
\Phi - \hat{\epsilon} \in \left( \mathcal{I}_1 \mathcal{I}_2 \mathcal{I}_2 \mathcal{I}_1 \right), \tag{87}
\]

where each block refers to a super-operator in the appropriate class of operator ideal. These convergence conditions imply that the super-symplectic form \( \Omega_s \) we defined in the finite dimensional setting makes sense. The trace class conditions are important to write down moment maps, but we will ignore it for this work. Hence we have an infinite dimensional phase space,

\[
OSp_1(\mathcal{H}|\mathcal{H})/U(\mathcal{H}_+|\mathcal{H}_+), \quad \Omega_s = \frac{i}{4} \text{Str} \Phi d\Phi \wedge d\Phi. \tag{88}
\]

The reader can now see how this is related to our system, from the experience we have in the previous cases. In our problem we have a free action which has bosons and fermions,

\[
S_0 = \int dx^+ dx^- \left( \frac{1}{2} \phi^T (-2 \partial_- \phi + \frac{1}{2} i 2 \sqrt{2} \psi^T \partial_+ \psi - \frac{1}{2} i m_B^2 \phi^T \phi - \frac{1}{2} \sqrt{2} m_F^2 \psi^T \frac{1}{i \partial_-} \psi \right). \tag{89}
\]

This action is written in the standard light-cone frame and one of the components of the Majorana field has been eliminated in favor of the other. The transpose refers to the color indices for the gauge group \( SO(N_c) \). As it stands this does not require the full content of the super-geometry, but as we have seen the interaction terms, given by the proper bilinears of field operators, makes the use of super geometry most convenient: when we reformulate our theory in terms of bilinears, we need the combinations which can only be expressed in terms of odd operators. We will now see that the Poisson algebra of these bilinears can only be formulated as a super-two form. Moreover a simple iterative solution of the constraint equation reveals that the bosonic operators should be given as an infinite series of products of odd operators, this is why we think it is most natural to use the full content of the Berezin’s super-analysis. (We hope to come back to the more mathematical aspects of our system in a future publication).

In our theory we have a super-symplectic form and a super-quadratic form which is in the standard representation given by

\[
\omega_s = \begin{pmatrix} -2 \partial_- & 0 \\ 0 & i 2 \sqrt{2} \end{pmatrix} I_c, \quad Q_s = \begin{pmatrix} m_B^2 & 0 \\ 0 & -\sqrt{2} m_F^2 i \partial_-^{-1} \end{pmatrix} I_c, \tag{90}
\]

where we have the identity \( I_c \) in the color space. The relevant operator is

\[
\omega_s^{-1} Q_s = \begin{pmatrix} -\frac{m_B^2}{2} \partial_-^{-1} & 0 \\ 0 & -\frac{m_F^2}{2} \partial_-^{-1} \end{pmatrix} I_c. \tag{91}
\]
The complex structure becomes (we drop the identity in the color space),

\[ J_s = \left[ \left( -\omega_s^{-1} \right) Q_s \right]^{-1/2} \omega_s Q_s = \begin{pmatrix} -(-\partial^2)^{1/2} \partial^{-1} & 0 \\ 0 & -(-\partial^2)^{1/2} \partial^{-1} \end{pmatrix}. \]  

(92)

Clearly we can use the Fourier decomposition of our fields to diagonalize this and the frequency operator \( K_s \),

\[ \phi^\alpha(x^-) = \int \frac{[dp]}{\sqrt{2|p|}} w^\alpha(p) e^{-ipx^-} \psi^\alpha(x^-) = \int \frac{[dp]}{2^{3/4}} s^\alpha(p) e^{-ipx^-}, \]

(93)

here we should think of \( w^\alpha \) as even and \( \zeta^\alpha \) odd elements of the Grassmann algebra defined by a series (an infinite one) of unspecified generators \( \theta^\alpha(p) \). Apply the operator \( J_s \) to the vectors of this graded space,

\[ J_s \psi = J_s \left( \phi^\alpha \right) = \int [dp] e^{-ipx^-} (-i \text{sgn}(p)) \left( \frac{\sqrt{2|p|}}{2^{-3/4}} \right) \frac{w^\alpha(p)}{\zeta^\alpha(p)}, \]

(94)

which we should rewrite as,

\[ (J_s \psi)(x^-) = \int_0^\infty [dp] e^{-ipx^-} \left[ (-i) \left( \frac{\sqrt{2p}}{2^{-3/4}} \right) z^\alpha(p) + i \left( \frac{\sqrt{2p}}{2^{-3/4}} \right) \zeta^\alpha(p) \right], \]

(95)

defining the super-holomorphic coordinates, \( (z^\alpha(p), \zeta^\alpha(p)) \). If we assign now our creation and annihilation operators according to the sign of \( i \),

\[ \begin{pmatrix} z^\alpha(p) \\ \zeta^\alpha(p) \end{pmatrix} \mapsto \begin{pmatrix} a^{\alpha^\dagger}(p) \\ \chi^\alpha(p) \end{pmatrix}, \quad \begin{pmatrix} \alpha^\dagger(p) \\ \alpha(p) \end{pmatrix} \mapsto \begin{pmatrix} \zeta^\alpha(p) \\ \zeta(p) \end{pmatrix}, \]

(96)

we get the commutation/anticommutation relations for \( a^{\alpha^\dagger}(p), a^\beta(q) \) and \( \chi^{\alpha^\dagger}(p), \chi^\alpha(q) \) respectively, and the zero commutator between the two sets. These commutation/anticommutation relations have the operator \( \omega_s \) on the right hand-side, this is what determines the algebra. Hence we see that we are in the geometric setting we were describing. Our bilinears combined in the form of \( \Phi \) satisfy all the Lie algebra properties. In fact it is instructive to write down the super-symplectic form \( \Omega_s \) with \( \Phi \) expressed in terms of the bilinears \( B, F, C, C^\dagger \).

From the above discussion we again see the remarkable fact that the geometry which is defined by the complex structure \( J_s \) is independent of the parameters of the theory in this light-cone method. This means even though the masses change due to the interactions this will not change the representation of canonical commutation/anticommutation relations we started with, as a result the geometry stays the same.

Our bilinears will correspond to the generators of the automorphisms of this full algebra of commutation/anticommutation relations, and it is the restricted real \( OSp \) group (for finite dimensional automorphism groups see [34], for Bogoliubov automorphisms of quasi-free representations see [35, 36, 37, 38]). We can check that the bilinears we have satisfy the Lie algebra conditions and the implementability of these automorphisms will imply the convergence conditions. Therefore the evolution of the system in the large-\( N_c \) limit realizes all
automorphisms of the the quasi-free second quantization of this system when we think of it without the color part—the color part has been averaged out and reduced the system to this bilinears. We may give an argument using the super-coherent states[39, 40], similar to the ordinary cases: there is a central extension of the automorphism group $\text{OSP}_1$ which is realized by these bilinears on the full Fock space. When we think about the projective Fock space this descends to the $\text{OSP}_1$ group. The orbit of the vacuum under this group gives us a classical phase space albeit a more general one, with a super symplectic form. The large-$N_c$ limit provides this reduction to the space of super-coherent states. This is a natural classical phase space and the large-$N_c$ limit corresponds to this classical limit.

Before ending our discussions we would like to make a few comments of general nature. Let us write down a super-dynamical system in the Hamiltonian form

$$S_0 = \int dt \frac{1}{2} \Psi^* \omega_s \partial_t \Psi - \int dt \frac{1}{2} \Psi^* \dot{Q}_s \Psi. \quad (97)$$

We assume that the action is an element of the even part of the Grassmann algebra. If we want this to be real we demand $(\Psi^* Q_s \Psi)^* = \Psi^* \tilde{E} Q_s^* \Psi = \Psi^* Q_s \Psi$, that is $Q_s = \tilde{E} Q_s^*$, we are again using $\tilde{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the standard decomposition. For the first term it implies the same $\tilde{E} \omega_s^* = \omega$. If we further note that it should be invariant under the transpose, we get for the first term using an integration by parts for the time derivative, $\Psi^* \omega_s \partial_t \Psi = -\Psi^* \omega_s^T \tilde{E} \partial_t \Psi$, which implies $\omega_s = -\omega_s^T \tilde{E}$ and the second term requires $\Psi^* Q_s \Psi = \Psi^* Q_s^T \tilde{E} \Psi$, which means we should have $Q_s = Q_s^T \tilde{E}$. The equations of motion will give us,

$$\partial_t \Psi = \omega_s^{-1} Q_s \Psi. \quad (98)$$

This suggests that we should further investigate operator $\omega_s^{-1} Q_s$ which is a type $(1, 1)$ tensor, thus a true linear transformation. We note that $\omega_s^{-1} Q_s$ is real: $(\omega_s^{-1} Q_s)^* = (\omega_s^{-1})^* Q_s^* = \omega_s^{-1} \tilde{E} \tilde{E} Q_s = \omega_s^{-1} Q_s$, by using the conjugation properties of $\omega_s$ and $Q$. This operator is antisymmetric with respect to the form defined by $Q_s$:

$$Q_s^{-1} (\omega_s^{-1} Q_s)^* Q_s = Q_s^{-1} Q_s^T (\omega_s^T)^{-1} Q_s = -Q_s^{-1} Q_s \tilde{E} \omega_s^{-1} Q_s = -\omega_s^{-1} Q_s. \quad (99)$$

as well as under $\omega_s$. It would be most natural if we could use a generalization of the polar decomposition for $Q_s^{-1} \omega_s$, and write this operator as $\omega_s^{-1} Q_s = J_s K_s$, where $J_s^T J_s = 1$, and $K_s > 0, K_s^T = K_s$, with an appropriate transpose $^T$ and positivity is assumed to be given a meaning in this super-context. Then we could claim that the basis in which $J_s$ is diagonal, will tell us the separation of creation and annihilation operators in this full generality. This can be done in the simple case we looked at, when the operators involved only had body parts, and no Grassmann numbers. Unfortunately for the general case we do not have the proper mathematical machinery. If we could find a super-transformation $S$, such that $S^{-1} \omega_s^{-1} Q_s S$ is diagonal with each entry $(\pm i \lambda_k)$ for a pure number $\lambda_k$ we could postulate the quantization by means of canonical commutation/anticommutation relations. To the best of our knowledge there is no such theorem. We think these questions deserve further investigations.
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