BELINFANTE TENSORS INDUCED BY MATTER-GRAVITY COUPLINGS

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Abstract

We show that any generally covariant coupling of matter fields to gravity gives rise to a conserved, on-shell symmetric energy-momentum tensor equivalent to the canonical energy-momentum tensor of the flat-space theory. For matter fields minimally coupled to gravity our algorithm gives the conventional Belinfante tensor. We establish that different matter-gravity couplings give metric energy-momentum tensors differing by identically conserved tensors. We prove that the metric energy-momentum tensor obtained from an arbitrary gravity theory is on-shell equivalent to the canonical energy-momentum tensor of the flat-space theory.

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1 Introduction

It is well known that a field theory with a Lagrangian density $\mathcal{L}(\Phi, \partial \Phi)$ which doesn’t have explicit coordinate dependence has a canonical energy-momentum tensor [1]

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_\nu \mathcal{L},$$

which is conserved

$$\frac{\delta S}{\delta \Phi} \partial_\nu \Phi = -\partial_\mu T^\mu_\nu$$

as a consequence of the invariance under

$$x^\mu \to x^\mu + \epsilon^\mu, \quad \Phi \to \Phi + \epsilon^\mu \partial_\mu \Phi$$

with constant parameters $\epsilon^\mu$.

In what follows we will assume $d$-dimensional Minkowskian space-time with $d \geq 2$. Using tensor $T^\mu_\nu$ we may define generators of the space-time translations

$$P^\mu = \int d^d \mathbf{x} \ T^{\mu 0}.$$ (2)

Equations (1) and (2) are invariant under

$$T^\mu_\nu(x) \to T^\mu_\nu(x) = T^\mu_\nu(x) + \partial_\lambda C^{[\lambda\mu]}_\nu(x)$$ (3)

with an arbitrary tensor $C^{[\lambda\mu]}_\nu(x)$ which obeys appropriate boundary conditions at infinity. Therefore, equation (3) defines an equivalence relation for conserved energy-momentum tensors.

It is known that the canonical energy-momentum tensor is generally not symmetric on the equations of motion. Any on-shell symmetric energy-momentum tensor equivalent to the canonical one is called a Belinfante energy-momentum tensor. It is easy to see that such a tensor is not unique and one may consider further “improvements” [2]-[3]. In the case of tensor fields, the conventional choice for the Belinfante energy-momentum tensor [4]-[5] is given by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda A^{[\lambda\mu]}_{\nu},$$ (4)

with

$$A^{[\mu\nu]}_\rho = \frac{1}{2} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial (\partial_\rho \Phi)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi)} S^{\mu\rho} \Phi \right),$$

where $S^{\mu\rho}$ are generators of the Lorentz transformations. Using $\Theta^{\mu\nu}$ we may construct conserved currents

$$j^{\mu\nu} = \Theta^{\mu\nu} x^\rho - \Theta^{\mu\rho} x^\nu,$$
associated with the Lorentz transformations.

For a given generally covariant formulation of the theory the metric energy-momentum tensor is defined by

\[ \bar{T}_{\mu\nu} = -2 \frac{\delta S[\Phi,g]}{\delta g_{\mu\nu}} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}}, \]

which is a flat-space limit of the Einstein energy-momentum tensor. The metric energy-momentum tensor is symmetric for all field configurations (i.e. off-shell). However, it is the canonical energy-momentum tensor, not a metric one, which defines generators of the space-time translations.

In the next sections we will find a relation between the metric and canonical energy-momentum tensors, and show that any generally covariant formulation of a theory enables us to construct an on-shell symmetric energy-momentum tensor equivalent to \( T_{\mu\nu} \), i.e. a Belinfante tensor. In the case of minimally coupled tensor fields, it coincides with the conventional Belinfante tensor. We will find that metric energy-momentum tensors defined by various gravity theories are on-shell equivalent to the canonical energy-momentum tensor of the flat-space theory.

**Notations**

\( \Phi \) stands for all matter fields, and we use the notation

\[ [F] = \{ F, \partial_\mu F, \partial_\mu \partial_\nu F, \ldots \}. \]

The Euler-Lagrange derivative of a function \( f(x, \Phi(x), \partial_\mu \Phi(x), \ldots, \partial_{\mu_1} \ldots \partial_{\mu_n} \Phi(x)) \) is defined by

\[ \frac{\delta_{EL} f}{\delta \Phi} = \sum_{k=0}^{n} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} \left( \frac{\partial f}{\partial (\partial_{\mu_1} \ldots \partial_{\mu_k} \Phi)} \right). \]

We also assume a summation over the repeated indices.

### 2 The Conserved Current Induced by a Gauge Invariant Action

We will begin in a very general setting which is applicable not only to matter coupled to gravity, but also to matter coupled to gauge fields of arbitrary spin. Let a local action
\[ S[X] = \int dx L([X], x), \quad X = (\Phi, A) \] be invariant under the gauge symmetry
\[ \delta \Phi(x) = \sum_{n=0}^{N} t^{\mu_1 \cdots \mu_n}_\alpha(x) \partial_{\mu_1} \cdots \partial_{\mu_n} \epsilon^\alpha(x), \quad N < \infty, \]
\[ \delta A(x) = \sum_{k=0}^{M} \lambda^{\mu_1 \cdots \mu_k}_\alpha(x) \partial_{\mu_1} \cdots \partial_{\mu_k} \epsilon^\alpha(x), \quad M < \infty \]
with arbitrary functions \( \epsilon^\alpha(x) \), where \( \alpha \) is not necessarily a tensor index. We have
\[ \frac{\delta S}{\delta \Phi} \delta \Phi + \frac{\delta S}{\delta A} \delta A = -\partial_\mu J^\mu, \quad (5) \]
with
\[ \frac{\delta S}{\delta X} = \delta_{EL} L, \quad J^\mu(x) = \sum_{i=0}^{\max\{N,M\}-1} j^\mu_{\alpha}(x) \partial_{\nu_1} \cdots \partial_{\nu_i} \epsilon^\alpha(x). \]
The local functions \( t^{\mu_1 \cdots \mu_n}_\alpha(x) \), \( \lambda^{\mu_1 \cdots \mu_n}_\alpha(x) \), and \( j^{\mu}_{\alpha}(x) \) are assumed to be symmetric in \( (\mu_1 \ldots \mu_n) \). Suppose that for \( A(x) = A_0(x) \) we have
\[ \lambda^\alpha(x)|_{A(x)=A_0(x)} = 0. \]
Define an action
\[ \bar{S}[\Phi] = S[\Phi, A]|_{A(x)=A_0(x)} \]
and consider terms proportional to \( \epsilon^\alpha \) in the equation (5):
\[ \frac{\delta \bar{S}}{\delta \Phi} \bar{t}^\mu_\alpha = -\partial_\mu \bar{j}^\mu_\alpha, \quad (6) \]
where “bar” means that we set \( A(x) \) to \( A_0(x) \). Therefore, an action \( \bar{S}[\Phi] \) is invariant under a rigid symmetry
\[ \bar{\delta} \Phi = \bar{t}^\mu_\alpha \epsilon^\alpha, \quad \epsilon^\alpha = \text{const}, \quad (7) \]
left unbroken by the background \( A(x) = A_0(x) \), and \( \bar{j}^\mu_\alpha \) are the associated currents. We conclude that an action \( \bar{S}[\Phi, A] \) can be viewed as a gauge invariant extension of the action
\[ \bar{S}[\Phi] = \int dx \ L_0([\Phi], x), \quad L_0 = \bar{L} + \partial_\mu K^\mu \]
with some local functions \( K^\mu([\Phi], x) \). Let \( j^\mu_{0\alpha} \) be the canonical currents associated with a symmetry (7):
\[ \frac{\delta \bar{S}}{\delta \Phi} \bar{j}^\mu_\alpha = -\partial_\mu j^\mu_{0\alpha}. \]
Comparison with (6) gives
\[ \partial_\mu (j^\mu_{0\alpha} - \bar{j}^\mu_\alpha) = 0. \]
Now we need the following corollary of the algebraic Poincare lemma [6]-[7]
If \( \partial_\mu j^\mu = 0 \) (identically) for a local \( j^\mu(\Phi, x) \), then there exists a local \( \sigma^{[\nu\mu]}(\Phi, x) \) such that \( j^\mu = \partial_\nu \sigma^{[\nu\mu]} + C^\mu \), where \( C^\mu \) are constants. When \( j^\mu = j^\mu(\Phi) \) then there exists \( \sigma^{\nu\mu} = \sigma^{[\nu\mu]}(\Phi) \). For \( d \geq 2 \) we may absorb constants \( C^\mu \) in \( \sigma^{\nu\mu} \). In this case \( \sigma^{\nu\mu} \) will have explicit coordinate dependence.

It follows that

\[
\tilde{j}_\alpha^\mu = j_\alpha^\mu + \partial_\sigma N_\alpha^{[\sigma\mu]} 
\]

with some local tensor \( N_\alpha^{[\sigma\mu]}(\Phi, x) \). Let’s define the currents \( \tilde{j}_A^\mu (x) \)

\[
\int dx \tilde{j}_A^\mu(x) \partial_\mu \epsilon^\alpha(x) = - \int dx \frac{\delta S}{\delta A}(x) \delta A(x), \tag{8}
\]

where we neglect the boundary terms. LHS of the equation (8) is invariant under

\[
\tilde{j}^\mu_A \rightarrow \tilde{j}^\mu_A + \partial_\rho C^{[\rho\mu]}_\alpha 
\]

with an arbitrary tensor \( C^{[\rho\mu]}_\alpha \) and specifies \( \tilde{j}_A^\mu \) up to the equivalence transformation. From equation (5) it follows that

\[
\epsilon^\alpha \partial_\mu j_\alpha^\mu + \tilde{j}_A^\mu \partial_\mu \epsilon^\alpha = \frac{\delta S}{\delta \Phi} \sum_{n=1}^{N} \tilde{t}^{\mu_1\ldots\mu_n} \partial_{\mu_1} \ldots \partial_{\mu_n} \epsilon^\alpha + \text{total divergence}.
\]

Applying the Euler-Lagrange derivatives on both sides of this equation we have

\[
\partial_\mu \tilde{j}_\alpha^\mu - \partial_\mu \tilde{j}_A^\mu = \partial_\mu \sum_{n=0}^{N-1} (-)^n \partial_{\nu_1} \ldots \partial_{\nu_n} \left( \frac{\delta S}{\delta \Phi} \tilde{t}^{\mu_1\ldots\nu_n} \right),
\]

which implies

\[
\tilde{j}_A^\mu = j_\alpha^\mu + \sum_{n=0}^{N-1} (-)^n \partial_{\nu_1} \ldots \partial_{\nu_n} \left( \frac{\delta S}{\delta \Phi} \tilde{t}^{\mu_1\ldots\nu_n} \right) + \partial_\lambda C^{[\lambda\mu]}_\alpha, \tag{9}
\]

with some local \( C^{[\lambda\mu]}_\alpha(\Phi, x) \). Therefore, currents \( \tilde{j}_A^\mu \) are conserved and on-shell equivalent to the canonical currents \( j_\alpha^\mu \). In the next sections we will specialize to fields coupled to gravity and refine the relation (9).

### 3 Construction of a Belinfante Tensor from a Gravity Theory

Analysis of the previous section can be applied to any generally covariant extensions of a given flat-space action. In this section we will assume that such an extension is
independent of higher derivatives of matter fields and doesn’t involve explicit coordinate
dependence. Our conclusions, however, stay valid even if we relax these assumptions.

Consider a flat-space action
\[ \tilde{S}[\Phi] = \int dx \, L_0(\Phi, \partial_\mu \Phi) \]
and let an action\(^2\)
\[ S[\Phi, g] = \int dx \, L(\Phi, \partial_\mu \Phi, g, \partial_\nu g, \ldots, \partial_{\nu_1}, \ldots, \partial_{\nu_M} g) \tag{10} \]
be a generally covariant extension of the theory:
\[ S[\Phi, g]|_{g_{\mu\nu} = \eta_{\mu\nu}} = \tilde{S}[\Phi], \quad L_0(\Phi, \partial_\mu \Phi) = \tilde{L}(\Phi, \partial_\mu \Phi) + \partial_\nu K^\nu \]
with some local functions \( K^\nu \) and we use a notation
\[ \tilde{F}[\Phi] \equiv F[\Phi, g]|_{g_{\mu\nu} = \eta_{\mu\nu}}. \]
In (10) we allow for the most general local coupling of the fields \( \Phi \) to the gravitational
field, provided that \( L \) is a scalar density with weight one. Action \( S[\Phi, g] \) is invariant
under the diffeomorphisms generated by \( x^\mu \to x'^\mu = x^\mu - \epsilon^\mu(x) \) with arbitrary functions \( \epsilon^\mu(x) \). The corresponding transformations of fields are given by
\[ \delta \Phi = \sum_{k=0}^{N} s^\mu_1 \ldots \mu_k ([\Phi], [g]) \partial_{\mu_1} \ldots \partial_{\mu_k} \epsilon^\lambda, \quad N < \infty, \]
\[ \delta g_{\mu\nu} = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \epsilon^\lambda + g_{\nu\lambda} \partial_\mu \epsilon^\lambda \]
with local functions \( s^\mu_1 \ldots \mu_k ([\Phi], [g]) \) which are assumed to be symmetric in the upper indices. We also assume \( s_\lambda = \partial_\lambda \Phi \). For the variation of the action \( S \) we have
\[ \delta S[\Phi, g] = \int dx \, \partial_\mu (\epsilon^\mu L) = \int dx \left( \frac{\delta S}{\delta \Phi} \delta \Phi + \frac{\delta S}{\delta g} \delta g + \partial_\mu f^\mu \right), \]
where
\[ f^\mu = \frac{\partial L}{\partial (\partial_\mu \Phi)} \delta \Phi + \frac{1}{2} \sum_{n=0}^{M-1} a^{\mu_1 \ldots \mu_n} [\rho_1 \ldots \rho_n] \partial_{\rho_1} \ldots \partial_{\rho_n} \delta g_{\nu\sigma} \]
with some local functions \( a^{\mu_1 \ldots \mu_n} [\Phi] \) symmetric in \((\nu\sigma)\) and \((\rho_1, \ldots, \rho_n)\). Introducing
the Einstein tensor
\[ T^\mu_\nu = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}, \]
\(^2\)We absorb the integration measure \( \sqrt{|g|} \) into the definition of \( L \).
we obtain
\[
\frac{\delta \bar{S}}{\delta \Phi} \sum_{k=0}^{N} \bar{s}_{\lambda}^{\mu_1 \ldots \mu_k} \partial_{\mu_1} \ldots \partial_{\mu_k} \epsilon^\lambda = \partial_\mu \left( \epsilon^\mu \bar{\mathcal{L}} - \frac{\partial \bar{\mathcal{L}}}{\partial \partial_\mu \Phi} \sum_{k=0}^{N} \bar{s}_{\lambda}^{\mu_1 \ldots \mu_k} \partial_{\mu_1} \ldots \partial_{\mu_k} \epsilon^\lambda \right) - \sum_{n=0}^{M-1} \bar{a}^{\mu \nu \sigma \rho_1 \ldots \rho_n} \eta_{\nu \lambda} \partial_{\rho_1} \ldots \partial_{\rho_n} \partial_\sigma \epsilon^\lambda \right)
\]
for arbitrary functions \( \epsilon^\lambda(x) \). Taking the Euler-Lagrange derivatives of both sides in (11) gives
\[
\partial_\mu \bar{T}_{E \lambda} = \sum_{k=0}^{N} (-)^{k+1} \partial_{\mu_1} \ldots \partial_{\mu_k} \left( \frac{\delta \bar{S}}{\delta \Phi} \bar{s}_{\lambda}^{\mu_1 \ldots \mu_k} \right),
\]
which implies that \( \bar{T}_{E \nu} \) is conserved. Terms proportional to \( \epsilon^\lambda(x) \) in (11) give
\[
\frac{\delta \bar{S}}{\delta \Phi} \bar{s}_{\lambda} = -\partial_\mu \bar{T}_{E \mu \lambda}, \quad \bar{T}_{E \mu \lambda} = \frac{\partial \bar{\mathcal{L}}}{\partial \partial_\mu \Phi} \bar{s}_{\lambda} - \delta^\mu_\mu \partial_\mu \bar{\mathcal{L}}.
\]
Since both \( \mathcal{L}_0 \) and \( \bar{\mathcal{L}} \) are independent of higher derivatives of the fields \( \Phi \) and do not have explicit coordinate dependence, functions \( K^\nu \) can be chosen to have the following form
\[
K^\nu(x) = k^\nu(\Phi(x)) + k^\nu_{\mu} \epsilon^\mu(x)
\]
with some functions \( k^\nu(\Phi(x)) \) and constants \( k^\nu_{\mu} \). Therefore
\[
\bar{T}_{E \mu \lambda} = T_{0 \mu \lambda} + \partial_\sigma R_{\lambda}^{[\sigma \mu]} + \delta^\lambda_\lambda k^\rho_{\rho},
\]
where
\[
R_{\lambda}^{[\sigma \mu]}(\Phi) = 2k^{[\sigma}(\Phi)\delta^\mu_{\lambda]}
\]
and \( T_{0 \mu \lambda} \) is the canonical energy-momentum tensor in flat-space. Collecting terms with \( \partial_\rho \epsilon^\lambda(x) \), we obtain a relation between the canonical and metric energy-momentum tensors:
\[
\bar{T}_{E \lambda} = T_{0 \lambda} + \frac{\delta \bar{S}}{\delta \Phi} \bar{s}_\lambda + \partial_\mu \Sigma_\lambda^{\mu \rho} + \partial_\rho R_{\lambda}^{[\sigma \mu]} + \delta^\rho_\rho k^\rho_{\rho},
\]
where
\[
\Sigma_\lambda^{\mu \rho} = \frac{\partial \bar{\mathcal{L}}}{\partial \partial_\mu \Phi} \bar{s}_\lambda^\rho + \bar{a}^{\mu \nu \sigma} \eta_{\nu \lambda}.
\]
Using (12) and (13) we obtain
\[
\partial_\mu \partial_\nu \Sigma_\lambda^{\mu \rho} = \sum_{k=2}^{N} (-)^{k+1} \partial_{\mu_1} \ldots \partial_{\mu_k} \left( \frac{\delta \bar{S}}{\delta \Phi} \bar{s}_{\lambda}^{\mu_1 \ldots \mu_k} \right) + \partial_\mu D_{\lambda}^{[\mu \rho]}.
\]
Taking into account that a tensor \( \Sigma_\lambda^{\mu \rho} \) doesn’t have explicit coordinate dependence, the algebraic Poincare lemma gives
\[
\partial_\mu \Sigma_\lambda^{\mu \rho} = \sum_{k=1}^{N-1} (-)^{k+1} \partial_{\mu_1} \ldots \partial_{\mu_k} \left( \frac{\delta \bar{S}}{\delta \Phi} \bar{s}_{\lambda}^{\mu_1 \ldots \mu_k} \right) + \partial_\mu D_{\lambda}^{[\mu \rho]}.
\]
for some local tensor $D^{[\mu\rho]}_{\lambda} \Phi$. Let us introduce a tensor

$$\Theta^\mu_\nu [\Phi] = T^\mu_0 [\Phi] + \partial_\rho (R^\rho_{\mu\nu} [\Phi] + D^\rho_{\mu\nu} [\Phi]).$$

From equations (13) and (14) it follows that

$$\Theta^\mu_\nu = T^\mu_\nu + \sum_{k=0}^{N-1} (-1)^{k+1} \partial_{\rho_1} \ldots \partial_{\rho_k} \left( \frac{\delta S}{\delta \Phi} \delta^{[\mu\rho_1 \ldots \rho_k]} \eta^{\lambda\nu} \right) - \eta^\mu_\nu k^\rho, \quad (15)$$

which implies that the tensor $\Theta^\mu_\nu$ is symmetric on-shell. Thus, $\Theta^\mu_\nu$ is a Belinfante energy-momentum tensor. We note that the last term on the RHS of (15) can be written as

$$\eta^\mu_\nu k^\rho = \partial_\lambda C^{[\mu\lambda] \nu}, \quad C^{[\mu\lambda] \nu} = \frac{2}{d-1} \eta^{[\mu_\nu \lambda]} k^\rho.$$ 

Generators of translations are given by

$$P^\mu = \int d\vec{x} T^\mu_0 = \int d\vec{x} \Theta^\mu_0,$$

provided that we choose appropriate boundary conditions at infinity. Using $\Theta^\mu_\nu$ we may construct conserved currents

$$j^{\mu\nu\rho} = \Theta^{\mu\nu} x^\rho - \Theta^{\mu\rho} x^\nu, \quad \partial_\mu j^{\mu\nu\rho} = O \left[ \frac{\delta S}{\delta \Phi} \right].$$

Let $S_1[\Phi, g]$ and $S_2[\Phi, g]$ correspond to different generally covariant formulations of a given theory:

$$S_1[\Phi, \eta] = S_2[\Phi, \eta] = \bar{S}[\Phi].$$

Equation (12) implies that the corresponding metric energy-momentum tensors differ by an identically conserved tensor$^3$:

$$\bar{T}^\mu_\nu - \bar{T}^\mu_\nu = \partial_\rho \Lambda^{[\mu\rho] \nu},$$

with some $\Lambda^{[\mu\rho] \nu}$, such that $\partial_\rho \Lambda^{[\mu\rho] \nu}$ is symmetric in $(\mu \nu)$.

Thus, we conclude that any generally covariant generalization of a flat-space theory gives rise to an energy-momentum tensor $\Theta^\mu_\nu$ which is symmetric on-shell. Contrary to the canonical energy-momentum tensor, which has only the first-order derivatives of $\Phi$, the expression for $\Theta^\mu_\nu$ may involve higher derivatives. The tensors $T^\mu_0$ and $\Theta^\mu_\nu$ are on-shell equivalent to the metric energy-momentum tensor $\bar{T}^\mu_\nu$.

$^3$We assume that transformation laws for $\Phi$ in both theories are the same.
4 Minimal Coupling. Tensor Fields.

Consider matter fields $\Phi$ minimally coupled to gravity:

$$S[\Phi, g] = \int dx \ L(\Phi, \partial_\mu \Phi, g, \partial_\nu \Phi), \quad L_0(\Phi, \partial_\mu \Phi) = L(\Phi, \partial_\mu \Phi).$$

We will consider the case when the variations $\delta \Phi$ are independent of higher derivatives of the parameters $\epsilon^\lambda(x)$:

$$\delta \Phi = \epsilon^\lambda \partial_\lambda \Phi + s^\mu_\lambda(\Phi, [g]) \partial_\mu \epsilon^\lambda.$$

Equation (11) reduces to

$$\frac{\delta \bar{S}}{\delta \Phi} \left( \epsilon^\lambda \partial_\lambda \Phi + s^\mu_\lambda \partial_\mu \epsilon^\lambda \right) - \bar{T}^\mu_\nu \eta_{\mu\lambda} \partial_\nu \epsilon^\lambda = \partial_\mu \left( \epsilon^\mu \mathcal{L}_0 - \frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \Phi)} (\epsilon^\lambda \partial_\lambda \Phi + s^\sigma_\lambda \partial_\sigma \epsilon^\lambda) \right) - \frac{\overline{\partial \mathcal{L}}}{\partial (\partial_\mu g_{\nu\sigma})} (\eta_{\nu\lambda} \partial_\sigma \epsilon^\lambda + \eta_{\lambda\sigma} \partial_\nu \epsilon^\lambda).$$

Terms proportional to the second-order derivatives of $\epsilon^\lambda(x)$ give

$$\Sigma^\mu_\nu = -\Sigma^\nu_\mu$$

with

$$\Sigma^\mu_\nu = \frac{\partial \mathcal{L}_0}{\partial (\partial_\mu \Phi)} s^\nu_\lambda + 2 \frac{\overline{\partial \mathcal{L}}}{\partial (\partial_\mu g_{\nu\sigma})} \eta_{\sigma\lambda}.$$ 

Terms with the first-order derivatives of $\epsilon^\lambda(x)$ imply

$$T_0^\mu - \bar{T}_0^\mu = -\frac{\delta \bar{S}}{\delta \Phi} s^\nu_\mu - \partial_\sigma \Sigma^{[\sigma\nu]}.$$ 

If we define

$$\Theta^{\mu\nu} = T_0^{\mu\nu} + \partial_\sigma \Sigma^{[\sigma\mu\nu]},$$

then we have

$$\Theta^{[\mu\nu]} = -\frac{\delta \bar{S}}{\delta \Phi} s^{[\mu\nu]} - \partial_\mu \Theta^{\nu\mu} = O\left(\delta \bar{S} \over \delta \Phi\right).$$

Thus, $\Theta^{\mu\nu}$ is a conserved, on-shell symmetric energy-momentum tensor equivalent to $T_0^{\mu\nu}$, i.e. a Belinfante tensor.

Now we consider the case when the fields $\Phi$ are tensors. For simplicity we assume that the fields $\Phi$ have only lower indices. For the tensor fields

$$s^{[\mu\nu]} = -\frac{1}{2} s^{\mu\nu} \Phi,$$
so that
\[ \bar{s}_{a_1...a_n} = \sum_{i=1}^{n} \Phi_{a_1...a_{i-1}\lambda a_{i+1}...a_n} \eta^{\nu\lambda} \delta_{a_i}^\rho. \]

In the case of minimal coupling to gravity we have:
\[ \partial_\mu \Phi_{a_1...a_n} \rightarrow \nabla_\mu \Phi_{a_1...a_n} = \partial_\mu \Phi_{a_1...a_n} + \Omega_{\mu[a_1...a_n} \Phi_{b_1...b_n]}, \]
with \( \Omega_\mu \) defined by
\[ \Omega_{\mu[a_1...a_n} = -\sum_{i=1}^{n} \left( \Gamma_{\mu a_i}^{b_i} \prod_{j \neq i} \delta_{a_j}^{b_j} \right), \]
with the metric connection \( \Gamma_{\mu a_i}^{b_i} \). Therefore,
\[ \bar{\partial} \frac{\partial L}{\partial (\partial_\nu g_{\rho\sigma})} = \frac{\partial L_0}{\partial (\partial_\lambda \Phi)} \bar{\partial} \frac{\partial \Omega_\lambda}{\partial (\partial_\nu g_{\rho\sigma})} \Phi, \]

Using
\[ \bar{\partial} \frac{\partial \Phi_{b_1...b_n}}{\partial (\partial_\nu g_{\rho\sigma})} = -\frac{1}{2} \sum_{i=1}^{n} \left( \eta^{b_i} \delta_{\mu}^{\nu} \delta_{\rho}^{a_i} + \eta^{b_i} \delta_{\rho}^{\sigma} \delta_{\mu}^{a_i} - \eta^{b_i} \delta_{\rho}^{\nu} \delta_{\mu}^{a_i} \prod_{j \neq i} \delta_{a_j}^{b_j} \right), \]
we have
\[ \bar{\partial} \frac{\partial L}{\partial (\partial_\nu g_{\rho\sigma})} = \frac{1}{2} \bar{\partial} \frac{\partial L_0}{\partial (\partial_\sigma \Phi)} \bar{s}^{[\rho\nu]} + \frac{1}{2} \bar{\partial} \frac{\partial L_0}{\partial (\partial_\rho \Phi)} \bar{s}^{[\sigma\nu]} - \frac{1}{2} \bar{\partial} \frac{\partial L_0}{\partial (\partial_\nu \Phi)} \bar{s}^{[\rho\sigma]}, \]
and finally
\[ \Sigma^{\mu\nu\lambda} = \bar{s}^{[\rho\nu]} + \bar{s}^{[\sigma\nu]} + \bar{s}^{[\rho\sigma]} = A^{[\mu\nu\lambda]}. \]

Thus in the case of tensor fields minimally coupled to gravity \( \Theta^{\mu\nu} \) is the standard Belinfante tensor (4).

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References


