Abstract: We discuss recent results on one-loop contributions to the effective action in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions. Contributions with five external vector fields are compared with corresponding ones in open superstring theory in order to understand the relation with the $F^5$ terms that appear in the nonabelian generalization of the Born-Infeld action.

1 Introduction

This talk is based on the paper [1]. Scattering amplitudes of massless modes in string theories can be described in terms of effective Lagrangians. For the open string case the abelian Born-Infeld action represents a remarkable example of an effective field theory which contains string corrections to all orders in $\alpha'$ [2]. The contributions can be summed up to all orders in the case of a constant, abelian field strength. As soon as these two conditions are relaxed the problem becomes complicated and a complete action for such fields has not been obtained. A non-abelian generalization of the Born-Infeld action can be defined as suggested in [3] using a symmetrized trace operation over the gauge group matrices. However there are indications that this prescription might not be sufficient to include all the contributions that one would obtain from an open string approach [4, 5].

The best available way to construct the effective action for the non-abelian, non-constant curvature case, is to proceed order by order. One has to compute corrections with increasing number of derivatives and of field strengths. A well established result is up to the order $\alpha'^2$ [6]. As soon as one focuses on higher orders the calculations become difficult. The inclusion of supersymmetry seems to be quite useful to gain insights and it might set enough constraints to fix uniquely the form of the action. Several attempts in different directions are under consideration [5, 7].

In our paper [1] we have attacked the problem as follows: since we want to make contact with the ten-dimensional open superstring we have considered its four-dimensional field theory limit, i.e. the $\mathcal{N}=4$ supersymmetric Yang-Mills theory. Then we have computed at one-loop perturbatively: the idea is that in this way we construct an effective
action which is supersymmetric and generalizes the Yang-Mills theory. If supersymmetry determines the form of the allowed deformations [8] it should correspond to the non-abelian Born-Infeld theory. However the crucial open question is: to what extent is supersymmetry enough to fix the form of higher order corrections?

We present the calculation of the four- and five-point functions, and of the part of the six-point function which is needed for the covariantization of the lower order results. These computations allowed us to evaluate the complete gauge invariant structures for the $F^4$ and the $\nabla \nabla F^4$, $F^5$ contributions. Since the symmetric trace prescription would rule out $F^5$ terms, the nonvanishing result that we find suggests a richer form for the non-abelian Born-Infeld action. In fact the trace operation on the gauge group indices receives contributions from the symmetric as well as the antisymmetric part.

In the next section we briefly recall the $\mathcal{N} = 1$ superspace formulation of the $\mathcal{N} = 4$ Yang-Mills action and describe the main ingredients that enter the quantization via the background field approach. In section 3 we present the one-loop amplitude up to order $\frac{1}{\alpha'}$ in the low-energy expansion. We have extracted from the superfield result its component content and in particular have studied the bosonic contributions which contain the field strengths $F_{\mu \nu}$. We then compare our results with corresponding ones from open superstring theory [9] and with other results recently obtained with different techniques in [10, 11].

2 The $\mathcal{N} = 4$ Yang-Mills action and its quantization

The $\mathcal{N} = 4$ supersymmetric Yang-Mills classical action written in terms of $\mathcal{N} = 1$ superfields (we use the notations and conventions adopted in [12]) is given by

$$S = \frac{1}{g^2} \text{Tr} \left( \int d^4x \ d^4\theta \ e^{-V} \Phi_i e^V \Phi^i + \frac{1}{2} \int d^4x \ d^2\theta \ W^\alpha W_\alpha + \frac{1}{2} \int d^4x \ d^2\bar{\theta} \ W^\alpha \bar{W}_\alpha \right. \left. + \frac{1}{3!} \int d^4x \ d^2\theta \ i\epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{1}{3!} \int d^4x \ d^2\bar{\theta} \ i\epsilon_{ijk} \bar{\Phi}_i [\bar{\Phi}_j, \bar{\Phi}_k] \right)$$

where the $\Phi^i$ with $i = 1, 2, 3$ are three chiral superfields, and the $W^\alpha = i \bar{D}^2 (e^{-V} D^\alpha e^V)$ are the gauge superfield strengths. All the fields are Lie-algebra valued, e.g. $\Phi^i = \Phi_a^i T^a$, in the adjoint representation of the gauge group, with $[T_a, T_b] = i f_{abc} T_c$.

Since we are interested in computing field strength corrections to the Born-Infeld action, we will consider amplitudes with vector fields as external background. Moreover we will extract from them only the gauge field bosonic components using the relations

$$D_{(\alpha W\beta)}|_{\theta=0} = f_{\alpha\beta} = \frac{1}{2} (\sigma_{\mu\nu})_{\alpha\beta} F^{\mu\nu} \quad \bar{D}_{(\dot{\alpha} \bar{W}\dot{\beta})}|_{\theta=0} = \bar{f}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} F^{\mu\nu}$$

We have used the background field method [12]; after gauge-fixing the quadratic quantum $V$-action becomes

$$S \rightarrow -\frac{1}{2g^2} \text{Tr} \int d^4x \ d^4\theta \ V \left[ \Box - i\Gamma^{\gamma\dot{\gamma}} \partial_{\gamma\dot{\gamma}} - i \frac{1}{2} (\partial^{\gamma\dot{\gamma}} \Gamma_{\gamma\dot{\gamma}}) - \frac{1}{2} \Gamma^{\gamma\dot{\gamma}} \Gamma_{\gamma\dot{\gamma}} - i \bar{W}^\alpha (D_\alpha - i\Gamma_\alpha) - i \bar{W}_{\dot{\alpha}} (\bar{D}_{\dot{\alpha}} - i\Gamma_{\dot{\alpha}}) \right] V$$

The $\mathcal{N} = 4$ theory is particularly simple since the loops with the three chiral matter fields are cancelled by the three ghosts.

A one-loop $n$-point amplitude will contain $n$ external background fields from the interactions in (3), with at least two $W$ and two $\bar{W}$ in order to complete the $D$-algebra. We
set the external fields on-shell, i.e. we freely use the equations of motion \( \nabla^\alpha W_\alpha = 0 \) and \( \nabla^\alpha \bar{W}_\bar{\alpha} = 0 \)

We obtain the structures we are looking for if our one-loop diagram has produced products of fields \( D_\alpha W_\beta \) and their hermitian conjugate ones \( D_\bar{\alpha} \bar{W}_\bar{\beta} \).

In addition we have to deal with a loop-momentum integral which contains \( n \) scalar propagators and momentum factors directly from the vertices in (3) and/or from commutators of spinor derivatives produced while performing the \( D \)-algebra

\[
I_n = \int d^4k \frac{h_n(k, p_i)}{k^2(k + p_2)^2(k + p_2 + p_3)^2 \ldots (k + p_2 + \ldots + p_n)^2} \tag{4}
\]

We can rewrite (4) as an infinite series of local terms in a low-energy expansion with higher derivatives by introducing an IR mass \( M \) and expanding the propagators keeping the external momenta small as compared to \( M \) (see [1] for more details).

3 \( \mathcal{N} = 4 \) Yang-Mills at one loop up to order \( \frac{1}{M^6} \)

The four-point function corresponds to box-type diagrams as the one shown in Fig.1, and gives rise to a momentum integral

\[
I_4 = \int d^4k \frac{1}{k^2(k + p_2)^2(k + p_2 + p_3)^2(k + p_2 + p_3 + p_4)^2} \tag{5}
\]

and to a background field dependence

\[
\int d^4\theta \text{Tr}_{Ad}(W_\alpha(p_1)W_\alpha(p_2)\bar{W}_\bar{\alpha}(p_3)\bar{W}_\bar{\alpha}(p_4)) \tag{6}
\]

where in general we have defined \( \text{Tr}_{Ad}(AB \cdots) = A_a B_b \cdots g(a, b, \ldots) \), with the group colour factor \( g(a_1, a_2, \ldots, a_n) = f_{x_1 a_1 x_2} f_{x_2 a_2 x_3} \cdots f_{x_n a_n x_1} \).

We are interested in rewriting the superspace result in components and in particular we

want to study the bosonic gauge field contributions. If we do so we find that at the leading order in the local expansion the planar sector (large \( N \) limit) reproduces the symmetrized trace structure [3]

\[
\Gamma_4 \Rightarrow \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4} + \text{permutations}) \left\{ (F^4)_{a_1 a_2 a_3 a_4} - \frac{1}{4} (F^2)_{a_1 a_2} (F^2)_{a_3 a_4} \right\}
\]

\[
\equiv STr \left(F^4 - \frac{1}{4} (F^2)^2\right) \tag{7}
\]
The subleading contribution corresponds to terms with two derivatives:

$$\Gamma_4^{(\text{der})} = -\frac{1}{40} \left( \frac{\pi^2}{12 M^6} \right) \left[ 10(p_2)^2 + 8p_2 \cdot p_3 + 2p_2 \cdot p_4 \right] g(a_1, a_2, a_3, a_4)$$

$$\frac{1}{8} \left\{ \left[ 4(F^4)_{a1a2a3a4} + 4(F^4)_{a1a2a4a3} + 4(F^4)_{a1a3a2a4} - (F^2)_{a1a2}(F^2)_{a3a4} ight. \\
- (F^2)_{a1a3}(F^2)_{a2a4} - (F^2)_{a2a4}(F^2)_{a3a2} \left. \right] + \text{permutations } a_2, a_3, a_4 \right\} \quad (8)$$

These terms give rise to contributions that are not gauge invariant since from $p \rightarrow i\partial$ ordinary derivatives are produced. The correct covariantization of the result is obtained adding terms with one and two background connections $\Gamma_{\alpha\beta}$ from the five- and six-point functions respectively. They are shown in Fig.2b, Fig.3a and Fig.3b. By taking into account the contributions from these graphs we get the complete covariantization [1] of the derivative expression in (8). By further taking into account the properties of the trace operation and after some integrations by parts we obtain

$$\Gamma_4^{(\text{der})} = \frac{1}{20} \left( \frac{\pi^2}{12 M^6} \right) \Tr_{\text{Ad}} \left\{ (\nabla F_{\mu\nu} \nabla F_{\nu\rho} F_{\rho\sigma} F_{\sigma\mu} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\nu\rho} F_{\sigma\mu} + F_{\mu\nu} \nabla F_{\rho\sigma} \nabla F_{\nu\rho} F_{\sigma\mu} \right)$$

$$- \frac{1}{4} \left( \nabla F_{\mu\nu} \nabla F_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} + F_{\mu\nu} \nabla F_{\rho\sigma} \nabla F_{\mu\nu} F_{\rho\sigma} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) \right\}$$

$$- \frac{1}{20} \left( \frac{\pi^2}{12 M^6} \right) \Tr_{\text{Ad}} \left\{ -2 \nabla^2 F_{\mu\nu} F_{\nu\rho} F_{\rho\sigma} - 4 \nabla^2 F_{\mu\nu} F_{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right\} + \frac{1}{2} \nabla^2 F_{\mu\nu} F_{\mu\nu} F_{\rho\sigma} F_{\rho\sigma} \right\} \quad (9)$$

where it is understood that the indices on the two $\nabla$ are to be contracted.

Let us observe that quite generally we can rewrite terms which contain $\nabla^2$ and four $F$’s as $F^5$ contributions. Indeed, using the equations of motion $\nabla^2 F_{\mu\nu} = 0$ and the Bianchi identities $\nabla_{[\mu} F_{\nu\rho]} = 0$, we have the following identities (see [1])

$$\nabla^2 F_{\mu\nu} = 2i(F_{\mu\nu} F_{\sigma\rho} - F_{\nu\rho} F_{\mu\sigma}) \quad (10)$$

Thus in order to study corrections to the symmetrized trace prescription one has to consistently take into account derivative contributions. In particular the above relation clearly shows that for the case of two covariant derivatives one has to consider not only the antisymmetrized products $\nabla_{[\mu} \nabla_{\nu]}$, as it was already noticed [5], but also the symmetrized ones $\nabla_{(\mu} \nabla_{\nu)}$. Using (10), we can rewrite the last two lines of the 2-derivative result in (9) as $F^5$ and $F^2 F^3$ contributions.

Let us now consider the five-point function with two $W$ and three $\bar{W}$ vertices as in Fig.2a. After completion of the $D$-algebra in the loop we obtain terms with five scalar propagators and a typical background dependence of the form

$$\int d^4 \theta \ W^a(p_1, a_1) W_\alpha(p_2, a_2) \bar{W}^\alpha(p_3, a_3) D_\alpha \bar{W}^\beta(p_4, a_4) \bar{W}_\beta(p_5, a_5) \quad (11)$$

The final result at the leading order in the local expansion is:

$$\Gamma_5 = -\frac{1}{32} \left( \frac{\pi^2}{12 M^6} \right) g(a_1, a_2, a_3, a_4, a_5) \left[ -2(F^5)_{a1a2a3a4a5} - (F^5)_{a1a4a2a5a3} \right.$$

$$+ 3(F^5)_{a1a4a3a2a5} + 1/2(F^2)_{a2a4}(F^3)_{a1a3a5} - 1/2(F^2)_{a3a4}(F^3)_{a1a2a5} \right] \quad (12)$$
Let us now collect the results (9) and (12); making use of the identities given in Appendix C of [1] we can eliminate \( F_2 \) and \( F_3 \) in favor of \( F_5 \) terms. Our final result to the \( M^{-6} \) order is:

\[
\Gamma_{\text{tot}} = \frac{1}{20} \left( \frac{\pi^2}{12 M^6} \right) \text{Tr}_{\text{Ad}} \left\{ \left( \nabla F_{\mu\nu} \nabla F_{\nu\rho} F_{\rho\sigma} F_{\sigma\mu} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\sigma\rho} F_{\mu\nu} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\sigma\rho} F_{\mu\nu} \right) \\
- \frac{1}{4} \left( \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} + \nabla F_{\mu\nu} \nabla F_{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} \right) \right\} \\
+ \frac{1}{20} \left( \frac{\pi^2}{12 M^6} \right) \text{Tr}_{\text{Ad}} \left( -F_{\mu\nu} F_{\nu\rho} F_{\rho\sigma} F_{\sigma\tau} F_{\tau\mu} - \frac{7}{2} F_{\mu\nu} F_{\nu\rho} F_{\sigma\tau} F_{\rho\sigma} F_{\tau\mu} \\
- \frac{3}{2} F_{\mu\nu} F_{\rho\sigma} F_{\tau\mu} F_{\nu\rho} F_{\sigma\tau} + 2 F_{\mu\nu} F_{\rho\sigma} F_{\nu\tau} F_{\tau\mu} F_{\sigma\rho} \right) \right) \tag{13}
\]

Up to an overall numerical factor, the first two lines in (13) reproduce the corresponding result in ref. [9], formula (3.3), obtained from an open superstring scattering amplitude. Moreover they agree with the recent results recently obtained in [10, 11]. So, our first result is that, at the level of the four-point function, the supersymmetric Yang-Mills effective action exactly reproduces the structure of the non-abelian Born-Infeld theory, including the first derivative corrections. The terms \( F_5 \), which were also computed in [9], are more difficult to compare with ours: this is essentially due to the fact that the result quoted in [9] is not written in a canonical form and moreover it requires additional symmetrizations which we are not clear how to interpret unambiguously. However, they do not agree with the result found in [11].

We would like to conclude summarizing our results. We have considered the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory and computed at one-loop the four- and five-point
functions with external vector fields. From the superfield result we have extracted the part of the bosonic components which contain the field strengths $F_{\mu\nu}$. These non-local one-loop contributions have been expanded in a low energy approximation and expressed as a sum of an infinite series of local terms. We have argued that these local expressions reproduce contributions to the non-abelian Born-Infeld action, if supersymmetry has to determine its structure. However, the discrepancy with the result found in [11] seems to indicate that supersymmetry is not enough at this order.

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References