Remarks on the Calculations of Charged Open String Amplitudes: the 1-loop Tadpole

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Abstract: In string theory, there are various physical situations where the worldsheet fields have a shifted moding. For instance, this is the case for the twisted closed string in $\mathbb{Z}_N$ orbifold or for the charged open string in a constant electro-magnetic field. Because of this feature, it is quite challenging to give explicit formulae describing the string interaction, even for the bosonic case. In this note, we focus on the case of the charged open bosonic string and construct the 1-loop tadpole which is an object generating all 1-point functions from the annulus in the presence of an external field. In the operator formalism, this represents one of the basic building blocks for the construction of a general loop amplitude.

1 Introduction

In the last years we have learned that string theory has an even richer structures than was expected. Many interesting phenomena were discovered during the second string revolution and the concept of duality has become a central guideline in all the recent developments. Most of these new features of string theory are not manifest in the perturbative expansion and this is the main reason why they have been overlooked in previous analysis. However, even in this new context, well known techniques of string perturbation theory proved to be surprisingly useful. For instance, perturbative computations often represent a first check for conjectures based on expected duality relations between different theories. Even more important, open string perturbation theory acquired a new relevance. In general, we have learned that the boundary conditions for open strings play a central role in many different contexts: for instance, they give a quantitative description of various aspects of D-brane physics [1] and provide a natural setup for non–commutative theories [2].

In the light of these new ideas we reconsider the operator formalism for the construction
of string amplitudes and, in particular, the old technique of the “sewing procedure”. It
has already proven to be very useful in the study of open string theory in the presence
of a constant NS-NS $B$-field; in fact, by using this formalism, we were able to construct
multiloop amplitudes among neutral strings [3], and had shown that they are directly
related to the Feynman diagrams of non–commutative field theories. Our goal is to
extend this approach to the case of the charged string, where the boundary conditions are
different on the various borders of the worldsheet. It is known that this computation is
physically relevant in different contexts, such as the study of open strings in an external
electro-magnetic field [4] or the analysis of moving D-branes (which is a T-dual version of
the previous system) [5, 6], or even the study of closed string amplitudes in $Z_N$ orbifold
theories, where the string coordinates have a similar shifted moding. Here we consider
the simple case of bosonic string theory and derive the expression of the basic building
blocks that in principle can be sewn together to construct more complicated amplitudes.

In this note, we apply the approach of Refs. [7, 8] to the case of charged bosonic string
and construct an object generating all 1-point functions from the annulus in the presence
of an external field (the charged tadpole). Our main result is the derivation of this building
block both in the open channel (where the world–sheet looks like an annulus) and in the
closed one (where the surface looks like a cylinder). We finally comment about the possible
use of these tadpoles in the construction of higher loop charged string amplitudes.

2 Charged tadpoles

In the operator formalism, the string interaction is usually described by means of vertex
operators, like $V$ in Eq. (1), encoding the quantum number of the external emitted on-shell
state and depending on the Hilbert space of a virtual emitting string. This asymmetry is
cumbersome in multiloop computation and thus, for this purpose, it is better to use the
so–called Reggeon Vertex [9] $E\langle W\rangle$, where also the emitted string is described by means
of a Hilbert space.

$$V = \langle 1 \rangle^c \langle \langle kX(1) \rangle \; \; \longrightarrow \; \; E\langle W\rangle = \int dp \; E\langle p; 0 \rangle : e\{ -\frac{1}{2\alpha'} \oint dz X(z) \partial_z X + (\text{ghost}) \} :. \ (1)$$

One can think of the Reggeon Vertex as an ”off–shell” generalization of the usual vertex
operator. However, here off–shell does not have the usual meaning as in field theory. On
the contrary, off–shell just means that the external states have not been specified yet,
and thus $E\langle W\rangle$ can be seen as the generator of all possible three string interaction when
it is saturated with physical (on–shell) states. For instance, the 3-tachyon interaction is
proportional to $\langle k_1 \rangle E\langle W|k_2\rangle E\langle k_3 \rangle$. In this respect the Reggeon vertex is conceptually on
the same footing as the boundary states which generates all the 1-point functions on a
disk and thus the coupings and the classical field of the corresponding D-brane [10].

The key property of $E\langle W\rangle$ is the BRST invariance. Thus one can use a BRST invariant
propagator to identify couples of legs and construct objects with loops and more external
states. Again these new objects can be seen as the generators of more complicated string
amplitudes. The simplest example of this sewing procedure is to derive from $E\langle W\rangle$ the
1-loop tadpole [7, 8], that is an object generating all 1-point functions from the annulus.
This can be done simply by computing

$$E\langle T_O \rangle = \text{Tr} \left[ E\langle W\rangle \frac{b_0}{E_0} \right] = \int_0^1 \text{Tr} \left[ E\langle W\rangle b_0 k^L_0 \right] \frac{dk}{k}, \tag{2}$$
and the result is

\[ E(T_O) = \int \frac{dk}{k^2} \prod_{n=1}^{\infty} (1 - k^n)^{2-D} \int \frac{d^D p}{(2\pi)^D} \ E(0; 0) \ \exp \left\{ \frac{\sqrt{2\alpha'}}{\sqrt{n}} \sum_{n=1}^{\infty} \frac{a_n^E}{n} \right\} \]

\[ \times \exp \left\{ -\frac{1}{2} \sum_{r,s=1}^{\infty} a_s^E \left[ \sum_{m \neq 0} D_{sr}(\Gamma \rho k^m \rho) \right] a_r^E \right\} \ e^{2\alpha' \ln k} \times (\text{ghost}) . \] 

In the above formula \( \rho : z \to (1 - z) \) and \( \Gamma : z \to 1/z \) are projective transformations, \( a_n \) are the string modes satisfying the harmonic oscillator algebra, and, finally, the \( D \)-matrix form an infinite dimensional representation of the projective group of weight zero [11]: \( D_{nm}(\gamma) = \left( \sqrt{m/n} \right) \partial_z^n \gamma^n(z)|_{z=0} \), with \( n, m \geq 1 \) and \( D_{n0}(\gamma) = 1/\sqrt{n} \). From now on we will omit any reference to the ghost part and focus on the \( X \)-contribution, where all the novelties related to the external field are concentrated.

It is possible to derive the same tadpole of Eq. (3) by means of a different sewing procedure. One can start with a slight modification of the bosonic boundary state

\[ \langle 1, 1 | = \int \frac{d^4 p}{(2\pi)^4} \langle p, \ell' | \ 0 \ a p \ 0 | \ 0 \ \rangle \ \exp \left\{ -\sum_{n=1}^{\infty} \frac{a_n \cdot S \cdot \tilde{a}_n}{n} \right\} e^{i\ell' \cdot Y} \times \]

\[ \exp \left\{ -\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{a_n^{op}}{\sqrt{n}} \cdot p - \sum_{n=1,m=0}^{\infty} \left( \tilde{a}_n \cdot S D_{nm}(\rho) \cdot a_m^{op} + a_n \cdot D_{nm}(\Gamma \rho) \cdot a_m^{op} \right) \right\} , \]

which describes the interaction of one open and one closed string on a disk. For completeness Eq. (4) is written with mixed Dirichlet (\( \bot \)) and Neumann (\( | \)) boundary conditions, even if in the following we will need the expression with only Neumann directions, that is \( S_{\mu\nu} = \eta_{\mu\nu} \). In general, \( S_{\mu\nu} \) depends on the boundary conditions one wants to impose on the border of the disk and here we follow the convention of [10]. In order to construct the tadpole (3) starting from (4), one must propagate the closed state by means of the usual closed string propagator and finally saturate it with another boundary state so that a cylinder of finite length is formed. The result of this computation \( \langle d^2 r \langle 1, 1 | b_0 r^{L_0} \tilde{b}_0 \tilde{r}^{\bar{L}_0} | B, S = \eta \rangle \) is

\[ E(T_C) = \int \frac{dq}{q^2} \prod_{n=1}^{\infty} (1 - q^n)^{2-D} E(0; 0) \ \exp \left\{ -\frac{1}{2} \sum_{r,s=1}^{\infty} a_s^E \left[ \sum_{m \neq 0} D_{sr}(\Gamma \rho q^m \rho) \right] a_r^E \right\} . \] 

Formally the only difference from Eq. (3) is the absence of the Gaussian integration. In fact, by construction, the result (5) is written in the closed string channel where no momentum is exchanged when only Neumann directions are considered. Of course also the resulting parameterization of the surface is different and one has the usual relation \( |r|^2 = q = \exp(1/\ln k) \).

Let us make a remark about the relation of (3) and (5). In general one may wonder whether the tadpole written in the open channel and the one written in the closed channel are related by the usual 1-loop “modular” transformation. For this purpose, it is useful to rewrite the infinite products and sums in the above formulae (3) and (5) in terms of the \( \theta \)-function \( \theta_{11} \) and its derivatives. This is more easily done in the formulation of Eq. (7), where the exponential is written in terms of contour integrals. Since the properties of the \( \theta \)-function are well-known, the modular transformation can now be performed explicitly and one finds that, by transforming the tadpole \( T_C \), Eq. (3) is almost exactly recovered, apart from a single factor of \( \tau \): \( \langle T_O \rangle = \langle T_C \rangle \tau^{-1} \). This disagreement is due to a subtlety in
this check. In fact, as explained in [14] for the usual case of string amplitudes, in order to recover the correct results after a modular transformation, one need use both the BRST invariance (which ensures the cancellation of cuts in the parameter \( k \)) and the on-shell conditions for the external states (and this is necessary to get the right power of \( \tau = \ln k \)). For our tadpoles the first condition is satisfied by construction, once \( D \) is fixed to 26, as usual in the bosonic case. However, it is clear that the on-shell conditions cannot be imposed until the external legs are saturated with some physical states. Thus the modular transformation, by its nature, cannot be performed at the level of off-shell objects such as the tadpoles (3) and (5); and can be performed only at the level of amplitudes. And at this level, \( \langle T_c \rangle \) and \( \langle T_o \rangle \) generate results that are in perfect agreement with each other under modular transformation.

An advantage of working in the closed channel is that one can very easily introduce a constant field-strength in the boundary state, just by using \( S = \frac{1 + F}{1 - F} \). If, on the contrary, the matrix \( S \) appearing in \( \langle 1, 1 | \) is kept trivial \( (S = \eta) \), then the open string stretching between the two borders satisfies different boundary conditions at \( \sigma = 0 \) and \( \sigma = \pi \) (we usually refer to this case as the “charged string”). As is well known, in this case the modings of the open string coordinates are shifted (we use the conventions of [12]). If \( X \) is regarded as a complex function of the world-sheet coordinate \( z \), cuts are present and they are responsible for the difference between the boundary conditions on the two borders. As we will see, this makes the computation of the tadpole in the open channel (and thus of all multiloop amplitudes) quite challenging. On the contrary in the closed channel no modification on the \( X \)-expansion is present and the matrix \( S \) can be easily diagonalized, thus yielding only some phases in various steps of the computation. For sake of simplicity we think that \( F \) is non–vanishing only along the two directions \( x^1, x^2 \); in this case, in the coordinates \( x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm ix^2) \), \( S \) is diagonal: \( S = \{e^{2\pi i \epsilon}, e^{-2\pi i \epsilon}\} \), where \( F = \tan \pi \epsilon \). The result of the charged tadpole in the closed channel is

\[
\langle T_c, F \rangle = \frac{1}{\cos \pi \epsilon} \int dq q^2 \prod_{n=1}^{\infty} (1 - q^n)^{2-D} \frac{\prod_{n=1}^{\infty} (1 - q^n)^2}{\prod_{n=1}^{\infty} (1 - e^{2\pi i \epsilon} q^n)(1 - e^{-2\pi i \epsilon} q^n)} \times \langle 0, 0 | \exp \left\{ -\sum_{r,s=1}^{\infty} a_s \left[ \sum_{m>0} D_{sr}(\Gamma \rho q^m \rho) e^{-2\pi i m \epsilon} \right] a_r^+ - \ldots \right\} ,
\]

where the dots stand for the complex conjugate of the written exponent. Notice the overall factor \( \cos \pi \epsilon \) that is just the rewriting of the usual “Born-Infeld” normalization of the boundary state with a nontrivial \( F \) [13] or velocity [6].

It is interesting to notice that the above tadpole can be rewritten directly in terms of fields

\[
\langle T_c, F \rangle = \frac{1}{\cos \pi \epsilon} \int dq q^2 \prod_{n=1}^{\infty} (1 - q^n)^{2-D} \frac{\prod_{n=1}^{\infty} (1 - q^n)^2}{\prod_{n=1}^{\infty} (1 - e^{2\pi i \epsilon} q^n)(1 - e^{-2\pi i \epsilon} q^n)} \langle 0, 0 | \exp \left\{ -\frac{1}{2\alpha'} \int dzdw \partial X^-(z) \sum_{m>0} \left[ \ln \left( 1 - q^m \frac{1-w}{1-z} \right) \right] e^{-2\pi i m \epsilon} \partial X^+(w) - \ldots \right\} ,
\]

where the contour integral over \( z \) and \( w \) is around zero.

The modular transformation in the case of a non-vanishing \( F \) is clearly more complicated. In fact, the exponent in (7), cannot be simply related to the usual \( \theta \)-function because the phase encoding the external field enters in a non trivial way in the exponent and the sum of the log’s cannot be rewritten anymore as a log of product. However, there
is a natural guess for the form of this tadpole. In fact, the complications are concentrated in the exponent, while the measure can be written also in the open channel, obtaining the known result [4]. The remaining part of the expression can be constructed by analogy with Eq. (7). In fact, in the exponent one would expect the appearance of the Green function appropriate for the string which is propagating in the loop. So, in the case of the uncharged tadpole, or the tadpole in the closed channel the log is present. On the contrary, when the loop is described in terms of charged open strings, one would expect to find the hypergeometric functions that characterize their tree-level Green function. Thus our guess for the charged tadpole in the open channel is

\[ \langle T_O, F \rangle = F \int \frac{dk}{k^2} \tau^{-\frac{D}{2}} \prod_{n=1}^{\infty} (1-k^n)^{2-D} \frac{\tau e^{\pi \epsilon^2 \tau}}{\sinh \epsilon \tau} \prod_{n=1}^{\infty} (1-k^{n+\epsilon})(1-k^{n-\epsilon}) \langle 0, 0 \rangle \] (8)

In the second line, we have used the Green function \( \langle X^-(z)X^+(w) \rangle \) and the hypergeometric function arises from the sum over the shifted modes \( \sum_{n=0}^{\infty} \frac{1}{n+\epsilon} \left( \frac{w}{z} \right)^{n+\epsilon} \), while in the last line the correlator \( \langle X^+(z)X^-(w) \rangle \) has been used. We have performed a few checks on the above formula and it always yielded consistent results. For instance, in the \( \epsilon \to 0 \) limit it reproduces Eq. (3), including the terms coming from the gaussian integration over \( p \). Another check is possible in the case \( \epsilon = 1/2 \), where the modular transformation is doable. In this case the tadpoles of (8) and that of (7) are consistently mapped into each other under the usual map \( \tau_o \to -1/\tau_c \). Finally the 2-tachyon amplitude constructed from (6) and (8) agree with each other for all values of \( \epsilon \).

### 3 Discussion

Clearly the main problem of (8) is its complication. In particular, if one rewrites it in terms of modes, the matrices contracting the oscillators (analogous to the \( D \)'s of Eq. (3)) do not seem to be representation of the projective group or of any other group. This is a major obstacle in the multiloop computation where one has to compute products of these infinite-dimensional matrices. On the contrary the tadpole in the closed channel displays the usual structure related to the projective group. Thus, using Eq. (7) as a starting point for the sewing procedure, it is not difficult to construct higher loops amplitudes which appear automatically written in the Schottky parameterization, as usual in this approach. In particular the results are analytic in the Schottky multipliers \( q_o \), which are the multiloop generalization of the parameter \( q \) describing the length of the cylinder in Eq. (5). This has a clear physical explanation. In fact, the poles of the string amplitudes (when they are regarded as functions of the \( q_o \)'s) are, by unitarity, related to the mass of the particles exchanged in the various propagators. In the closed channel these are the masses of the closed states that are not modified by the presence of the external electro-magnetic field. On the contrary, in the open channel one expects some non-analytic behaviour, because the Virasoro constraint \( L_0 \) and thus mass-shell condition for the open strings are deformed by \( F \). This feature is already present at the level of the
1-loop partition function \([4]\), where only the mass spectrum of the theory is relevant. The tadpoles presented here are a starting point towards a multiloop generalization of the above result. In the multiloop case also the interaction among charged and uncharged strings must play a role. From the above considerations, it is clear that, even if the goal is to compute the multiloop interaction of the charged string in the open channel, it is easier to use the building blocks written in closed variables to construct the amplitude and then do the modular transformation on the final result in order to rewrite it in the desired channel. We will pursue this approach in a forthcoming paper.

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References


