IFUM 693/FT

Algebraic Properties of BRST Coupled Doublets.

Andrea Quadri [1]

Università degli Studi di Milano
and
INFN, Sezione di Milano
via Celoria 16, I20133 Milano - Italy

Abstract

We characterize the dependence on doublets of the cohomology of an arbitrary nilpotent differential s (including BRST differentials and classical linearized Slavnov-Taylor (ST) operators) in terms of the cohomology of the doublets-independent component of s. All cohomologies are computed in the space of local integrated formal power series. We drop the usual assumption that the counting operator for the doublets commutes with s (decoupled doublets) and discuss the general case where the counting operator does not commute with s (coupled doublets). The results are purely algebraic and do not rely on power-counting arguments.

[1] `andrea.quadri@mi.infn.it`

1 Introduction

A couple of variables (z, w) is said a doublet under the nilpotent differential s if

$$sz = w\,, \qquad sw = 0\,. \tag{1}$$

In the physically relevant cases s is usually identified with the classical BRST differential or the linearized classical Slavnov-Taylor (ST) operator $\mathcal{S}_0$ [1, 2].

In this paper we provide a comprehensive discussion of the dependence of the cohomology of s on doublets in the space of local integrated formal power series. We first review the standard result referring to the case of decoupled doublets, where the counting operator for the set of doublets (z_k, w_k)

$$\mathcal{N} \equiv \int d^4x \sum_k \left(z_k \frac{\delta}{\delta z_k} + w_k \frac{\delta}{\delta w_k} \right) \tag{2}$$

commutes with the differential s. In this case it can be shown [2, 3] that the contribution of the doublets (z_k, w_k) to the cohomology of s is always trivial: if $\mathcal{I}[z, w, \varphi]$ is any s-invariant local integrated formal power series in its arguments and their derivatives, depending on the set of doublets $z = \{z_k\}$, $w = \{w_k\}$ and on a set of other fields and external sources collectively denoted by φ, then there exists a local integrated formal power series $\mathcal{G}[z, w, \varphi]$ such that

$$\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi] = s\mathcal{G}[z, w, \varphi]\,. \tag{3}$$

From the point of view of Renormalization theory, eq.(3) implies that the anomalies of a model, to be identified with the non-trivial cohomology classes of the classical linearized ST operator $\mathcal{S}_0$ in the sector with Faddeev-Popov (FP) charge $+1$, are not affected by the introduction of a set of decoupled doublets.

In Algebraic Renormalization decoupled doublets play for instance an essential rôle in the off-shell formulation of the Equivalence Theorem [4].

We wish to point out that eq.(3) holds regardless the FP charge of $\mathcal{I}$: it may also happen that $\mathcal{I}$ is a sum of terms with different FP charges, although this does not happen in Renormalization theory.

We then relax the assumption that s and $\mathcal{N}$ commute and discuss the more general case of coupled doublets, for which

$$[s, \mathcal{N}] \neq 0\,. \tag{4}$$

Under suitable assumptions, in some cases [5] it is known that one can reduce the problem of handling coupled doublets to the decoupled case by means of a properly chosen change of coordinates.

It is the purpose of this letter to give a full characterization of the dependence of the cohomology of s on coupled doublets.

In order to discuss the problem on general grounds it is convenient to decompose the nilpotent differential s according to the degree induced by $\mathcal{N}$:

$$
s = \sum_{j=0}^{\infty} s^{(j)} \tag{5}
$$

where $s^{(j)}$ is the component of s of degree j. Moreover we split the zero-th order operator $s^{(0)}$ as

$$
s^{(0)} = \bar{s}^{(0)} + \int d^4x \, w_k \frac{\delta}{\delta z_k} \tag{6}
$$

where $\bar{s}^{(0)}$ is (z, w)-independent and only acts on the variables φ. From the nilpotency of s we get that $s^{(0)}$ is nilpotent. This implies together with eq.(6) that $\bar{s}^{(0)}$ is also nilpotent.

We will prove along the lines of homological perturbation theory [6, 7, 8] that the cohomology of the nilpotent differential s in the space of local integrated formal power series in $\{\varphi, z, w\}$ and their derivatives is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space of local integrated formal power series which depend only on φ and their derivatives.

To this extent this result establishes the independence of the cohomology of s on (generally coupled) doublets.

2 Decoupled doublets

In this section we review the standard result [2] showing that the contribution of doublets to the cohomology of the differential s is trivial if

$$
[s, \mathcal{N}] = 0 \,. \tag{7}
$$

The counting operator $\mathcal{N}$ has been defined in eq.(2).

If eq.(7) is verified, we say that we are dealing with "decoupled doublets". This definition is motivated by the observation that if eq.(7) holds true, then no doublet (z_k, w_k) can appear in the s transformation of any other field.

We follow [9] and introduce the operator

$$
\mathcal{K} = \int_0^1 dt \, \sum_i z_i \lambda_t \frac{\delta}{\delta w_i} \,, \tag{8}
$$

where the operator λ_t acts as follows

$$
\lambda_t X[z, w, \varphi] = X[tz, tw, \varphi] \,. \tag{9}
$$

In the previous equation φ denotes any set of fields and external sources other than (z,w) on which the local integrated formal power series X might depend. By explicit computation it can be verified that $\mathcal{K}$ is a contracting homotopy for s, since it fulfills the following equation

$$\{s,\mathcal{K}\}X = \iota X \,. \tag{10}$$

ι is the projector on the orthogonal complement to the kernel of $\mathcal{N}$:

$$\iota = 1|_{C_0^\perp} \oplus 0|_{C_0} \,, \tag{11}$$

where C_0 denotes the kernel of the counting operator $\mathcal{N}$ in eq.(2) and $C_0^\perp$ its orthogonal complement.

Assume now that $s\mathcal{I} = 0$. $\mathcal{I}$ can depend on (z,w) and φ. We apply eq.(10) to $\mathcal{I}$ and obtain

$$\{s,\mathcal{K}\}\mathcal{I}[z,w,\varphi] = s(\mathcal{K}\mathcal{I}) = \iota\mathcal{I}[z,w,\varphi] = \mathcal{I}[z,w,\varphi] - \mathcal{I}[0,0,\varphi] \,. \tag{12}$$

In the previous equation we have used the fact that $\mathcal{I}[z,w,\varphi]$ is s-invariant. From eq.(12) we conclude that the (z,w)-dependent part $\mathcal{I}[z,w,\varphi]-\mathcal{I}[0,0,\varphi]$ of $\mathcal{I}$ is a s-exact term.

3 Coupled doublets

Let us now move to the "coupled" case where

$$[s,\mathcal{N}] \neq 0 \,. \tag{13}$$

In this case the operator $\mathcal{K}$ defined in eq.(8) is no more a contracting homotopy for s.

We denote by $\varphi = \{\varphi_i\}$ all the variables different from (z,w) on which s may act. We also adopt a compact notation and assume that the integration over d^4x is implicit in the sum over repeated indices, i.e. we write

$$\varphi_i \frac{\delta X}{\delta \varphi_i} \equiv \int d^4x \sum_i \varphi_i(x) \frac{\delta X}{\delta \varphi_i(x)} \,. \tag{14}$$

Then we get the following decomposition for s:

$$s = g_i \frac{\delta}{\delta \varphi_i} + w_i \frac{\delta}{\delta z_i} \,. \tag{15}$$

$g_i[\varphi,z,w]$ is the s-variation of φ_i.

We can decompose s according to the degree induced by $\mathcal{N}$:

$$s = \sum_{j=0}^{\infty} s^{(j)} \tag{16}$$

where $s^{(j)}$ is the component of s of degree j. By comparison with eq.(15) we have explicitly

$$
\begin{aligned}
s^{(0)} &= g_i^{(0)} \frac{\delta}{\delta \varphi_i} + w_i \frac{\delta}{\delta z_i} \\
s^{(j)} &= g_i^{(j)} \frac{\delta}{\delta \varphi_i} \,, \qquad\qquad j = 1, 2, \dots
\end{aligned}
\tag{17}
$$

In the above equation $g_i^{(j)}$ denotes the component of g_i in the eigenspace of eigenvalue j for the counting operator $\mathcal{N}$:

$$
\mathcal{N} g_i^{(j)} = j g_i^{(j)} \,. \tag{18}
$$

$g_i^{(0)}$ is the (z, w)-independent component of g_i. By comparison with eq.(6) we see that

$$
\bar{s}^{(0)} = g_i^{(0)} \frac{\delta}{\delta \varphi_i} \,. \tag{19}
$$

In most cases s is to be identified with the classical linearized ST operator $\mathcal{S}_0$. We denote by $S[\phi_i, \phi_i^*, z, w]$ the classical action from which we define the symplectic gradient

$$
(S, \cdot) \equiv \frac{\delta S}{\delta \phi_i} \frac{\delta}{\delta \phi_i^*} + \frac{\delta S}{\delta \phi_i^*} \frac{\delta}{\delta \phi_i} \,. \tag{20}
$$

$S[\phi_i, \phi_i^*, z, w]$ depends on both the fields ϕ_i and the associated antifields ϕ_i^* as well as on the doublets (z_k, w_k). The full classical linearized ST operator is then

$$
\mathcal{S}_0 = (S, \cdot) + w_k \frac{\delta}{\delta z_k} = \frac{\delta S}{\delta \phi_i} \frac{\delta}{\delta \phi_i^*} + \frac{\delta S}{\delta \phi_i^*} \frac{\delta}{\delta \phi_i} + w_k \frac{\delta}{\delta z_k} \,. \tag{21}
$$

Nilpotency of $\mathcal{S}_0$ follows from the ST identity for S [2]:

$$
\frac{1}{2}(S, S) + w_k \frac{\delta S}{\delta z_k} = 0 \,. \tag{22}
$$

The symplectic structure in eq.(21) is not essential to prove the results of the present section. In particular, we will only rely on eq.(15), which is more general than eq.(21).

However, the results presented in this section can be derived in an effective geometrical way [10] if the nilpotent differential s is given by the classical linearized ST operator $\mathcal{S}_0$ in eq.(21).

We will prove that the cohomology of the nilpotent differential s in the space of local integrated formal power series in $\{\varphi, z, w\}$ and their derivatives

is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space of local integrated formal power series which depend only on φ and their derivatives.

Our first task is to show that if $\mathcal{I}[z,w,\varphi]$ is a s-closed local integrated formal power series such that $s\mathcal{I}[0,0,\varphi]=0$ then its (z,w)-dependent part $\mathcal{I}[z,w,\varphi]-\mathcal{I}[0,0,\varphi]$ is s-exact.

Lemma 3.1 *Let $\mathcal{I}[z,w,\varphi]$ be a local integrated formal power series closed under the nilpotent differential s, i.e. it fulfills the Wess-Zumino consistency condition*

$$s\mathcal{I}=0\,. \tag{23}$$

Moreover let us assume that $s\mathcal{I}[0,0,\varphi]=0$. Then we have

$$\mathcal{I}[z,w,\varphi]-\mathcal{I}[0,0,\varphi]=s\mathcal{G}[z,w,\varphi] \tag{24}$$

for some local integrated formal power series $\mathcal{G}(z,w,\varphi)$.

Proof. The condition in eq.(23) implies

$$w_i\frac{\delta\mathcal{I}}{\delta z_i}=-g_i\frac{\delta\mathcal{I}}{\delta\varphi_i}\,. \tag{25}$$

Differentiating eq.(25) with respect to w_k we get

$$\frac{\delta\mathcal{I}}{\delta z_k}=(-1)^{\epsilon(w_k)+1}s\left(\frac{\delta\mathcal{I}}{\delta w_k}\right)-\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\,, \tag{26}$$

so that eq.(26) becomes (z_k and w_k have opposite statistics)

$$\frac{\delta\mathcal{I}}{\delta z_k}=(-1)^{\epsilon(z_k)}s\left(\frac{\delta\mathcal{I}}{\delta w_k}\right)-\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\,. \tag{27}$$

In the previous equations we have denoted by $\epsilon(X)$ the Grassmann parity of X ($\epsilon(X)=0$ if X is bosonic, $\epsilon(X)=1$ if X is fermionic).

We apply to both sides of eq.(27) the operator $\int_0^1 dt\, z_k\lambda_t$, where the action of λ_t on the local integrated formal power series X is defined by eq.(9).

We get:

$$\int_0^1 dt\, z_k\lambda_t\frac{\delta\mathcal{I}}{\delta z_k}=\int_0^1 dt\,\left((-1)^{\epsilon(z_k)}z_k\lambda_t s\left(\frac{\delta\mathcal{I}}{\delta w_k}\right)-z_k\lambda_t\frac{\delta g_i}{\delta w_k}\frac{\delta\mathcal{I}}{\delta\varphi_i}\right)\,. \tag{28}$$

On the other hand

$$\int_0^1 dt\,(-1)^{\epsilon(z_k)}z_k\lambda_t s\left(\frac{\delta\mathcal{I}}{\delta w_k}\right) \quad = \quad \int_0^1 dt\,\left((-1)^{\epsilon(z_k)}z_k s\left(\lambda_t\frac{\delta\mathcal{I}}{\delta w_k}\right)\right.$$

$$-(-1)^{\epsilon(z_k)} z_k [s, \lambda_t] \frac{\delta \mathcal{I}}{\delta w_k} \Big)$$

$$= \int_0^1 dt \left(s \left(z_k \lambda_t \frac{\delta \mathcal{I}}{\delta w_k} \right) - w_k \lambda_t \frac{\delta \mathcal{I}}{\delta w_k} - (-1)^{\epsilon(z_k)} z_k [s, \lambda_t] \frac{\delta \mathcal{I}}{\delta w_k} \right) . \quad (29)$$

Substituting in eq.(28) yields

$$\int_0^1 dt \left(z_k \lambda_t \frac{\delta \mathcal{I}}{\delta z_k} + w_k \lambda_t \frac{\delta \mathcal{I}}{\delta w_k} \right) = \int_0^1 dt \left(s \left(z_k \lambda_t \frac{\delta \mathcal{I}}{\delta w_k} \right) - z_k \left((-1)^{\epsilon(z_k)} [s, \lambda_t] \frac{\delta \mathcal{I}}{\delta w_k} + \lambda_t \frac{\delta g_i}{\delta w_k} \frac{\delta \mathcal{I}}{\delta \varphi_i} \right) \right) . \quad (30)$$

The L.H.S. in eq.(30) gives $\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi]$, so that

$$\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi] = \int_0^1 dt \left[s \left(z_k \lambda_t \frac{\delta \mathcal{I}}{\delta w_k} \right) - z_k \left((-1)^{\epsilon(z_k)} [s, \lambda_t] \frac{\delta \mathcal{I}}{\delta w_k} + \lambda_t \frac{\delta g_i}{\delta w_k} \frac{\delta \mathcal{I}}{\delta \varphi_i} \right) \right] . \quad (31)$$

Notice that

$$[s, \lambda_t] = [g_i \frac{\delta (\cdot)}{\delta \varphi_i}, \lambda_t] . \quad (32)$$

Let us define now

$$\mathcal{I}_1 = - \int_0^1 dt \, z_k \left((-1)^{\epsilon(z_k)} [s, \lambda_t] \frac{\delta \mathcal{I}}{\delta w_k} + \lambda_t \frac{\delta g_i}{\delta w_k} \frac{\delta \mathcal{I}}{\delta \varphi_i} \right) . \quad (33)$$

By using $s\mathcal{I} = 0$, $s\mathcal{I}[0, 0, \varphi] = 0$ and the fact that $s^2 = 0$ we get from eq.(31)

$$s\mathcal{I}_1 = 0 . \quad (34)$$

Moreover we see from eq.(33) that $\mathcal{I}_1$ satisfies the condition

$$\mathcal{I}_1[z, w, \varphi] \big|_{z=w=0} = 0.$$

Thus in particular

$$s\mathcal{I}_1[0, 0, \varphi] = 0 . \quad (35)$$

Hence we can repeat the argument used for $\mathcal{I}$ and write $\mathcal{I}_1$ in the form

$$\mathcal{I}_1[z, w, \varphi] = \mathcal{I}_1[z, w, \varphi] - \mathcal{I}_1[0, 0, \varphi] = \int_0^1 dt \left[s \left(z_k \lambda_t \frac{\delta \mathcal{I}_1}{\delta w_k} \right) - z_k \left((-1)^{\epsilon(z_k)} [s, \lambda_t] \frac{\delta \mathcal{I}_1}{\delta w_k} + \lambda_t \frac{\delta g_i}{\delta w_k} \frac{\delta \mathcal{I}_1}{\delta \varphi_i} \right) \right] . \quad (36)$$

By taking into account eq.(32) we see that the second term in the R.H.S. of eq.(36) contains at least two z's.

The construction can be iterated. Assume that $\mathcal{I}_n$, $n \geq 1$ contains the product of at least n z's and assume that $s\mathcal{I}_n = 0$. Moreover, we assume that $\mathcal{I}_n[z, w, \varphi]|_{z=w=0} = 0$, which implies in particular $s\mathcal{I}_n[0, 0, \varphi] = 0$. Then we can write

$$\begin{aligned} \mathcal{I}_n[z, w, \varphi] &= \mathcal{I}_n[z, w, \varphi] - \mathcal{I}_n[0, 0, \varphi] \\ &= s\left[\int_0^1 dt \left(z_k \lambda_t \frac{\delta \mathcal{I}_n}{\delta w_k}\right)\right] + \mathcal{I}_{n+1}\,. \end{aligned} \tag{37}$$

In the previous equation we have defined

$$\mathcal{I}_{n+1} = -\int_0^1 dt\, z_k \left((-1)^{\epsilon(z_k)}[s, \lambda_t]\frac{\delta \mathcal{I}_n}{\delta w_k} + \lambda_t \frac{\delta g_i}{\delta w_k}\frac{\delta \mathcal{I}_n}{\delta \varphi_i}\right)\,. \tag{38}$$

We now notice that $\mathcal{I}_{n+1}$ contains the product of at least $(n+1)$ z's. Moreover $s\mathcal{I}_{n+1} = 0$ and $\mathcal{I}_{n+1}(z, w, \varphi)|_{z=w=0} = 0$.

Thus we obtain that $\mathcal{I}$ is s-exact up to a term containing $(n+1)$ z's.

By the previous arguments we can therefore conclude that $\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi]$ is s-exact as a local integrated formal power series in z. This ends the proof.

In particular, if $\mathcal{I}[0, 0, \varphi] = 0$ the previous Lemma implies that the whole s-invariant $\mathcal{I}[z, w, \varphi]$ is s-exact: any s-closed local formal power series vanishing at $z = w = 0$ is s-exact.

We remark that the condition

$$s\mathcal{I}[0, 0, \varphi] = 0 \tag{39}$$

is also necessary if the (z, w)-dependence of the s-invariant $\mathcal{I}[z, w, \varphi]$ has to be cohomologically trivial. This is true independently of the coupled or decoupled nature of the doublets under investigation. Indeed assume that there exists a local formal power series $\mathcal{G}[z, w, \varphi]$ such that

$$\mathcal{I}[z, w, \varphi] - \mathcal{I}[0, 0, \varphi] = s\mathcal{G}[z, w, \varphi]\,. \tag{40}$$

Then by the nilpotency of s and the s-invariance of $\mathcal{I}[z, w, \varphi]$ we conclude that

$$s\mathcal{I}[0, 0, \varphi] = 0\,. \tag{41}$$

Therefore eq.(41) is both a necessary and sufficient (by virtue of Lemma **3.1**) condition in order that the dependence of the s-invariant $\mathcal{I}[z, w, \varphi]$ on the doublets (z, w) is cohomologically trivial in the sense of eq.(3).

Notice that in the case of decoupled doublets eq.(41) is actually a consequence of the fact that $\mathcal{I}[z,w,\varphi]$ is s-invariant, i.e. of the equation

$$s\mathcal{I}[z,w,\varphi] = 0 \tag{42}$$

once one takes the zero-th order (with respect to the grading induced by $\mathcal{N}$) component of eq.(42).

Lemma **3.1** turns out to be a useful tool in many practical applications, like for instance in the discussion of the on-shell case of the Equivalence Theorem [4]. We now show by using Lemma **3.1** that the following Theorem, characterizing the full dependence of the cohomology of s on (generally coupled) doublets, holds true.

Theorem 3.1 *The cohomology of the nilpotent differential s in the space Σ of local integrated formal power series in $\{\varphi, z, w\}$ and their derivatives is isomorphic to the cohomology of $\bar{s}^{(0)}$ in the space Σ_0 of local integrated formal power series which only depend on φ and their derivatives:*

$$H(s,\Sigma) \approx H(\bar{s}^{(0)},\Sigma_0)\,. \tag{43}$$

Proof. We will explicitly construct an isomorphism Φ between $H(s,\Sigma)$ and $H(\bar{s}^{(0)},\Sigma_0)$. Let $\mathcal{I} \in \Sigma$ be such that

$$s\mathcal{I} = 0\,. \tag{44}$$

We decompose $\mathcal{I}$ according to the degree induced by $\mathcal{N}$, i.e.

$$\mathcal{I} = \sum_{k=0}^{\infty} \mathcal{I}^{(k)} \tag{45}$$

where $\mathcal{I}^{(k)}$ belongs to the eigenspace of eigenvalue k of the counting operator $\mathcal{N}$:

$$\mathcal{N}\mathcal{I}^{(k)} = k\mathcal{I}^{(k)}\,. \tag{46}$$

We notice that

$$\bar{s}^{(0)}\mathcal{I}^{(0)} = 0 \tag{47}$$

i.e. $\mathcal{I}^{(0)}$ belongs to the cohomology of $\bar{s}^{(0)}$ in Σ_0. This follows from eq.(44) once we consider its zero-th order in the expansion based on the grading induced by $\mathcal{N}$:

$$(s^{(0)} + s^{(1)} + \ldots)(\mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \ldots) = 0 \Rightarrow s^{(0)}\mathcal{I}^{(0)} = 0\,. \tag{48}$$

The dots stand for term of higher order with respect to the grading induced by $\mathcal{N}$. Since $\mathcal{I}^{(0)}$ is independent of (z,w) the R.H.S. of eq.(48) entails eq.(47).

Let us now set

$$\Phi[\mathcal{I}] = [\mathcal{I}^{(0)}] \tag{49}$$

where $[\mathcal{I}]$ stands for the cohomology class of $\mathcal{I}$ in $H(s, \Sigma)$ and $[\mathcal{I}^{(0)}]$ stands for the cohomology class of $\mathcal{I}^{(0)}$ in $H(\bar{s}^{(0)}, \Sigma_0)$.

The map in eq.(49) is well-defined in cohomology since if

$$\mathcal{I} = s\mathcal{G} \tag{50}$$

then also

$$\mathcal{I}^{(0)} = \bar{s}^{(0)}\mathcal{G}^{(0)} \,. \tag{51}$$

This follows by expanding eq.(50) according to the grading induced by $\mathcal{N}$ once we look at the zero-th order terms:

$$\begin{aligned} \mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \ldots = (s^{(0)} + s^{(1)} + \ldots)(\mathcal{G}^{(0)} + \mathcal{G}^{(1)} + \ldots) \\ \Rightarrow \mathcal{I}^{(0)} = s^{(0)}\mathcal{G}^{(0)} \,. \end{aligned} \tag{52}$$

We recover eq.(51) from the above equation since $\mathcal{G}^{(0)}$ is independent of (z, w).

We now show that Φ is an isomorphism.

Let us first prove that Φ is surjective. Given any cocycle $\mathcal{I}^{(0)}$ of $\bar{s}^{(0)}$ we show that it can be completed to a cocycle of s. So we look for the coefficients $\mathcal{I}^{(1)}, \mathcal{I}^{(2)}, \ldots$ in such a way that

$$\mathcal{I} \equiv \mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \ldots \tag{53}$$

fulfills

$$s\mathcal{I} = (s^{(0)} + s^{(1)} + s^{(2)} \ldots)(\mathcal{I}^{(0)} + \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \ldots) = 0 \,. \tag{54}$$

The proof is a recursive one. At order zero eq.(54) is true since $\mathcal{I}^{(0)}$ is a $\bar{s}^{(0)}$-cocycle and $\mathcal{I}^{(0)}$ is (z, w)-independent, so that

$$0 = \bar{s}^{(0)}\mathcal{I}^{(0)} = s^{(0)}\mathcal{I}^{(0)} \,. \tag{55}$$

We next move to the first order. From the nilpotency of s we get

$$s^{(1)}s^{(0)} + s^{(0)}s^{(1)} = 0 \,. \tag{56}$$

Since $s^{(0)}\mathcal{I}^{(0)} = 0$ we get from the above equation

$$s^{(0)}s^{(1)}\mathcal{I}^{(0)} = 0 \,. \tag{57}$$

$s^{(0)}$ is a nilpotent differential with respect to which (z, w) form a set of decoupled doublets. By using the standard results on the independence of

the cohomology of decoupled doublets reported in Sect. 2 we conclude from eq.(57) that there must exist a local integrated formal power series $\mathcal{I}^{(1)}$ such that

$$s^{(1)}\mathcal{I}^{(0)} = s^{(0)}(-\mathcal{I}^{(1)})\,. \tag{58}$$

This in turn implies that eq.(54) is fulfilled at order one since from eq.(58) we have

$$s^{(1)}\mathcal{I}^{(0)} + s^{(0)}\mathcal{I}^{(1)} = 0\,. \tag{59}$$

The construction can be iterated to all orders. Assume that $\mathcal{I}$ has been constructed up to order $n-1$ by assigning the coefficients $\mathcal{I}^{(0)}, \mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(n-1)}$, fulfilling

$$\sum_{j=0}^{m} s^{(j)}\mathcal{I}^{(m-j)} = 0\,, \qquad m = 0, 1, \ldots, n-1\,. \tag{60}$$

Then by eq.(60) we see that

$$s\sum_{k=0}^{n-1}\mathcal{I}^{(k)}$$

starts with a coefficient of order n, let us call it $\Delta^{(n)}$:

$$s\sum_{k=0}^{n-1}\mathcal{I}^{(k)} = \Delta^{(n)} + \ldots \tag{61}$$

One has explicitly

$$\Delta^{(n)} = \sum_{j=1}^{n} s^{(j)}\mathcal{I}^{(n-j)}\,. \tag{62}$$

Since s is nilpotent we get from eq.(61)

$$s^2\sum_{k=0}^{n-1}\mathcal{I}^{(k)} = s(\Delta^{(n)} + \ldots) = 0\,, \tag{63}$$

so that

$$(s^{(0)} + s^{(1)} + \ldots)(\Delta^{(n)} + \ldots) = 0\,. \tag{64}$$

We look at the lowest order contribution to the above equation, i.e. at order n, and get

$$s^{(0)}\Delta^{(n)} = 0\,. \tag{65}$$

By using again the results on decoupled doublets of Sect. 2 we can show that there exists a local integrated formal power series $\mathcal{I}^{(n)}$ such that

$$\Delta^{(n)} = -s^{(0)}\mathcal{I}^{(n)} . \tag{66}$$

By taking into account eq.(62) we get from eq.(66)

$$s^{(0)}\mathcal{I}^{(n)} + \sum_{j=1}^{n} s^{(j)}\mathcal{I}^{(n-j)} = \sum_{j=0}^{n} s^{(j)}\mathcal{I}^{(n-j)} = 0 , \tag{67}$$

so that eq.(54) is fulfilled at order n.

We now show that Φ is also one-to-one by proving that

$$\ker \Phi = \{[0]\} . \tag{68}$$

For that purpose let us take a s-invariant $\mathcal{I}$ such that

$$\Phi([\mathcal{I}]) = [0] . \tag{69}$$

This means that the zero-th order component $\mathcal{I}^{(0)}$ of $\mathcal{I}$ fulfills

$$\mathcal{I}^{(0)} = \bar{s}^{(0)}\mathcal{G}^{(0)} \tag{70}$$

for some local formal power series $\mathcal{G}^{(0)}$ independent of (z, w). Then

$$\begin{aligned} \mathcal{I} - s\mathcal{G}^{(0)} &= \left(\mathcal{I}^{(0)} - \bar{s}^{(0)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(1)} - s^{(1)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(2)} - s^{(2)}\mathcal{G}^{(0)}\right) + \dots \\ &= \left(\mathcal{I}^{(1)} - s^{(1)}\mathcal{G}^{(0)}\right) + \left(\mathcal{I}^{(2)} - s^{(2)}\mathcal{G}^{(0)}\right) + \dots \end{aligned} \tag{71}$$

In the second line of the above equation we have used eq.(70). Therefore $\left(\mathcal{I} - s\mathcal{G}^{(0)}\right)\Big|_{z=w=0} = 0$. Moreover

$$\mathcal{I} - s\mathcal{G}^{(0)}$$

is s-invariant since $s\mathcal{I} = 0$ and $s^2 = 0$. Thus $\mathcal{I} - s\mathcal{G}^{(0)}$ fulfills the assumptions of Lemma **3.1** and we conclude that

$$\mathcal{I} - s\mathcal{G}^{(0)} = s\mathcal{H} \tag{72}$$

for some local formal power series $\mathcal{H}$. Hence we see that

$$\mathcal{I} = s(\mathcal{G}^{(0)} + \mathcal{H}) . \tag{73}$$

From eq.(73) we conclude that $\mathcal{I}$ is a s-coboundary. Therefore eq.(68) is verified. This ends the proof of the Theorem.

4 Conclusions

In the present paper we have discussed on general grounds the dependence of nilpotent differentials on doublets, both in the decoupled and the coupled case.

We have explicitly constructed an isomorphism between the cohomology of s in the space of local integrated formal power series depending on the set of doublets (z, w) and on φ and their derivatives and the cohomology of $\bar{s}^{(0)}$ in the space of local integrated formal power series only depending on φ and their derivatives.

To this extent the cohomology of any nilpotent differential s in the space of local integrated formal power series is independent of doublets both in the decoupled and the coupled case.

As a final point we remark that the whole analysis was purely algebraic. No use was made of power-counting arguments.

Acknowledgements

Useful discussions with Prof. Ruggero Ferrari and Prof. Raymond Stora are gratefully acknowledged. The author wishes to thank Prof. Marc Henneaux for many suggestions and his valuable help in the proof of Theorem **3.1**.

References

[1] C. Becchi, A. Rouet and R. Stora, *Phys. Lett.* **B32** (1974) 344; L.V. Tyutin, Lebedev preprint FIAN n.39 (1975).

[2] P. Piguet and S.P. Sorella, *Algebraic Renormalization*, Lecture Notes in Physics m28, Springer-Verlag (1995).

[3] G. Barnich, F. Brandt and M. Henneaux, *Phys. Rept.* **338** (2000) 439.

[4] R. Ferrari, M. Picariello, A .Quadri, *"An Approach to the Equivalence Theorem by the Slavnov-Taylor Identities"*, in preparation.

[5] F. Brandt, *Commun. Math. Phys.* **190** (1997) 459.

[6] M. Henneaux and C. Teitelboim, *"Quantization Of Gauge Systems,"* Princeton, USA: Univ. Pr. (1992) 520 p.

[7] J. M. Fisch and M. Henneaux, *Commun. Math. Phys.* **128** (1990) 627.

[8] J. M. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, *Commun. Math. Phys.* **120** (1989) 379.

[9] B. Zumino, UCB-PTH-83/16 *Lectures given at Les Houches Summer School on Theoretical Physics, Les Houches, France, Aug 8 - Sep 2, 1983*.

[10] R. Stora, private communication.

[11] C. Becchi,A. Rouet and R. Stora, *Comm.Math.Phys.* **42** (1975) 127-162.

[12] R. Ferrari and P.A. Grassi, *Phys.Rev.* **D60**:065010 (1999).

[13] R. Ferrari, P.A. Grassi and A. Quadri, *Phys. Lett.* **B472** (2000) 346-356.

[14] I.V. Tyutin, *"Once again on the Equivalence Theorem"*, hep-th/0001050.