An analysis of CPN models is given in terms of general coordinates or arbitrary interpolating fields. Only closed expressions made from simple functions are involved. Special attention is given to CP2 and CP4. In the first of these the retrieval of stereographic coordinates reveals the hermitian form of the metric. A similar analysis for the latter case allows comparison with the Fubini-Study metric.

**Key words:** Kahler, stereographic, complex projective

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**Introduction**

Despite the central importance of CPN models [1], and their recent revival as arising in supersymmetric from from minimized linear models [2] following the revision of the underlying supersymmetry algebra of densities to include central terms, there appears to be no treatment of them in general coordinates which would allow arbitrary field redefinitions for the interpolating Goldstone Bosons. In this article just such an analysis is presented. The next section explains how this is achieved by embedding the necessary structure into a more complicated one. Strangely, perhaps, nothing is needed but simple functions, and a completely general solution is found in closed form. In the following section the special case of CP2 and stereographic coordinates is presented. Then the corresponding step is made for CP4 allowing the connection to the Fubini-Study metric. Finally, there are brief conclusions and suggestions are made for future work.

**General Framework**

Curiously this section begins by consideration of the embedding of the structure needed for the current problem into that of a larger system which has previously been solved in general coordinates leading to a closed form involving only simple functions [3]. The embedding is unique. Thus the starting
point is a review of this established larger system and its solution, in which
the liberty of changing notation (slightly) for convenience has been taken.

Consider then the Lie algebra of $SU_n$ specified by taking as a basis the set of
$(n^2 - 1)$ traceless hermitian $n \times n$ matrices $\lambda_i$ with the product law

$$\lambda_i \lambda_j = \frac{2}{n} \delta_{ij} + (d_{ijk} + i f_{ijk}) \lambda_k$$

as specified by Gell Mann [4]. When specific values of the structure constants
$f_{ijk}$, or the symmetric $d_{ijk}$ tensors or the $\lambda_i$ matrices themselves are needed
then the notation of reference [4] will be assumed. An element of the group
$SU_n$, $g(\theta)$ is specified in exponential form by a set of $(n^2 - 1)$ real parameters
$\theta_i$, so that in infinitesimal form the transformations

$$q_A \rightarrow q_A - \frac{i}{2} \theta^i (\lambda_i)_{AB} q_B$$

and

$$M_i \rightarrow M_i + \theta_k f_{ikj} M_k$$

specify the behaviour of the basic spinor $q_A$ (quark) and adjoint vector $M_i$
fields. Now define a traceless matrix $M$ by

$$M_{KL} = M_i (\lambda_i)_{KL}$$

so that

$$M_i = \frac{1}{2} Tr.(M \lambda_i)$$

then a group element $g(\theta)$ which induces a unitary transformation

$$q_A \rightarrow U(\theta)_{AB} q_B$$

on the basic spinors clearly induces an orthogonal transformation

$$M_i \rightarrow R_{ij} M_j = \frac{1}{2} Tr.(U^{-1} \lambda_i U \lambda_j) M_j$$

on the adjoint representation.
The algebra of $SU_n \times SU_n$ is spanned by two sets of $(n^2 - 1)$ orthogonal elements $L_i$ and $R_i$ satisfying the commutation relations

\begin{align}
[L_i, L_j] &= i f_{ijk} L_k \\
[R_i, R_j] &= i f_{ijk} L_k \\
[L_i, R_j] &= 0
\end{align}

and the linear combinations

\begin{align}
V_i &= L_i + R_i \\
A_i &= L_i - R_i
\end{align}

are frequently used. Obviously the $V_i$ generate a $SU_n$ subgroup which is parity conserving. An element of the $SU_n \times SU_n$ group may be specified by two sets of $(n^2 - 1)$ real parameters, and the alternative expressions

\begin{align}
g &= \exp(-i [\theta_i^V V_i + \theta_i^A A_i]) \\
\text{and} \\
g &= \exp(-i \theta_i^L L_i) \exp(-i \theta_i^R R_i)
\end{align}

will prove useful with

\begin{align}
\theta_i^L &= \theta_i^V + \theta_i^A \\
\theta_i^R &= \theta_i^V - \theta_i^A
\end{align}

specifying the correspondence. Every element of the group can also be decomposed into a product of the form

\begin{align}
g &= \exp(-i \phi_i A_i) \exp(-i \theta_i V_i)
\end{align}

which is unique in a neighbourhood of the identity element and this will play a crucial role in the general nonlinear realization scheme. The linear transformation laws are best specified by giving the quarks a Dirac spinor index in the usual manner and taking

\begin{align}
q \rightarrow q - \frac{i}{2} \theta_i^L \lambda_i \frac{(1 + \gamma_5)}{2} q - \frac{i}{2} \theta_i^R \lambda_i \frac{(1 - \gamma_5)}{2} q
\end{align}
as the concrete infinitesimal form.

Since the matrices

\[ P_L = \frac{(1 + \gamma_5)}{2} \]  

(19)

and

\[ P_R = \frac{(1 - \gamma_5)}{2} \]  

(20)

act as a standard set of projection operators, the treatment of linearly transforming multiplets of \( SU_n \times SU_n \) now follows trivially.

To treat the nonlinear realizations of \( SU_n \times SU_n \) in full generality the \((n^2 - 1)\) hermitian components \( M_i \) of the adjoint vector of \( SU_n \) must be considered in more detail. In the terminology of Michel and Radicati [5], the vector is said to be generic (or to belong to the generic stratum) if all eigenvalues of \( M \) are distinct. For the generic case the minimal polynomial for the matrix is the characteristic polynomial satisfying the equation

\[ \prod_{A=1}^{n} (M - m_A) = 0 \]  

(21)

where the \( m_A \) are the eigenvalues which satisfy

\[ \sum_{m=1}^{n} m_A = 0 \]  

(22)

if the matrix is traceless. Thus the \((n - 1)\) vectors with components given by powers of the matrix in the form

\[ M^{\alpha_i} = \frac{1}{2} Tr.([M]^{\alpha} \lambda_i) \quad [\alpha = 1, 2, \ldots, (n - 1)] \]  

(23)

are a linearly independent set, and the quantities

\[ T_A = Tr.([M]^A) = \sum_{B=1}^{n} [m_B]^A \equiv \sum_{B=1}^{n} m_{AB} \]  

(24)

are \((n - 1)\) independent \( SU_n \) invariants. \((S_1 \) is identically zero\.) At once it is clear that the general vector which can be constructed from the \( M_i \) has the
where the $F_\alpha$ are functions of the $(n-1)$ independent $SU_n$ invariants. This freedom has been discussed at length by Gasiorowicz and Geffen [6]. From the point of view of field theory it corresponds to freedom of choice of interpolating fields. Provided that $F_1(0)$ is taken to be unity, and parity is respected, then all $\xi_i$ so defined are equally good interpolating fields. From a geometrical viewpoint the $\xi_i$ may be regarded as coordinates of points of the $(n^2-1)$ dimensional coset space manifold formed by the quotient of $SU_n \times SU_n$ by the vector $SU_n$ subgroup. The freedom is then viewed as the ability to change coordinates within a local patch near the origin.

An arbitrary point on the manifold is parameterized by

$$
\exp(-i\xi_i A_i) \equiv L(M) \equiv \exp \left( -i\phi M_i \lambda_i [P_L - P_R] \right) \quad (26)
$$

where the first form corresponds with equation (17) and the second form represents the appropriate expression when equation (25) has been used so that the $M_i$ are regarded as the coordinates, and $\phi^2 = M_i M_i$.

The general theory is well described by Coleman, Wess and Zumino [7] and Callan, Coleman, Wess and Zumino [8], and the geometrical approach by Isham [9]. With the decomposition given in equation (17) the action of a general element $g$ of the full group may be written as

$$
g \exp(-i\xi_i A_i) = \exp(-i\xi'_i A_i) \exp(-i\eta V_i) \equiv L(M') \exp(-i\eta V_i) \quad (27)
$$

where $M'_i$ and $\eta$ both depend on $M_i$ and $g$. Then the primary result of the general theory is that

$$
g : M_i \longrightarrow M'_i \quad (29)
$$

gives a nonlinear realization of the algebra which is linear on the $SU_n$ vector subgroup. Moreover if $h$ is an element of the vector subgroup and

$$
h : \Psi_\Omega \longrightarrow D(h)_{\Omega \Gamma} \Psi_\Gamma \quad (30)
$$
is a linear (unitary) representation of that subgroup, then
\[
g : \Psi_\Omega \longrightarrow D[\exp(-i\eta_i\Psi_i)]_{\Omega\Gamma}\Psi_\Gamma
\]  
(31)
gives a realization of the full group. Notice that this latter transformation is
linear in \(\Psi\) but nonlinear (through \(\eta\)) in the \(M_i\) when \(g\) is not in the vector
subgroup. Fields which transform according to equation (31) are called stan-
dard fields, and it is important to understand that by a suitable redefinition
of coordinates any nonlinear realization of \(SU_n \times SU_n\) which is linear on the
vector subgroup can be brought into this standard form. In practice the most
useful result is that, if one has a linear irreducible (unitary) representation of
\(SU_n \times SU_n\) such that
\[
g : N_\Omega \longrightarrow D[g]_{\Omega\Gamma}N_\Gamma
\]  
(32)
then
\[
\Psi_\Omega(M) = D[L^{-1}(M)]_{\Omega\Gamma}N_\Gamma
\]  
(33)
transform as the components of standard fields.

It is now clear that there are just three classes of fields to consider:

(i) Linear representations which may be built up in the usual way as mul-
tispinors with transformation laws defined by equation (18). These will
not be treated in more detail.

(ii) Vectors \(M_i\) transforming as the adjoint representation of \(SU_n\) with a
nonlinear transformation law under chiral action specified by equation
(27). These will allow a description of the massless Goldstone Bosons
(pions etc.) corresponding to the axial degrees of freedom spontaneously
violated. The specification of invariants constructed (nonlinearly) from
these is most important and will be exhibited later.

(iii) Standard fields which appear linearly in their transformation laws, but
with nonlinear functions of the \(M_i\) induced according to equations (31)
and (28). These are important in describing matter (eg nucleons) in-
teracting with the Goldstone Bosons as chiral matter. Once more, the
specification of the corresponding invariants is most important and will
be given later.

The technical problem of finding the invariants is solved in reference [3]. A

\[
[M]^A = [m_B]^A P_B \equiv m_{AB} P_B
\]  
(34)

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where the $P_B$ are $n$ hermitian matrices, each $n \times n$, with the properties

$$P_A P_B = \delta_{AB} P_B \quad \text{(no sum)}$$

$$\text{Tr}.(P_A) = 1$$

and

$$\sum_{A=1}^{n} P_A = 1$$

where this 1 is the unit $(n \times n)$ matrix. Although the $P_A$ are not in general diagonal, the above projection operator properties make calculations tractable. Now define

$$P_{Ai} = \frac{1}{2} \text{Tr}.(P_A \lambda_i)$$

and

$$(P_A)_{MN} = P_{Ai}(\lambda_i)_{MN} + \frac{1}{n} \delta_{MN}$$

where because the $P_A$ are complete it follows that

$$\sum_{A=1}^{n} P_{Ai} = 0$$

and, introducing

$$p_{Ai} = \sqrt{2}[P_{Ai} - (1 + \sqrt{n})^{-1}]P_{ni}$$

with

$$\sqrt{2}P_{Ai} = p_{Ai} + \frac{1}{\sqrt{n}} p_{ni}$$

establishes that $p_{\mu i}$ for $\mu = 1, 2, ..., (n - 1)$ are orthonormal.

The second-rank tensors defined by the $M_i$ are conveniently handled by an extension of these ideas, and fall into two classes. One such class is formed by the $n(n - 1)$ independent tensors defined by

$$(P_{AB})_{ij} \equiv P_{Ai Bj} \equiv \frac{1}{2} \text{Tr}.(P_A \lambda_i P_B \lambda_j) \quad (A \neq B)$$
and

\[ I_{ij} = \frac{1}{2} \text{Tr}(P_A \lambda_i P_A \lambda_j), \quad (44) \]

which have the properties

\[ II = I \quad (45) \]

\[ IP_{AB} = 0 = P_{AB}I \quad (46) \]

and

\[ P_{AB}P_{CD} = \delta_{AC}\delta_{BD}P_{AB} \quad \text{(no sum)} \quad (47) \]

in terms of the matrix notation of the last section. Moreover, these are all hermitian matrices and the trace of each \( P_{AB} \) is unity. Since it is easy to show also that

\[ \sum_{A \neq B} P_{AB} = 1 - I \quad (48) \]

where the sum is over all A and B but excluding terms with \( A = B \), this gives a projection operator resolution in one sector of the space of these second-rank tensors and so \( I \) (with trace \([n - 1]\)) will decompose further. The second class of tensors may be identified with the \((n - 1)^2\) independent matrices with components

\[ (p_{\alpha\beta})_{ij} \equiv p_{\alpha i}p_{\beta j} \quad (49) \]

which span the subspace of \((n^2 - 1) \times (n^2 - 1)\) matrices projected out on multiplication by \( I \) from both sides and which are therefore orthogonal to the subspace in which the \( P_{AB} \) lie. Since the \( p_{\alpha i} \) are orthonormal, the multiplication law for the \( p_{\alpha\beta} \) is

\[ p_{\alpha\beta}p_{\gamma\delta} = \delta_{\beta\gamma}p_{\alpha\delta} \quad (50) \]

It has been established by Barnes and Delbourgo [10] that all the independent second-rank tensors which can be constructed from the \( M_i \) are spanned by the \((n - 1)(2n - 1)\) independent \( p_{\alpha\beta} \) and \( P_{AB} \).
The most general unitary unimodular matrix $U$ constructed from the $M_i$ may be written in the form

$$U = U_A P_A = \exp \left[ -\frac{i}{2} \theta_A \right] P_A \quad \text{where} \quad \sum_{A=1}^{n} \theta_A = 0 \quad (51)$$

but the $\theta_A$ are otherwise completely arbitrary independent functions of the independent $SU_n$ invariants $S_A$ subject to the considerations of parity and weak field limits as mentioned before. These $(n-1)$ effective arbitrary functions of the $(n-1)$ invariants are characteristic of the general solution and will persist throughout this work.

It has been conventional to define

$$\sqrt{2} \phi_A = m_A - (1 + \sqrt{n})^{-1} m_n \quad (52)$$

with

$$m_A = \sqrt{2}(\phi_A + n^{-\frac{1}{2}} \phi_n) \quad (53)$$

so that, extending the notation used previously,

$$M_i = \phi_\alpha p_{\alpha i} \quad (54)$$

and

$$\phi_{\alpha,i} = p_{\alpha i} \quad (55)$$

follow immediately. Similarly, defining

$$\sqrt{2} \psi_A = \theta_A - (1 + n^{\frac{1}{2}})^{-1} \theta_n \quad (56)$$

with

$$\theta_A = \sqrt{2}(\psi_A + n^{-\frac{1}{2}} \psi_n) \quad (57)$$

the $\psi_\alpha$ may be treated as $(n-1)$ independent (arbitrary) functions of the $\phi_\alpha$ which then serve as the $(n-1)$ independent invariants.

The transformation laws for all realizations are now given in reference [3] in closed form and in terms of simple functions. Restricting attention to first-order derivatives of the fields with respect to space and time, and also restricting attention to a study of the Goldstone Boson fields $M_i$ and the standard
fields the results can be given in terms of the general analysis of references [7] and [8]. There are two important results. First, although $\partial_{\mu} M_i$ and $\partial_{\mu} \Psi_\Gamma$ do not transform as standard fields, the covariant derivatives

$$D_\mu M_i = A_{\mu i}$$

and

$$D_\mu \Psi_\Gamma = \partial_{\mu} \Psi_\Gamma - i\nu_{\mu i} (T_i)_{\Gamma\Omega} \Psi_\Omega$$

where under $SU_n$

$$\Psi_\Gamma \rightarrow \Psi_\Gamma - i\theta_i (T_i)_{\Gamma\Omega} \Psi_\Omega$$

and where

$$2iL^{-1}(M)\partial_\mu L(M) = \exp(-i\xi_iA_i)\partial_\mu \exp(-i\xi_jA_j) = \nu^i V_i + a^i A_i$$

have precisely this property. Secondly, they show that the most general Lagrangian of the type under consideration maybe written as a function of the standard fields $\Psi$, $D_\mu \Psi$ and $D_\mu M_i$ only; that is the $M_i$ will not appear explicitly, and the Goldstone Bosons will be massless. It then follows that the Lagrangian so formed will be invariant under $SU_n \times SU_n$ if and only if it is constructed to be invariant under the $SU_n$ vector subgroup. This latter requirement is, of course, achieved by index saturation once more.

The result given in reference [3] (now dropping the chiral projectors and normalizing for this problem) takes the concrete form

$$D_\mu M_i = \left\{ \frac{\partial\psi_\beta}{\partial\phi_\gamma} (P_{\gamma\beta})_{ik} + \sum_{A \neq B} \frac{\sqrt{2}}{(\phi_A - \phi_B)} \sin \left[ \frac{\psi_A - \psi_B}{\sqrt{2}} \right] (P_{AB} + P_{BA})_{ik} \right\} (\partial_\mu M_k)$$

and represent a complete specification of the required Lagrangian in simple closed form. Using the geometric formulation of Isham [9] gives the coset space metric in the form related to the covariant derivatives as

$$g_{ij}(\partial_\mu M_i)(\partial^\mu M_j) = (D_\mu M_i)(D^\mu M_j)$$

and we have normalized $g_{ij}$ to $\delta_{ij}$ in the limit of zero fields. In matrix notation this yields

$$g = \frac{1}{4} \left\{ p_{\beta \lambda} \frac{\partial\psi_\alpha}{\partial\phi_\beta} \frac{\partial\psi_\alpha}{\partial\phi_\lambda} + \sum_{A \neq B} \frac{2}{(\phi_A - \phi_B)^2} (P_{AB} + P_{BA}) \sin^2 \left[ \frac{\psi_A - \psi_B}{\sqrt{2}} \right] \right\}$$
immediately because of the orthonormality.

At last it is time to see how this structure is related to $CP_n$. Returning to the $SU_n \times SU_n$ action given in equations (27) and (28), consider the restriction of $\xi_i$ to the subset of dimension $2(n - 1)$ given by

$$A_{(n-1)^2}, A_{(n-1)^2+1}, \ldots, A_{n^2-2}$$

and similarly the restriction of $V_i$ to the subset of dimension $(n - 1)^2$ given by

$$V_1, V_2, \ldots, V_{(n-1)^2-1} = V_{n(n-2)} \quad \text{and} \quad V_{n^2-1} = V_{(n+1)(n+1-2)}$$

which restrictions are overall obviously unique. The remaining $V_i$ after the restriction clearly generate $SU_{n-1} \times U_1$, and the remaining $A_i$ combine with the $V_i$ to yield the whole $SU_n$ in which the former are uniquely embedded. This gives the manifold $(SU_n/(SU_{n-1} \times U_1)$ which is of dimension $2(n-1)$ and forms the basis for $CP_n$. All the previous results now apply to this embedded space simply by applying the same restrictions.

It is still necessary to interpret the information thus obtained in terms of the $CP_n$ structure. From this viewpoint the $V_1, V_2, \ldots, V_{n^2-1}$ and $V_{n(n+2)}$ generate an $SU_{n-1} \times U_1$ under which the $A_\mu$ transform linearly as a complex $2(n - 1)$ dimensional multiplet.

Recall that the $2(n-1)$ $\xi_i$ are the generalized coordinates or interpolating fields for the massless Goldstone Bosons. We can combine these into $(n - 1)$ complex coordinates by taking

$$z_0 = \xi_0 - i\xi_1, z_1 = \xi_2 - i\xi_3, \ldots, z_{n-2} = \xi_{2(n-1)} - i\xi_{2(n-1)+1}$$

by a judicious choice of labels.

We can see that with the new labels then

$$M = \sum_{\mu=0}^{n-2} \left[ z_\mu \frac{[\lambda_{2\mu+(n-1)^2} + i\lambda_{2\mu+(n-1)^2+1}]}{2} \right. $$

$$\left. + \bar{z}_\mu \frac{[\lambda_{2\mu+(n-1)^2} - i\lambda_{2\mu+(n-1)^2+1}]}{2} \right]$$

having only non-zero entries going from $z_0$ to $z_{n-2}$ down the final right hand column from the top, and going from $\bar{z}_0$ to $\bar{z}_{n-2}$ across the final row from the left. There are zeros in the top $(n - 3) \times (n - 3)$ left hand block, and a zero in the bottom right hand corner. Each of the complex $z'$s gives two vectors.
in the coset space. The corresponding lengths can be expressed in terms of the independent \((n - 2)\) invariants \(\phi_\alpha\) out of which the \((n - 2)\) independent functions \(\theta_\alpha\) (used in constructing the \(z's\)) are formed.

The Special Cases of CP2 and CP4

The CP2 case has previously been called the chiral 2-sphere by Barnes, Generowicz and Grimshare [11] when it has been described in some detail. In the present notation \(M\) takes the form

\[
M = \frac{1}{2}(z + \bar{z})\sigma_1 + \frac{1}{2}i(z - \bar{z})\sigma_2 = M_A\sigma_A
\]  

(67)

where \(z_0\) is written as \(z\) and where \(\phi^2 = z\bar{z}\), and writing \(M_A = \phi n_A\) gives

\[
(P_{12} + P_{21})_{AB} = \delta_{AB} - n_A n_B
\]

(68)

Thus putting

\[
\psi_1 - \psi_2 = \sqrt{2}\theta
\]

(69)

and

\[
\phi_1 - \phi_2 = \sqrt{2}\phi
\]

(70)

one finds immediately that

\[
g_{AB} = \frac{1}{4} \left[ \left( \frac{d\theta}{d\phi} \right)^2 n_A n_B + \frac{\sin^2 \theta}{\phi^2} (\delta_{AB} - n_A n_B) \right]
\]

(71)

Note that in this example where there is only a single arbitrary function \(\theta\) of a single invariant \(\phi\), the notation of the \(\delta U_2\) description does not need adapting for the \(SU_2/U_1\) coset space.

The condition to find hermitian form is obviously

\[
\left( \frac{d\theta}{d\phi} \right)^2 = \frac{\sin^2 \theta}{\phi^2}
\]

(72)
with the solution

\[ \phi = \cot \left( \frac{\theta}{2} \right) \]  

(73)

where \( c \) is a constant, being the one conventionally chosen. This is the coordinate system usually known as stereographic. Obviously equation (71) now yields

\[ g_{AB} = \frac{\delta_{AB}}{[1 + z\bar{z}]^2} \]  

(74)

where

\[ z_0 = M_1 + iM_2, \text{ when } c = 1 \]  

(75)

and hence it follows that

\[ \mathcal{L}_2 = \frac{1}{2} g_{AB} (\partial_\mu M_A)(\partial^\mu M_B) = \frac{1}{[1 + z\bar{z}_0]^2} \frac{(\partial_\mu z_0)(\partial^\mu \bar{z}_0)}{2} \]  

(76)

in obvious hermitian form in these stereographic coordinates, and sometimes this is written as

\[ \mathcal{L}_2 = \frac{(\partial_\mu \xi)(\partial^\mu \bar{\xi})}{2[1 + \xi\bar{\xi}]^2} \]  

(77)

where \( z_0 = \xi r \) is used to emphasize the constant radius \( r \) of the 2-sphere.

The \( CP^4 \) case has

\[ M = M_\mu \lambda_\mu = \frac{(z_0 + \bar{z}_0)}{2} \lambda_4 + i \frac{(z_0 - \bar{z}_0)}{2} \lambda_5 + \frac{(z_1 + \bar{z}_1)}{2} \lambda_6 + i \frac{(z_2 - \bar{z}_2)}{2} \lambda_7 \]  

(78)

This is perhaps a suitable place to note that if the functions \( z_0 \) and \( z_1 \) are not chosen carefully then \( M \) will not be generic and the degree of the equation satisfied by it will be less than the maximum.

The Goldstones Bosons of this scheme are the octet of pseudo scalar mesons described by the \( M_i \). In general there are two \( SU_3 \) invariants which maybe constructed from the \( M^i \). These can be denoted

\[ X = M^i M_i \]  

(79)
and

\[ Y = d_{ijk} M^i M^j M^k \]  \hfill (80)

where the determinantal inequality

\[ 3Y^2 \leq X^3 \]  \hfill (81)

ensures that the norm of an arbitrary vector constructed from the \( M^i \) shall be positive definite. Now define \( \phi \) and \( \delta \) by

\[ \phi = X^{\frac{1}{2}} \]  \hfill (82)

and

\[ \phi^3 \sin \delta = \sqrt{3} Y \]  \hfill (83)

as the basic invariants.

It is straightforward to show [12] that, if

\[ N_i = d_{ijk} M^j M^k \]  \hfill (84)

then

\[ \hat{m}_i = \phi^{-1} M_i \]  \hfill (85)

and

\[ \hat{r}_i = \phi^{-2} \sec \delta \left( \sqrt{3} N_i - \phi M^i \sin \delta \right) \]  \hfill (86)

are an orthonormal base for the independent vectors.

It has also been shown that the vectors

\[ q_i = \hat{r}_i \cos \alpha + \hat{m}_i \sin \alpha \]  \hfill (87)

and

\[ s_i = (-) \hat{r}_i \sin \alpha + \hat{m}_i \cos \alpha \]  \hfill (88)
with

$$3\alpha = \delta - 2A\pi \quad (A = 1, 2, 3) \quad (89)$$

are respectively charge and special vectors in the sense of Michel and Radicati [5]. Apart from their orthonormality these vectors also have the properties

$$(-)\sqrt{3}d_{ijk}q^jq^k = q_i = \sqrt{3}d_{ijk}s^js^k \quad (90)$$

$$\sqrt{3}d_{ijk}s^jq^k = s_i \quad (91)$$

and

$$f_{ijk}s^jq^k = 0 \quad (92)$$

so that a single pair \(q^i\) and \(s^i\) represent a useful alternative to working with the three \(P_i^A\) which are linearly dependent. Adopting the choice \(A = 3\) for the set of standard \(q^i\) and \(s^i\) it follows that

$$\sqrt{2}p_1^i = q_i + s_i \quad (93)$$

and

$$\sqrt{2}p_2^i = q_i - s_i \quad (94)$$

are the orthonormal basis vectors introduced previously.

The second rank tensors which may be constructed from the \(M^i\) are spanned by the six projection operators \((P_{AB})_{ij}\) and the four \((p_{\alpha\beta})_{ij}\), all of which are taken to be hermitian in the matrix sense.

It is standard to introduce projection operators with a cyclic notation in the form

$$(S_1)_{ij} = (P_{23})_{ij} + (P_{32})_{ij} \quad (95)$$

$$(S_2)_{ij} = (P_{13})_{ij} + (P_{31})_{ij} \quad (96)$$

$$(S_3)_{ij} = (P_{12})_{ij} + (P_{21})_{ij} \quad (97)$$

The first term in equation (64) can be treated by making the substitutions (where lower and upper Greek indices take the ranges 4-5 and 6-7 respectively)

$$p_1^\mu \Rightarrow n_1^\mu, \quad p_2^\Gamma \Rightarrow n_2^\Gamma \quad (98)$$
\[
\frac{\partial \psi^\beta}{\partial \mu_1} = \sqrt{2} \frac{\partial \psi^\beta}{\partial \omega_1} = \sqrt{2} \frac{\partial \psi^\beta}{\partial \omega_1} \frac{\partial M}{\partial \omega_1} \tag{99}\]

\[
\frac{\partial \psi^\beta}{\partial \Gamma_2} = \sqrt{2} \frac{\partial \psi^\beta}{\partial \omega_2} = \sqrt{2} \frac{\partial \psi^\beta}{\partial \omega_2} \frac{\partial M}{\partial \omega_2} \tag{100}\]

and similarly, using equations (95), (96) and (97), the \( S_A \) can be brought to the forms

\[
(S_1)_{\mu\nu} = \delta_{\mu\nu} - n'_\mu n'_\nu \tag{101}\]

\[
(S_2)_{\Gamma\Omega} = \delta_{\Gamma\Omega} - n^2_{\Gamma} n^2_{\Omega} \tag{102}\]

and \( S_3 \) vanishes because \( n^3 \) lies inside the \( SU_2 \times U_1 \) subspace rather than in the coset space. [This explains why the range of summation is reduced in future.]

It follows that

\[
4g_{\mu\nu} = 2 \frac{\partial \psi^\beta}{\partial \omega_1} \frac{\partial \psi^\beta}{\partial \omega_1} M_{\mu} M_{\nu} \frac{(\omega_1)^2}{(\omega_1)^2} + 2(S_1)_{\mu\nu} \sin^2 \left( \frac{\theta_1 + 2\theta_2}{2} \right) \tag{103}\]

\[
4g_{\Gamma\Omega} = 2 \frac{\partial \psi^\beta}{\partial \omega_2} \frac{\partial \psi^\beta}{\partial \omega_2} M_{\Gamma} M_{\Omega} \frac{(\omega_1)^2}{(\omega_2)^2} + 2(S_2)_{\Gamma\Omega} \sin^2 \left( \frac{\theta_2 - 2\theta_1}{2} \right) \tag{104}\]

and

\[
4g_{\mu\Omega} = 2 \frac{\partial \psi^\beta}{\partial \omega_1} \frac{\partial \psi^\beta}{\partial \omega_1} M_{\mu} M_{\Omega} \frac{(\omega_1)^2}{(\omega_1)^2} \omega_2 \tag{105}\]

Noting that \( M_{\mu} \equiv \omega_1 n^1_\mu \), it appears that the hermiticity conditions on the diagonal components are

\[
\frac{\partial \psi^\beta}{\partial \omega_1} \frac{\partial \psi^\beta}{\partial \omega_1} = 2 \frac{2}{(\omega_1)^2} \sin^2 \left( \frac{\theta_1 + 2\theta_2}{2} \right) \tag{106}\]

and

\[
\frac{\partial \psi^\beta}{\partial \omega_2} \frac{\partial \psi^\beta}{\partial \omega_2} = 2 \frac{2}{(\omega_2)^2} \sin^2 \left( \frac{\theta_2 - 2\theta_1}{2} \right) \tag{107}\]

These conditions can be imposed by a slight generalisation of the method used in the CP2 case. Obviously it will be advantageous to introduce the
abbreviations

\[ D = \frac{\partial \psi_1}{\partial \omega_1}, \quad D' = \frac{\partial \psi_2}{\partial \omega_1}, \quad d = \frac{\partial \psi_1}{\partial \omega_2} \quad \text{and} \quad d' = \frac{\partial \psi_2}{\partial \omega_2} \]  

(108)

It is simple to see that from equations (106) and (107) it follows that

\[ D^2 + D'^2 = \frac{2}{(\omega_1)^2} \sin^2 \left( \frac{\theta_1 + 2\theta_2}{2} \right) \]  

(109)

and

\[ d^2 + d'^2 = \frac{2}{(\omega_2)^2} \sin^2 \left( \frac{\theta_2 - 2\theta_1}{2} \right) \]  

(110)

These two results ensure the hermiticity constraints on \( g_{\mu\nu} \) and \( g_{\Gamma\Omega} \), which then takes forms

\[ 4g_{\mu\nu} = \frac{1}{(\omega_1)^2} \delta_{\mu\nu} \sin^2 \left( \frac{\theta_1 + 2\theta_2}{2} \right) \]  

(111)

and

\[ 4g_{\Gamma\Omega} = \frac{1}{(\omega_2)^2} \delta_{\Gamma\Omega} \sin^2 \left( \frac{\theta_2 - 2\theta_1}{2} \right) \]  

(112)

where the normalization has again been adjusted. The other components take the forms

\[ 4g_{\mu\Omega} = (Dd + D'd') \frac{n_1^1 n_2^2}{2} \]  

(113)

and

\[ 4g_{\Gamma\nu} = (Dd + D'd') \frac{n_1^1 n_2^2}{2} \]  

(114)

and as these are off diagonal it is necessary to show hermiticity makes them zero.

Now put

\[ \omega_1 = [c^2 + (\omega_2)^2]^{\frac{1}{2}} \tan \left( \frac{\theta_1 + 2\theta_2}{4} \right) \]  

(115)
to show that
\[
\frac{\partial \left( \frac{\theta_1 + 2\theta_2}{2} \right)}{\partial \omega_1} = \frac{\sin \left( \frac{\theta_1 + 2\theta_2}{2} \right)}{\omega_1} = \frac{2[c^2 + (\omega_2)^2]^{\frac{1}{2}}}{[c^2 + (\omega_1)^2 + (\omega_2)^2]}.
\]  
\text{(116)}

Similarly, put
\[
\omega_2 = [c'^2 + (\omega_1)^2]^{\frac{1}{2}} \tan \left( \frac{\theta_2 - 2\theta_1}{4} \right)
\]  
\text{(117)}

to show that
\[
\frac{\partial \left( \frac{\theta_2 - 2\theta_1}{2} \right)}{\partial \omega_2} = \frac{\sin \left( \frac{\theta_2 - 2\theta_1}{2} \right)}{\omega_2} = \frac{2[c'^2 + (\omega_1)^2]^{\frac{1}{2}}}{[c'^2 + (\omega_1)^2 + (\omega_2)^2]}.
\]  
\text{(118)}

It is straightforward to see from equations (106) and (107) that
\[
\left( \frac{\partial \theta_1}{\partial \omega_1} \right)^2 + \left( \frac{\partial \theta_1}{\partial \omega_1} \right) \left( \frac{\partial \theta_2}{\partial \omega_1} \right) + \left( \frac{\partial \theta_2}{\partial \omega_1} \right)^2 = \frac{1}{(\omega_1)^2} \sin^2 \left( \frac{\theta_1 + 2\theta_2}{2} \right)
\]  
\text{(119)}

and
\[
\left( \frac{\partial \theta_1}{\partial \omega_2} \right)^2 + \left( \frac{\partial \theta_1}{\partial \omega_2} \right) \left( \frac{\partial \theta_2}{\partial \omega_2} \right) + \left( \frac{\partial \theta_2}{\partial \omega_2} \right)^2 = \frac{1}{(\omega_2)^2} \sin^2 \left( \frac{\theta_2 - 2\theta_1}{2} \right)
\]  
\text{(120)}

so that, using the left hand parts of equations (116) and (118), it follows that
\[
c' = (\pm)c
\]  
\text{(121)}

is required since the c and c' are constants independent of the \(\omega_1\) and \(\omega_2\) variables, and the expressions on the right hand sides result simply from using a trigonometric substitution to evaluate the integral. Hence, the forms
\[
g_{\mu\nu} = \frac{\delta_{\mu\nu} [c^2 + (\omega_2)^2]}{[c^2 + (\omega_1)^2 + (\omega_2)^2]^2}
\]  
\text{(122)}

and
\[
g_{\Gamma\Omega} = \frac{\delta_{\Gamma\Omega} [c^2 + (\omega_1)^2]}{[c^2 + (\omega_1)^2 + (\omega_2)^2]^2}
\]  
\text{(123)}
are revealed.

Now consider

\[ D^2 + D'^2 = d^2 + d'^2 \]  

(124)

which follows from equations (109) and (110) by using the left hand parts of

equations (116) and (118), now that \( c' = (\pm)c \), reveals that

\[ dD + d'D' = 0 \]

(125)

From equations (113) and (114), it is now evident that

\[ g_{\mu\Omega} = 0 = g_{\Gamma\psi} \]

(126)

and this completes the specification of the metric through the hermiticity

conditions. It is perhaps worth repeating that the forms of the metric (given

for example in equation (103) and (104)) are in general coordinates before

the hermiticity conditions are applied. However, the forms given in equations

(122) and (123) are in hermitian form and maybe directly compared with the

classic results of Fubini [13] and Study [14]. These authors apply scaling by

using the conformal symmetry of the metric, and this is directly equivalent to

setting \( c = 1 \) in the present notation. Reverting to complex notation reveals

the invariant

\[ L_4 = \frac{dz_1 d\bar{z}_1 [1 + z_2 \bar{z}_2] + dz_2 d\bar{z}_2 [1 + z_1 \bar{z}_1]}{[1 + z_1 \bar{z}_1 + z_2 \bar{z}_2]^2} \]

(127)

retrieving the Fubini-Study form.

Conclusions

It appears that the CPN metric has been found in general coordinates (in

principle) for all \( N \), and that in the cases of \( N = 2 \) and \( N = 4 \) well known

forms are recovered in the hermitian limit. Obviously, the algebraic effort re-

quired does rise with \( N \) but only simple functions ever appear.

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