On the Nature of Angular Momentum Transport in Nonradiative Accretion Flows

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The principles underlying a proposed class of black hole accretion models are examined. The flows are generally referred to as “convection-dominated,” and are characterized by inward transport of angular momentum by thermal convection and outward viscous transport, vanishing mass accretion, and vanishing local energy dissipation. In this paper we examine the viability of these ideas by explicitly calculating the leading order angular momentum transport of axisymmetric modes in magnetized, differentially rotating, stratified flows. The modes are destabilized by the generalized magneto-rotational instability, including the effects of angular velocity and entropy gradients. It is explicitly shown that local convective modes, defined by wavenumbers that would be stable in the absence of a destabilizing entropy gradient, transport angular momentum outwards. General inward transport by convection is a property only of hydrodynamical disks; magnetized disks behave quite differently and are dominated by field line tension. Moreover, very general thermodynamic principles prohibit the complete recovery of irreversible dissipative energy losses, a central feature of convection-dominated models. Our results are in good agreement with global MHD simulations, which find significant levels of outward transport and energy dissipation, whether or not destabilizing entropy gradients are present.

Subject headings: accretion — accretion disks — black hole physics — convection — instabilities — (magnetohydrodynamics:) MHD — turbulence
1. Introduction

Originally developed to be powerful luminosity sources, black hole accretion models are now confronted by an embarrassing plethora of underluminous X-ray emitters, the best known of which is the Galactic Center source Sgr A* (Falcke & Melia 2001). These low luminosity objects are thought to be prime candidates for a class of theoretical accretion models that has been intensely studied in recent years, which we shall refer to generically as nonradiative accretion flows. Accretion generally requires significant energy loss, and the absence of radiative losses in these flows means that determining the ultimate fate the gas is less than straightforward.

Much of the recent interest in nonradiative flows was sparked by the work of Narayan & Yi (1994), who examined a series of one-dimensional, self-similar, steady accretion models. In the absence of radiative losses, these solutions were said to be “advection-dominated accretion flows,” or ADAFs for short. In these models, an $\alpha$ viscosity was invoked to allow angular momentum transport. The resulting flows were substantially sub-Keplerian, and had to have an adiabatic index less than $5/3$. Dissipative heating increases toward the flow center, and an inwardly increasing entropy profile develops, which we shall refer to as “adverse.”

Nonradiative accretion flows are amenable to numerical simulation. Hydrodynamical simulations carried out with with large assumed $\alpha$ values ($>0.3$) showed some similarities to ADAFs, but smaller values of $\alpha$ behaved differently (Stone, Pringle, & Begelman 1999; Igumenschev & Abramowicz 1999, 2000). Substantial turbulence developed. The net inward mass accretion rate was smaller than anticipated, and the density distribution far less centrally peaked. Both inward and outward mass fluxes were observed at different times and different locations within the flow, and they nearly cancelled.

These findings were given the following interpretation by Narayan, Igumenschev, & Abramowicz (2000; hereafter NAI), Quataert & Gruzinov (2000; hereafter QG), and Abramowicz et al. (2000; hereafter AIQN). The adverse entropy gradient triggers an instability, and significant levels of convection result; the global solutions were said to be “convection-dominated accretion flows” (CDAFs). The next step in the the argument is key: invoking the findings of other hydrodynamical simulations (Stone & Balbus 1996, Igumenschev, Abramowicz, & Narayan 2000), the angular momentum transport generated by the convective turbulence was claimed to be inward. This secondary transport was envisioned to be sufficiently great to cancel the primary outward angular momentum transport by whatever process the $\alpha$ viscosity was modeling—presumably the magnetorotational instability, or MRI. (Balbus & Hawley 1991).

The stated consequences of this behavior were novel and numerous. With the instigation of a second source of turbulence, the CDAF scenario holds that the vanishing of the angular momentum flux directly implies that the $R\phi$ component of the stress tensor responsible for accretion also vanishes. This has the further implication, it was argued, that there is essentially no mass accretion and no turbulent dissipation, despite the presence of vigorous turbulence throughout the bulk of the flow. The only region where there is any mass accretion in the model is at the very inner edge of the flow. All of the energy associated with the turbulent heating in this region would then be transported outward to infinity, with no dissipative losses. For this reason, CDAFs are often put forth as natural candidates to explain under-luminous X-ray sources. These models have been elaborated upon, becoming influential and widely-cited. Since black hole accretion models are central
to our understanding of much of X-ray astronomy, the theoretical support for CDAFs deserves careful scrutiny.

In this work, we demonstrate that CDAF models have two major inconsistencies. An explicit and representative example of magnetized, rotationally and convectively unstable gas is studied. Convective disturbances do not transport angular momentum inwards in the presence of the MRI. They transport angular momentum outwards, and are essentially indistinguishable from standard MRI modes. This latter point has been emphasized elsewhere (Hawley, Balbus, & Stone 2001; Balbus 2001), but here we demonstrate it quantitatively by explicitly calculating the leading order angular momentum transport associated with unstable modes. Convective modes transport angular momentum outwards when a magnetic field is present, and inwards when it is not. Indeed, the direction of angular momentum transport is not really a property of convection at all, it is a property associated with background medium: is it magnetized or not? That is what matters.

The second difficulty is even more fundamental, affecting magnetic and nonmagnetic models alike. By relying upon dissipated heat energy to trigger a secondary convective instability that supposedly renders the flow dissipation-free, CDAFs run afoul of thermodynamic principles.

We are led to a much more standard picture of turbulent accretion flow, though one at odds with the tenets of CDAF theory. The turbulent stress tensor in magnetized differentially rotating gas does not vanish. There is vigorous local turbulent dissipation. There is mass accretion. The near cancellation of inward and outward mass fluxes is a property of any turbulent flow with large rms fluctuations, and not a superposition of the contributions from two distinct sources of mass flux with opposite signs.

In the following sections, we present (§2) the details of the angular momentum calculation showing outward transport; (§3) an explanation of important thermodynamic inconsistencies evident in the formulation CDAF theory; (§4) a brief review of numerical simulations; and (§5) a concluding summary.

2. Radial Angular Momentum Transport

2.1. Local WKB Modes

Consider a disk with radially decreasing outward entropy and pressure gradients. We use standard cylindrical coordinates \((R, \phi, Z)\). The square of the Brunt-Väisälä frequency \((N^2)\) is thus negative, and absent rotation would be unstable by the Schwarzschild criterion. In what follows, it is convenient to work with the positive quantity

\[ N^2 \equiv -N^2 = \frac{3}{5} \frac{\partial P}{\partial R} \frac{\partial \ln \rho_P^{-5/3}}{\partial R} > 0. \]  

(1)

The disk is differentially rotating with decreasing outward angular velocity \(\Omega(R)\), and
epicyclic frequency
\[ \kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d \ln R} = \frac{1}{R^3} \frac{d R^4 \Omega^2}{d R} > 0. \] (2)

A vertical magnetic field \( \mathbf{B} = B \mathbf{e}_z \) threads the disk. Its associated Alfvén velocity is
\[ v_A^2 = B^2 / 4\pi \rho, \]
where \( \rho \) is the gas density. Axisymmetric WKB plane wave displacements of the form
\[ \xi(R, Z, t) = \xi \exp(i \mathbf{k} \cdot \mathbf{r} - i \omega t), \] (3)
where \( \mathbf{k} \) and \( \omega \) are respectively the wavenumber vector and the angular frequency, lead to the dispersion relation (Balbus & Hawley 1991)
\[ \tilde{\omega}^4 + \tilde{\omega}^2 (N^2 - \kappa^2) - 4\Omega^2 (k v_A)^2 = 0, \] (4)
where
\[ \tilde{\omega}^2 = \omega^2 - (k v_A)^2. \] (5)

Let \( \gamma = -i \omega \). Then, the unstable branch of the dispersion relation (4) is
\[ \gamma^2 = -(k v_A)^2 + \frac{1}{2} \left[ N^2 - \kappa^2 + \sqrt{(N^2 - \kappa^2)^2 + 16\Omega^2 (k v_A)^2} \right]. \] (6)

It is straightforward to show that these unstable modes must have
\[ (k v_A)^2 < N^2 - \frac{d\Omega^2}{d \ln R} = N^2 + \left| \frac{d\Omega^2}{d \ln R} \right|, \] (7)
and that the maximum growth rate is
\[ \gamma_{\text{max}} = \frac{\Omega}{4} \left( \frac{N^2}{\Omega^2} + \left| \frac{d \ln \Omega^2}{d \ln R} \right| \right), \] (8)
which is attained for wavenumbers satisfying
\[ (k v_A)^2_{mg} = \Omega^2 \left( 1 - \frac{(N^2 - \kappa^2)^2}{16\Omega^4} \right). \] (9)

Those wavenumbers that satisfy the condition
\[ \left| \frac{d\Omega^2}{d \ln R} \right| < (k v_A)^2 < N^2 + \left| \frac{d\Omega^2}{d \ln R} \right| \] (10)
are unstable only because of the presence of an adverse entropy gradient. In this sense, these wavenumbers correspond to purely convective modes, and it seems reasonable to adopt this as our definition. We shall show that these wavenumbers transport angular momentum outward, contrary to the assertion that convection always transports angular momentum inward.
2.2. Stress Calculation

The angular momentum flux is directly related to the $R\phi$ component of the stress tensor

$$T_{R\phi} = \rho (\delta v_R \delta v_\phi - \delta v_{AR} \delta v_{A\phi}),$$

where $\delta$ denotes an Eulerian perturbation, and

$$\delta v_A = \frac{\delta B}{\sqrt{4\pi \rho}}.$$  \hspace{1cm} (12)

For the local WKB modes we consider here, the angular momentum flux is $R\Omega T_{R\phi}$. Hence, the sign of the transport is simply the sign of $T_{R\phi}$.

The needed expressions can be written down immediately from the equations (2.3c–g) of Balbus & Hawley (1991). In terms of $\gamma$, they are

$$\delta v_\phi \delta v_R = (\delta v_R^2) \frac{\Omega}{D\gamma} \left( \frac{(kv_A)^2}{\gamma^2} \frac{d \ln \Omega}{d \ln R} - \frac{\kappa^2}{2\Omega^2} \right),$$

$$-\delta v_{A\phi} \delta v_{AR} = (\delta v_{AR})^2 \frac{2\Omega}{D\gamma} = (\delta v_R)^2 \left( \frac{kv_A}{\gamma} \right)^2 \frac{2\Omega}{D\gamma},$$

where

$$D = 1 + \frac{(kv_A)^2}{\gamma^2}.$$ \hspace{1cm} (15)

The problem with a loosely defined notion like “convection transports angular momentum inward,” is that neither of these two expressions knows anything about convection. There is no explicit dependence upon $N$ anywhere. Indeed, these equations are general beyond our simple example, holding in the presence of both vertical and radial entropy gradients. For a given growth rate, angular momentum transport is simply a matter of rotation and magnetic tension. This is the physics of the MRI; there is no distinct convective behavior.

In the absence of a magnetic field, all axisymmetric disturbance governed by the above equations will transport angular momentum inwards. This is why hydrodynamic simulations consistently find inward transport. There is nothing special about the physics of convection per se. In the presence of a magnetic field, however, no such generality is possible.

Consider, for example, the following Keplerian disk with an exaggeratedly large, adverse entropy gradient, $N^2 = 2\Omega^2$. Modes with $(kv_A)^2 < 5\Omega^2$ are unstable in such a disk, and those with $(kv_A)^2 > 3\Omega^2$ are unstable only because of the entropy gradient. In this sense they are unambiguously convective. The mode $(kv_A)^2 = 4\Omega^2$ fits this description, and with $\gamma^2 = 0.531\Omega^2$ has

$$\left( \frac{(kv_A)^2}{\gamma^2} \frac{d \ln \Omega}{d \ln R} - \frac{\kappa^2}{2\Omega^2} \right) = 10.8 > 0$$ \hspace{1cm} (16)

We need not even compute the manifestly positive Maxwell contribution, this is a convective disturbance propagating angular momentum outward.
By combining equations (6), (13), and (14), we may calculate the full range of wavenumbers that transport angular momentum outwards in our model:

$$(k v_A)^2 > \frac{1}{16} \left( 4 - \left| \frac{d \ln \Omega^2}{d \ln R} \right| \right) \left( 2 \mathcal{N}^2 - \kappa^2 \right).$$

(17)

It is apparent that not only will the convective modes (10) generally transport angular momentum outward, unless $\mathcal{N}$ surpasses and is sustained at a value of $\kappa/\sqrt{2}$, every mode, convective or otherwise, will transport angular momentum outward. In the next section we will see why there are very good reasons to believe that this is how turbulent accretion behaves.

3. Theoretical Implications

In contrast to the CDAF scenario, the stability of black hole accretion is not regulated by two separate processes, one producing viscous-like transport, the other producing inward transport by perturbations labeled “convective.” Even the notion of of a convective instability is not unambiguous, and needs to be defined: we have used equation (10) as our definition, since they are destabilized only because of the presence of $\mathcal{N}$, and hence seem closest to the CDAF concept. There is just one instability involved here, and its criteria depend simultaneously upon both thermal and dynamical gradients (Balbus 1995, 2001). Their violation generates turbulence, driving the accretion that characterizes the flow. In simulations, the relevant gradients are entropy and angular velocity. (In real flows, temperature may be more important than entropy.) The classical Høiland criteria used in CDAF theory are incorrect for a magnetized gas, place a very misleading emphasis on adverse entropy gradients (because the angular momentum gradients are never destabilizing), and ignore the physics of the MRI altogether. Magnetic tension in any accretion flow produces outward angular momentum transport, and trying to characterize a perturbation by anything other than its wavenumber is not a useful concept here.

The statements made in the literature concerning the dissipative properties CDAFs are very striking. In light of the fact that there is no convective domination of the transport in a nonradiative accretion flow, they merit scrutiny.

A CDAF requires the following behavior. Energy losses, which are caused by a viscous-like instability in the fluid, are dissipated as heat, and in the process generate a secondary convective instability. The triggering and ongoing maintenance of the secondary instability by this heat input recovers the same heat that is causing the secondary instability itself, and returns this heat energy, in full, to the fluid in the form of work (inward angular momentum and outward energy fluxes). When both the primary and secondary instabilities are active, the total angular momentum flux supposedly drops to zero, as does the volume-specific energy dissipation rate $Q^+$ (NAI, AIQN).

Clearly, this is a violation of the second law of thermodynamics, whether the system is hydrodynamical or magnetohydrodynamical: the onset of convection is caused by irreversible heat dissipation, and this energy cannot be fully recovered in the form of work. Even if, for some reason, $T_{R\phi}$ did vanish (e.g. isotropic turbulence), the identification by
as the fundamental expression for the volume specific dissipative energy loss rate is incorrect. The fundamental definition of the energy loss must be in terms of the dissipation coefficients themselves. With \( \eta_v \) equal to the viscous diffusivity, \( \eta_B \) the resistivity, and \( \delta J \) the fluctuating current density, the energy dissipation rate per unit volume is

\[
Q^+ \equiv \sum_i \langle \eta_v |\nabla \delta v_i|^2 + \eta_B \delta J^2 \rangle,
\]

where the sum is over vector components. These losses never vanish in a turbulent cascade, and a secondary source of turbulent fluctuations can never fully recover the energy that is dissipated in the process of triggering the creation of the same source. It amounts to reversing diffusion, a thermodynamic impossibility.

The sources and sinks for turbulent fluctuations may be read off from Balbus & Hawley (1998), eq. (89):

\[
-T_{R\phi} \frac{d\Omega}{d \ln R} + P \nabla \cdot \delta v - Q^+. \tag{20}
\]

The expression (18) happens to be equal to dissipative energy losses \( Q^+ \) under narrowly defined conditions that may be inferred from the above: steady, local turbulence in which the work done by pressure is negligible. This is, in fact, a good description of the behavior of turbulence in magnetized differentially rotating flows. An immediate consequence is that the time-averaged value of \( T_{R\phi} \) must be \( > 0 \), i.e., the angular momentum flux is outward. It is also possible, at least in principle, to have a vanishing stress tensor and still have steady dissipative disk turbulence—the source is then \( P \nabla \cdot \delta v \), as in a convective cascade. The vanishing of the stress tensor hardly implies dissipation-free turbulence. Conversely, a positive stress tensor does not necessarily imply that dissipation must be occurring. Indeed, the stress-by-strain expression (18) is the key to the principle of conservation of wave action, an entirely dissipation-free concept (Lighthill 1978).

The bottom line is that there is always a turbulent angular flux in an MRI unstable flow; the stress tensor component does not, and cannot vanish. Its associated \( \alpha \) value (i.e., the stress normalized to some fiducial pressure) is generally \( \sim 0.1 \) in the simulations. Dissipation is always present.

### 4. Comparison With Simulations

The point of contact of CDAFs with numerical simulations is the power law scaling of the density, \( \rho(r) \), where \( r \) is spherical radius. The argument runs that since the CDAF energy flux is conserved (no dissipation), \( \rho v^3 r^2 \) is a constant. With \( v \sim r^{-1/2} \), this gives \( \rho \sim r^{-1/2} \) as well. By way of contrast, constancy of the mass flux \( \sim \rho vr^2 \) would give a much steeper \( \rho \sim r^{-3/2} \) power law, which is associated with an ADAF solution. The power law scaling \( \rho \sim r^{-1/2} \) has been interpreted as evidence in support of convection-dominated flows.
Since gas accreting into a black hole is almost certainly magnetized at some level, there is very little to be gained by hydrodynamical simulations: the stability and transport properties of magnetized and unmagnetized gases are simply too different from one another. In general, MHD simulations have not been supportive of CDAFs (Stone & Pringle 2000; Hawley, Balbus, & Stone 2001; Hawley & Balbus 2002). The one exception cited is that of Machida, Matusmoto and Mineshige (2001). This simulation follows the evolution of a magnetized torus in a Newtonian potential. “Convective motions” are observed in the subsequent accretion flow. However, these authors draw no distinction between what they refer to as convective motions and turbulence in general. Turbulence is, of course, inevitable in an MHD accretion flow; it is hardly the defining signature of a CDAF. The only quantitative basis for the claimed agreement with CDAFs was the observation that at one point in time the the density exhibited a $r^{-1/2}$ power law behavior. However, in fact, the $r$ dependency was different at a later times in what was a highly time-dependent flow.

On should not, in any case, lose sight of the fact that any agreement reached between theory and simulation in a radial scaling law would not be an answer to fundamental dynamical and thermodynamical inconsistencies. Indeed, verification should run in the opposite direction: in the course of producing MHD turbulence, any numerical simulation should show on average a positive value for $T_{R\phi}$, and significant levels of energy dissipation.

5. Conclusion

We have shown that there is no basis for the claim that convection transports angular momentum inwards in an MRI-unstable flow. In fact, the direction of angular momentum transport depends entirely on whether the medium is magnetized or not, not upon the classification of the perturbation. Since inward convective transport is the fundamental principle behind CDAF models, this approach is untenable, as are the claims of vanishing stress, vanishing energy dissipation, and vanishing energy flux divergence. In particular, the absence of energy dissipation in a system driven by irreversible heating is a thermodynamic impossibility.

Instead, black hole accretion features far more conventional fluid dynamics. Underluminous sources are much more likely to be a consequence of low densities and lower than anticipated temperatures, as opposed to dissipation-free turbulence. MHD turbulence leads to outward angular momentum transport and to a positive $R\phi$ stress component, irreversible dissipative heating, mass accretion, and significant mass outflow as well. These are all in evidence in numerical MHD simulations. The development of the spectral and energetic properties will test these model flows more stringently.

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