Quantum Anomaly of the Transverse Ward-Takahashi Relation for the Axial-Vector Vertex

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Abstract

We study the possible quantum anomaly for the transverse Ward-Takahashi relations in four dimensional gauge theories based on the method of computing the axial-vector and the vector current operator equations. In addition to the well-known anomalous axial-vector divergence equation (the Adler-Bell-Jackiw anomaly), we find the anomalous axial-vector curl equation, which leads to the quantum anomaly of the transverse Ward-Takahashi relation for the axial-vector vertex. The computation shows that there is no anomaly for the transverse Ward-Takahashi relation for the vector vertex.

Keywords: anomaly, axial-vector current, transverse Ward-Takahashi relation.

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In quantum field theory symmetries lead to relations among the Green functions of the theory, which are generally known as the Ward-Takahashi (WT) identities [1]. They play an important role in providing a consistency condition in the perturbative approach as well as in the nonperturbative approach of any quantum field theory. In this regard, an very interesting problem relates to the nonperturbative studies of gauge field theories using the Dyson-Schwinger equation (DSE) approach [2][3]. The DSEs are an infinite set of coupled integral equations which relate the n-point Green function to the (n+1)-point function; at its simplest, propagators are related to three-point vertices and so on. Therefore, we must find some way to truncate this set of equations. If we can express the three-point vertices in terms of the propagators, these equations will form a closed system for the propagators. Naturally, the basic approach of solving this problem is to use the WT identities. But the normal WT identities specify only the longitudinal parts of Green functions, leaving the transverse parts undetermined [4]. How to solve exactly the transverse parts of vertices and so the full vertex functions then become a crucial problem [3][4]. Obviously the key point is to study the constraint on the transverse parts of vertices imposed by the symmetry of the system, i.e. the transverse WT relations, as in the case of the longitudinal parts of vertices. In Ref.[5], we studied the transverse WT relation for the vector vertex. It showed that in order to obtain the complete solution for the vector vertex we need to build simultaneously the WT relations for the axial-vector and the tensor vertices. Up to the effects of quantum anomaly, this problem was solved by He[6] who presented the transverse WT relations for the vector, the axial-vector and the tensor vertices in four dimensional gauge theories, and found that, indeed, the full vector and the full axial-vector vertex functions are expressed only in terms of the fermion propagators in the chiral limit with massless fermions.

However, the symmetry in the classical theory may be destroyed in quantum theory by the quantum anomaly arising from the loop diagram. The well-known example is the anomalous nonconservation of the four-dimensional axial current, expressed by the anomalous axial-vector divergence equation, which is known as the Adler-Bell-Jackiw (ABJ) anomaly [7]. Hence the normal WT identity for the axial-vector vertex is modified by the ABJ anomaly. Therefore, in order to obtain the universally satisfied transverse WT relations and the full vertex functions, we need to take into account the quantum anomaly for them.

In this Letter, we study the possible quantum anomaly for the transverse WT relations based on the method of computing the axial-vector and the vector current operator equations [8], respectively. In addition to the well-known anomalous axial-vector divergence equation, we find the anomalous axial-vector curl equation, which leads to the quantum anomaly in the transverse WT relation for the axial-vector vertex. The computation shows that there is no anomaly for the transverse WT relation for the vector vertex.

In a canonical quantum field theory, the transverse WT relation for the axial-vector vertex is related to the curl of the time-ordered products of the three-point function involving the axial-vector current operator [5][6]. In order to perform the curl operation, it is convenient to introduce the bilinear covariant current operator, 
\[ V^\lambda_{\mu
u}(x) = \frac{i}{2} \bar{\psi}(x) \left[ \gamma^\lambda, \sigma^{\mu\nu} \right] \gamma_5 \psi(x) = i [g^{\lambda\mu} j_5^\nu(x) - g^{\lambda\nu} j_5^\mu(x)], \]
where \( j_5^\mu(x) = \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) \). Thus the curl of the axial-vector current is given by \( \partial^\lambda V^\lambda_{\mu
u}(x) \), where \( \partial^\lambda \) denotes the derivative operator with respect to the argument \( x \). To study the possible quantum anomaly for the transverse WT relation for the axial-vector vertex, let us study the operator equation for the curl of the axial-vector current following the method of deriving the anomalous axial-vector divergence equation [8]. To this
end, we define the bilinear current operator by placing the two fermion fields at distinct points separated by a small distance $\epsilon$ and then carefully taking the limit as the two fields approach each other. Explicitly, we define

$$V^{\lambda\mu}_5(x) = \text{Symm} \lim_{\epsilon \to 0} \{ \bar{\psi}(x + \epsilon/2) \frac{1}{2} [\gamma^\lambda, \sigma^{\mu\nu}] \gamma_5 U P(x + \epsilon/2, x - \epsilon/2) \psi(x - \epsilon/2) \}. \quad (1)$$

Here the Wilson line $U_P(x', x) = P \exp(-ig \int_{x'}^{x} dy A_\rho(y))$ is introduced in order that the operator be locally gauge invariant, where $A_\mu$ is the gauge fields. In the QED case, $g = e$ and $A_\rho$ is the photon field. In the QCD case, $A_\rho = A^a_\rho T^a$, $A^a_\rho$ is the non-Abelian gluon field and $T^a$ are the generators of $SU(3)_c$ group. ”Symm” in Eq.(1) means that the limit $\epsilon \to 0$ should be taken symmetrically

$$\text{Symm} \lim_{\epsilon \to 0} \{ \frac{e^\mu}{\epsilon^2} \} = 0, \quad \text{Symm} \lim_{\epsilon \to 0} \{ \frac{e^\mu e^\nu}{\epsilon^2} \} = \frac{1}{d} g^{\mu\nu}, \quad (2)$$

where $d$ denotes the time-space dimension. In this work, we consider four dimensional gauge field theories, i.e. $d=4$.

Now we compute the curl of the axial-vector current. We have

$$\partial_\lambda V^{\lambda\mu}_5 = i[\partial^\mu j^\nu_5(x) - \partial^\nu j^\mu_5(x)]$$

$$= \text{Symm} \lim_{\epsilon \to 0} \{ \partial_\lambda \bar{\psi}(x + \epsilon/2) \frac{1}{2} [\gamma^\lambda, \sigma^{\mu\nu}] \gamma_5 U P(x + \epsilon/2, x - \epsilon/2) \psi(x - \epsilon/2) \}$$

$$+ \bar{\psi}(x + \epsilon/2) \frac{1}{2} [\gamma^\lambda, \sigma^{\mu\nu}] \gamma_5 U P(x + \epsilon/2, x - \epsilon/2) \partial_\lambda \psi(x - \epsilon/2)$$

$$+ \bar{\psi}(x + \epsilon/2) \frac{1}{2} [\gamma^\lambda, \sigma^{\mu\nu}] \gamma_5 U P(x + \epsilon/2, x - \epsilon/2) (-ig e^\rho \partial_\lambda A_\rho) \psi(x - \epsilon/2) \}. \quad (3)$$

To reduce this equation, we use the equations of motion for fermions with mass $m$ :

$$(i \not\! D - m) \psi = 0, \quad \bar{\psi}(i \not\! \bar{D} + m) = 0, \quad (4)$$

where $\not\! D_\mu = \not\! \partial + ig A_\mu$ and $\not\! \bar{D}_\mu = \bar{\partial}_\mu - ig A_\mu$ are covariant derivatives with $A_\mu$ being the gauge fields. Using Eq.(4), and keeping terms up to order $\epsilon$, we can reduce Eq.(3) to

$$\partial^\mu j^\nu_5(x) - \partial^\nu j^\mu_5(x)$$

$$= \lim_{x' \to x} \{ \partial^\mu \bar{\psi}(x') \gamma_\rho U P(x', x) \psi(x) \}$$

$$+ \text{Symm} \lim_{\epsilon \to 0} \{ \bar{\psi}(x + \epsilon/2) [\gamma^\mu \gamma_5 F^{\mu\rho}(x) - \gamma^\rho \gamma_5 F^{\nu\mu}(x)] \epsilon_\rho \} \psi(x - \epsilon/2) \}, \quad (5)$$

where $F^{\mu\rho} = \partial^\mu A^\rho - \partial^\rho A^\mu$.

Noticing the fact that the product of the fermion operators is singular, we must compute the singular terms in the operator product of the two fermion fields in the limit $\epsilon \to 0$. Making the operator product expansion of the two fermion fields in the presence of a background gauge field, we can perform the above mentioned computation. We obtain that the leading term is given by contracting the two operators using a free-field propagator, which gives zero contribution when trace with $\gamma^\mu \gamma^5$. The contribution from second term in the expansion of the product of operators leads to[8]
\[ \langle \bar{\psi}(x + \epsilon/2)\gamma^\mu\gamma_5\psi(x - \epsilon/2) \rangle = 2g\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}(x)\left( -\frac{i}{8\pi^2} \frac{\epsilon_\nu}{\epsilon^2} \right), \]  
(6)

where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). Substituting this expression into Eq.(5) and then taking limit in four dimensions, we find

\[ \partial^\mu j_5^\mu(x) - \partial^\nu j_5^\nu(x) \]
\[ = \lim_{x \to x'} i(\partial^x - \partial^x') \varepsilon^{\lambda\mu\nu\rho} \bar{\psi}(x')\gamma_\rho U_p(x', x)\psi(x) \]
\[ + \frac{g^2}{16\pi^2} [\varepsilon^{\alpha\beta\mu\rho} F_{\alpha\beta}(x) F_\rho^\nu(x) - \varepsilon^{\alpha\beta\nu\rho} F_{\alpha\beta}(x) F_\rho^\mu(x)]. \]  
(7)

This is the anomalous axial-vector curl equation, where the last term is the anomaly term for the curl of the axial-vector current.

The ABJ anomaly, i.e. the axial anomaly, is expressed by the anomalous axial-vector divergence equation

\[ \partial_\mu j_5^\mu(x) = -\frac{g^2}{16\pi^2} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}(x) F_{\mu\nu}(x) \]  
(8)

for the case of massless fermions. Comparing Eq.(7) and Eq.(8), we find that the anomaly term for the curl of the axial-vector current is different from the ABJ anomaly term, and hence is a type of new one which may be called the transverse axial anomaly.

It is well-known that the normal WT relation for the axial-vector vertex is modified by the axial anomaly. As a consequence of the transverse axial anomaly, the transverse WT relation for the axial-vector vertex in coordinate space (see Eq.(8) of Ref[6]) is also modified as

\[ \partial_x^\mu \left\langle 0 \left| T j_5^\mu(x)\bar{\psi}(x_1)\psi(x_2) \right| 0 \right\rangle - \partial_x^\nu \left\langle 0 \left| T j_5^\nu(x)\bar{\psi}(x_1)\psi(x_2) \right| 0 \right\rangle \]
\[ = i\sigma^{\mu\nu}\gamma_5 \left\langle 0 \left| T\bar{\psi}(x_1)\psi(x_2) \right| 0 \right\rangle \delta^4(x_1 - x) - i \left\langle 0 \left| T\bar{\psi}(x_1)\psi(x_2) \right| 0 \right\rangle \varepsilon^{\lambda\mu\nu\rho} \bar{\psi}(x')\gamma_\rho U_p(x', x)\psi(x_1)\psi(x_2) \]
\[ + \lim_{x \to x'} i(\partial^x - \partial^x') \varepsilon^{\lambda\mu\nu\rho} \left\langle 0 \left| T\bar{\psi}(x')\gamma_\rho U_p(x', x)\psi(x_1)\psi(x_2) \right| 0 \right\rangle \]
\[ + \frac{g^2}{16\pi^2} \left\langle 0 \left| T\bar{\psi}(x_1)\psi(x_2) \varepsilon^{\alpha\beta\rho\nu} F_{\alpha\beta}(x) F_\rho^\nu(x) - \varepsilon^{\alpha\beta\nu\rho} F_{\alpha\beta}(x) F_\rho^\mu(x) \right| 0 \right\rangle, \]  
(9)

where the last term arises from the anomaly term given in Eq.(7). Accordingly, the transverse WT relation for the axial-vector vertex in momentum space (see Eq.(11) of Ref.[6]) is modified as

\[ i\gamma^\mu \Gamma^\nu_A(p_1, p_2) = i\gamma^\mu \Gamma^\nu_A(p_1, p_2) \]
\[ = S_F^{-1}(p_1) \sigma^{\mu\nu}\gamma_5 - \sigma^{\mu\nu}\gamma_5 S_F^{-1}(p_2) + (p_1^\lambda + p_2^\lambda)\varepsilon^{\lambda\mu\nu\rho} \Gamma_{V\rho}(p_1, p_2) \]
\[ + \frac{g^2}{16\pi^2} K^{\mu\nu}(p_1, p_2), \]  
(10)

where the last term is the anomalous contribution, and \( K^{\mu\nu} \) is defined by

\[ \int d^4x d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 - p_2 \cdot x - q \cdot x)} \left\langle 0 \left| T\bar{\psi}(x_1)\psi(x_2) \varepsilon^{\alpha\beta\rho\nu} F_\rho^\nu(x) - \varepsilon^{\alpha\beta\nu\rho} F_\rho^\mu(x) \right| 0 \right\rangle \]
\[ = (2\pi)^4 \delta^4(p_1 - p_2 - q) iS_F(p_1) K^{\mu\nu}(p_1, p_2) iS_F(p_2). \]  
(11)
Above results are given in the Abelian case. For the axial currents of the non-Abelian QCD, the anomaly equation should be the Abelian result, supplemented by the appropriate group theory factors[8]. After reading the group theory factors for the anomaly from the fermion loop diagrams, we find that the curl of flavour non-singlet axial currents of QCD is unaffected by the anomaly of QCD. For the flavour singlet-axial current of QCD, the curl of the axial current has the anomaly and its form can be obtained from the corresponding result in the case of QED by multiplying a factor of $n_f/2$, where $n_f$ is the number of flavors. Thus, both longitudinal and transverse WT relations for the flavour non-singlet axial-vector vertex and the corresponding vertex function are unaffected by the anomaly of QCD. However, both longitudinal and transverse WT relations for the flavour-singlet axial-vector vertex and the corresponding vertex function have the anomalous contributions. Their forms can be written from the corresponding results in QED by multiplying a factor of $n_f/2$.

By the parallel procedure, we can perform the computation for the curl of the vector current operator. We find

$$\partial^\mu j^\nu(x) - \partial^\nu j^\mu(x) = \lim_{x' \to x} i(\partial^\lambda x - \partial^\lambda x')\epsilon^{\lambda\mu\rho\lambda} \bar{\psi}(x') \gamma_\rho \gamma_5 U_P(x', x) \psi(x) + \text{Symm} \lim_{\epsilon \to 0} \{ \bar{\psi}(x + \epsilon/2)[-ig(\gamma^\nu F^{\mu\rho}(x) - \gamma^\mu F^{\nu\rho}(x))\epsilon^\rho] \psi(x - \epsilon/2) \} \quad (12)$$

for the case of massless fermions. Applying the method of obtaining Eq.(7) to compute the last term of Eq.(12), we find that the contribution of this term disappears. It shows that there is no anomaly for the vector curl equation, and hence the transverse WT relation for the vector vertex has no anomaly.

In summary, we have computed the operator equations for the curl of the axial-vector and the vector currents, respectively, in four dimensional gauge theories. In addition to the well-known anomalous axial-vector divergence equation, we find the anomalous axial-vector curl equation, which may be called the transverse axial anomaly. It shows that both normal (longitudinal) and transverse WT relations for the axial-vector vertex have anomalies. The computation shows that the transverse WT relation for the vector vertex has no anomaly. As a consequence of anomalies, the full axial-vector as well as the full vector vertex functions given by Ref.[6] will be modified in such way that the additional terms arising from anomalies should be added in the vertex functions[9]. Applying the full vector vertex function including the anomalous contribution to the Dyson-Schwinger equations for the propagators will imply that the quantum anomaly may play a significant role in the nonperturbative studies of gauge theories using Dyson-Schwinger equation formalism, which involves a deeper aspect of the gauge theories and will be studied further.

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REFERENCES

[9] Because the vector and the axial-vector vertices are coupled each other through the transverse WT relations for them (see Ref.[6]), therefore not only the axial-vector vertex funcion but also the vector vertex function will be modified by the axial and the transverse axial anomalies.