Quantum universal variable-length source coding

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Abstract

We construct an optimal quantum universal variable-length code that achieves the admissible minimum rate, i.e., our code is used for any probability distribution of quantum states. Its probability of exceeding the admissible minimum rate exponentially goes to 0. Our code is optimal in the sense of its exponent. In addition, its average error asymptotically tends to 0.

1 Introduction

As was proven by Schumacher [1], and Jozsa and Schumacher [2], we can compress the unknown source state into the length $nH(\rho_p)$ with a sufficiently small error when the source state on $n$ quantum systems obeys the $n$-i.i.d. distribution of the known probability $p$, where $\rho_p := \sum \rho p(\rho) \rho$ and $H(\rho)$ is the von Neumann entropy $-\text{Tr} \rho \log \rho$. Jozsa and Schumacher’s protocol depends on the mixture state $\rho_p$. Concerning the quantum source coding, there are two criteria: blind coding, in which the input is the unknown quantum state, and visible coding, in which the input is classical information which determines the quantum state that we want to send, i.e., the encoder knows the input quantum state. In this paper, we treat only blind coding. In our setting, we allow mixed states as input states.

In blind coding, Koashi and Imoto [3] proved that even if we allow mixed states as input states, the minimum admissible length is $nH(\rho_p)$ without trivial redundancies. Depending only on the coding length $nR$, Jozsa et al. [4] constructed a code which is independent of the distribution which the input obeys. In their protocol, if the minimum admissible length of the distribution $p$ is less than $R$, we can decode with a sufficiently small error. This kind code is called a quantum universal fixed-length source code.

In the classical system, depending on the input state, the encoder can determine the coding length. Such a code is called a variable-length code. Using this type code, we can compress any information without error. When we suitably choose a variable-length code for the probability distribution $p$ of the input, the coding length is less than $nH(p)$, except for a small enough probability. In particular, Lynch [5] and Davisson [6] proposed a variable-length code with no error, in which the coding length is less than $nH(p)$ except

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for a small enough probability under the distribution $p$. Such a code is called a universal variable-length source code. Today, their code can be regarded as the following two-stage code: at the first step, we send the empirical distribution (i.e., the type) which indicates a subset of data, and in the second step, we send information which indicates every sequence belonging to the subset [7].

In this paper, in the quantum case, we treat codes in which the encoder determines the coding length, according to the input state. In order to make this decision, he must measure the input quantum system. After this measurement, depending on the data, the encoder compresses the final state of this measurement and sends its data and the compressed state. This type code is called a quantum variable-length source code. However, in general, the encoder knows only that the input state is written as a separable state $\rho_{x_1} \otimes \rho_{x_2} \otimes \cdots \otimes \rho_{x_n}$. Therefore, it is impossible to determine the coding length without destruction of the input state.

In particular, independently of the probability distribution $p$, we construct the code satisfying the following conditions: the average error concerning to Bures distance tends to 0. The probability that the coding length is greater than $nH(\rho_p)$, tends to 0. Such a code is called a quantum universal variable-length source code. In our construction, similarly to Keyl and Werner [8], an essential role is played by the representation theory of the special unitary group and the symmetric group on the tensored space. In our code, the encoder performs a quantum measurement closely related to irreducible decomposition of two groups, and resulting data can be approximately regarded as a quantum analogue of type. Thus, our code can be regarded as a quantum analogue of Lynch-Davisson code. Note that our criteria are different from Krattenthanler and Slater’s criteria [9] and Schumacher and Westmoreland’s criteria [10].

In this paper, we discuss the universality for the probability family $\mathcal{P}$ consisting of predicted probabilities on $S(\mathcal{H})$. For any probability family $\mathcal{P}$ on $S(\mathcal{H})$, we define universality of quantum variable-length source code and evaluate the exponents of the probability that the coding length is greater than the minimum admissible length. However, unfortunately, in our approach, it is difficult to construct such a code. In the visible coding case, it is possible to construct a quantum universal variable-length source code whose error exponentially tends to 0. This topic will be discussed in another paper.

We summarize quantum fixed-length source coding in section 2. After this summary, we state our mathematical setting and the main results in section 3. Our proofs and our construction of code are given in sections 4 and 5. Moreover, as is demonstrated in section 6, in the 2-dimensional case, a naive code destroys the state and is not used as a quantum universal variable-length source code.

## 2 Summary of quantum fixed-length source coding

Let $\mathcal{H}$ be a finite-dimensional Hilbert space that represents the physical system of interest and let $S(\mathcal{H})$ be the set of density operators on $\mathcal{H}$. Any quantum fixed-length code with the length $R_n$ on $\mathcal{H}^\otimes n$ is described by a triplet of a Hilbert space $\mathcal{H}_n$ with the dimension $e^{R_n}$, a trace-preserving completely positive (TP-CP) map $E^n$ from $S(\mathcal{H}^\otimes n)$ to $S(\mathcal{H}_n)$ and a TP-CP map $D^n$ from $S(\mathcal{H}_n)$ to $S(\mathcal{H}^\otimes n)$. When the input state $\rho := \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$ obeys the i.i.d. distribution $p^n$ of the probability $p$ on $S(\mathcal{H})$, which is regarded as a
probability on $S(\mathcal{H}^{\otimes n})$, the average error is given by

$$
\epsilon_{n,p}(\mathcal{H}_n, E^n, D^n) := \sum_{\tilde{\rho}_n \in S(\mathcal{H}^{\otimes n})} p^n(\tilde{\rho}_n) b^2(\tilde{\rho}_n, D^n \circ E^n(\tilde{\rho}_n)),
$$

where Bures distance is defined as

$$
b(\rho, \sigma) := \sqrt{1 - \text{Tr} \left| \sqrt{\rho} \sqrt{\sigma} \right|}.\]

Note that the support of $p$ does not necessarily consist of pure states. The minimum admissible rate $R_p$ of $p$ is defined by

$$
R_p := \inf \left\{ \limsup \frac{1}{n} \log \dim \mathcal{H}_n \mid \exists \{(\mathcal{H}_n, E^n, D^n)\}, \epsilon_{n,p}(E^n, D^n) \to 0 \right\}.
$$

When the source has no trivial redundancy in the sense following, it is calculated as

$$
R_p = H(\overline{p}_p) := -\text{Tr} \overline{\rho}_p \log \overline{\rho}_p,
$$

where $\overline{\rho}_p := \sum_{\rho \in S(\mathcal{H})} p(\rho) \rho$. The direct part was proven by Schumacher [1], and Jozsa and Schumacher [2], and the converse part was proven by Barnum et al. [11] in the pure state case. In the mixed case, Koashi and Imoto [3] discussed this problem as follows. We can consider the source to have trivial redundancy if the support of $p$ satisfies the following. The Hilbert space $\mathcal{H}$ is decomposed as

$$
\mathcal{H} = \bigoplus_l \mathcal{H}_{J,l} \otimes \mathcal{H}_{K,l},
$$

which satisfies the following two conditions: any element $\rho$ of the support of $p$ is commutative with $P_l$, where $P_l$ denotes the projection to the subspace $\mathcal{H}_{J,l} \otimes \mathcal{H}_{K,l}$. The state $\frac{\text{Tr}_{\mathcal{H}_{J,l}} P_l \rho P_l}{\text{Tr} P_l \rho}$ is independent of $\rho$. Precisely, we should state that these conditions hold almost everywhere for $p$. In this case, without loss of information, we can transform $\rho$ to $\sum_l \text{Tr}_{\mathcal{H}_{K,l}} P_l \rho P_l$. Koashi and Imoto proved the equation

$$
R_p = H \left( \sum_{\rho} p(\rho) \sum_l \text{Tr}_{\mathcal{H}_{K,l}} P_l \rho P_l \right).
$$

Following their proof, we can understand that

$$
R_p = \inf \left\{ \liminf \frac{1}{n} \log \dim \mathcal{H}_n \mid \exists \{(\mathcal{H}_n, E^n, D^n)\}, \epsilon_{n,p}(E^n, D^n) \to 0 \right\}. \quad (1)
$$

We can derive the optimal error exponent in the pure state case, which is treated in another paper [12].

3 Quantum universal variable-length source coding

In the variable-length case, we must treat a quantum measurement with state evolution, which is described by an instrument consisting of a decomposition $E = \{E_\omega\}_{\omega \in \Omega}$, by CP
maps from $S(H)$ to $S(H_{\omega})$ under the condition $\sum_{\omega \in \Omega} \text{Tr} E_\omega(\rho) = 1, \ \forall \rho \in S(H)$. When we perform the instrument $E = \{E_\omega\}_{\omega \in \Omega}$ for an initial state $\rho$, we get the data $\omega$ and the final state $\frac{E_\omega(\rho)}{\text{Tr} E_\omega(\rho)}$ with the probability $\text{Tr} E_\omega(\rho)$. Note that, in general, the Hilbert space $H_{\omega}$ depends on the data $\omega$. A pair of an instrument $E^n = \{E^n_\omega\}_{\omega \in \Omega_n}$ and a set of CP maps $D^n = \{D^n_\omega\}_{\omega \in \Omega_n}$, whose element is a TP-CP map from $S(H_{\omega_n})$ to $S(H^{\otimes n})$, is called a quantum variable-length source code on $H^{\otimes n}$.

The coding length is described by $\log |\Omega_n| + \log \dim H_{\omega_n}$, which is a random variable obeying the probability $P^n(\omega_n) := \text{Tr} E^n_\omega(\rho_n)$ when the input state is $\rho_n$. Of course, any quantum variable-length source code can be regarded as a quantum fixed-length source code whose length is the maximum of $\log |\Omega_n| + \log \dim H_{\omega_n}$. The error of decoding is evaluated by Bures distance as follows:

$$
\sum_{\omega_n \in \Omega_n} \text{Tr} E^n_\omega(\rho_n)b^2 \left( \rho_n, D_{\omega_n} \left( \frac{E^n_\omega(\rho_n)}{\text{Tr} E^n_\omega(\rho_n)} \right) \right).
$$

When the input state $\rho_n$ obeys the i.i.d. distribution $p^n$ of the probability $p$ on $S(H)$, the average error is given by

$$
\epsilon_{n,p}(E^n, D^n) := \sum_{\rho_n \in S(H^{\otimes n})} p^n(\rho_n) \sum_{\omega_n \in \Omega_n} \text{Tr} E^n_\omega(\rho_n)b^2 \left( \rho_n, D_{\omega_n} \left( \frac{E^n_\omega(\rho_n)}{\text{Tr} E^n_\omega(\rho_n)} \right) \right).
$$

In this case, the data $\omega_n$ obeys the probability $P^p(\omega_n) := \sum_{\rho_n \in S(H^{\otimes n})} p^n(\rho_n) \text{Tr} E^n_\omega(\rho_n) = \text{Tr} E^n_\omega(\overline{\rho}_p)$. A sequence $\{(E^n, D^n)\}$ of quantum variable-length source codes on $H^{\otimes n}$ is called universal for a probability family $P$ on $S(H)$ if

$$
\epsilon_{n,p}(E^n, D^n) \rightarrow 0
$$

for any probability $p \in P$.

**Theorem 1** If a sequence $\{(E^n, D^n)\}$ of quantum variable-length source codes on $H^{\otimes n}$ is universal for a family $P$, then

$$
\limsup -\frac{1}{n} \log P^p(E^n) \left\{ \frac{1}{n}(\log |\Omega_n| + \log \dim H_{\omega_n}) \geq R \right\} \leq \inf_{q \in P : H_q > R} D(\overline{\rho}_q || \overline{\rho}_p),
$$

(2)

where $D(\rho || \sigma)$ is quantum relative entropy $\text{Tr} \rho (\log \rho - \log \sigma)$.

**Theorem 2** For any family $P$, there exists a quantum variable-length source code $\{(E^n, D^n)\}$ on $H^{\otimes n}$ which satisfies the condition that $\epsilon_{n,p}(E^n, D^n)$ tends to 0 uniformly for $p \in P$ and that

$$
\lim -\frac{1}{n} \log P^p(E^n) \left\{ \frac{1}{n}(\log |\Omega_n| + \log \dim H_{\omega_n}) \geq R \right\} = \inf_{q \in P : H(\overline{\rho}_q) \geq R} D(\overline{\rho}_q || \overline{\rho}_p).
$$

(3)

4 Proof of Theorem 1

We prove Theorem 1 by contradiction. Let $p$ and $q$ be an arbitrary elements of $P$ and $R$ be arbitrary real number such that $R < R_q$. Assume that there exists a sequence
\((\{E^n, D^n\})\) of quantum variable-length source codes on \(\mathcal{H}^\otimes n\) satisfying
\[
\limsup_{n} \frac{-1}{n} \log D^n_{p^n} \left\{ \frac{1}{n} (\log |\Omega_n| + \log \dim \mathcal{H}_{\omega_n}) \geq R \right\} > D(\overline{p}_q || \overline{p}_p) \tag{4}
\]
\[
\lim \epsilon_{n,q}(E^n, D^n) = 0.
\]

Then, the equation
\[
\liminf_{n} \frac{1}{n} (\log |\Omega_n| + \log \dim \mathcal{H}_{\omega_n}) \geq R \tag{5}
\]
follows from the strong converse part of the quantum Stein’s lemma proven by Ogawa and Nagaoka [13]. Deforming the encoder \(E^n\), we define the encoder \(E^{R,n}\) as follows. When the data \(\omega_n\) satisfies
\[
\log |\Omega_n| + \log \dim \mathcal{H}_{\omega_n} \geq nR, \tag{6}
\]
we send classical information which indicates condition (6). Otherwise, we send the data \(\omega_n\) and the state \(E^n_{\omega_n}(\overline{p}_n)\). This code can be regarded as a fixed-length code whose length is less than \(nR\). From (5), we can check that the average error \(\epsilon_{n,p}(E^{R,n}, D^n)\) satisfies
\[
\liminf \epsilon_{n,q}(E^{R,n}, D^n) = 0.
\]

Note that equation (1) is equivalent to
\[
R_q = \inf \left\{ \limsup \frac{1}{n} \log \dim \mathcal{H}_n \left| \exists \{(\mathcal{H}_n, E^n, D^n)\}, \liminf \epsilon_{n,q}(E^n, D^n) = 0 \right\} \right. \tag{2}
\]
This implies that \(R \geq R_q\), which contradicts our assumption. Thus, the inequality
\[
\limsup_{n} \frac{-1}{n} \log D^n_{p^n} \left\{ \frac{1}{n} (\log |\Omega_n| + \log \dim \mathcal{H}_{\omega_n}) \geq R \right\} \leq D(\overline{p}_q || \overline{p}_p)
\]
holds for \(R < R_q\). Taking the infimum, we obtain (2).

\section{Proof of Theorem 2}
First, we construct a universal quantum variable-length source code that achieves the optimal rate (3) for the family of all probabilities on \(\mathcal{S}(\mathcal{H})\). This family is covariant for the actions of the \(d\)-dimensional special unitary group \(\text{SU}(d)\), and any \(n\)-i.i.d. distribution \(p^n\) is invariant for the action of the \(n\)-th symmetric group \(S_n\) on the tensored space \(\mathcal{H}^\otimes n\). Thus, our code should satisfy the invariance for these actions on \(\mathcal{H}^\otimes n\).

Now, we focus on the irreducible decomposition of the tensored space \(\mathcal{H}^\otimes n\) concerning the representations of \(S_n\) and \(\text{SU}(d)\), and define the Young index \(\mathbf{n}\) as,
\[
\mathbf{n} := (n_1, \ldots, n_d), \quad \sum_{i=1}^{d} n_i = n, n_i \geq n_{i+1},
\]
and denote the set of Young indices $n$ by $Y_n$. Young index $n$ uniquely corresponds to the unitary irreducible representation $V_n$ of $S_n$ and the unitary irreducible representation $U_n$ of $SU(d)$. The tensored space $\mathcal{H}^\otimes n$ is decomposed by

$$\mathcal{H}^\otimes n = \bigoplus_n W_n, \quad W_n := U_n \otimes V_n.$$ 

For details, see Weyl [14], Goodman and Wallch [15], and Iwahori [16].

For $\delta > 0$ we define a subset $Y_{\delta,n}$ of $\mathbb{Z}^d$ and an operator $M_{k,n,\delta}$ as

$$Y_{\delta,n} := \left\{ k \in \mathbb{Z}^d \left| \sum_{i=1}^d k_i = n, \exists n \in Y_n \cap U_{k,n\delta} \right. \right\}$$

$$M_{k,n,\delta} := \frac{1}{C_{1,d}(n\delta)} P_{k,n,\delta}$$

$$P_{k,n,\delta} := \sum_{n \in Y_n \cap U_{k,n\delta}} P_n$$

$$U_{p,\delta} := \{ q \in \mathbb{R}^d_+ \left| \| p - q \| \leq \delta \right. \} ,$$

where $P_n$ denotes the projection to $W_n$. Since

$$C_{1,d}(n\delta) = \# \left\{ k \in \mathbb{Z}^d \cap U_{n,n\delta} \left| \sum_{i=1}^d k_i = n \right. \right\}$$

for any $n \in Y_n$, the condition

$$\sum_{k \in Y_{\delta,n}} M_{k,n,\delta} = I$$

holds. The encoder $E_{k,n,\delta}$ is defined as an instrument, whose data set is $Y_{\delta,n}$, by

$$\mathcal{H}_{k,n,\delta} := \bigoplus_{n \in Y_n \cap \| n - k \| \leq \delta} W_n$$

$$E_{k,n,\delta}(\rho_n) := \sqrt{M_{k,n,\delta}} \rho_n \sqrt{M_{k,n,\delta}}, \quad \forall \rho_n \in \mathcal{S}(\mathcal{H}^\otimes n),$$

and the decoder $D_{k,n,\delta}$ is defined as the embedding from $\mathcal{H}_{k,n,\delta}$ to $\mathcal{H}^\otimes n$.

As is proven later, the quantum variable-length source code $(E_{k,n,\delta}, D_{k,n,\delta})$ on $\mathcal{H}^\otimes n$ satisfies

$$\epsilon_{n,p}(E_{k,n,\delta}, D_{k,n,\delta}) \leq 1 - \frac{C_{1,d}(n\delta_4)}{C_{1,d}(n\delta)} \left( 1 - (n + d + 1)^3 \exp \left( -nC_{2,d}(\delta - \delta_4) \right) \right)^{\frac{1}{2}}, \quad 0 < \delta_4 < \delta$$

(7)

$$\frac{-1}{n} \inf_{p_n} \left\{ \frac{1}{n} \left( \log |Y_{\delta,n}| + \log \dim \mathcal{H}_{k,n,\delta} \right) \geq R \right\}$$

$$\geq -\frac{5d}{n} \log(n + d) + \inf_{q \in \mathbb{R}^d_+: H(q) \geq R} \inf_{q' \in \mathbb{R}^d_+: \| q' - q \| \leq 2\delta} D(q' \| p_n(q)), \quad (8)$$

6
where

\[
C_{1,d}(x) := \# \left\{ \mathbf{k} \in \mathbb{Z}^d \left| \|\mathbf{k}\| \leq x, \sum_{i=1}^d k_i = 0 \right. \right\},
\]

\[
C_{2,d} := \min_{\mathbf{p} \in \mathbb{R}^d} \frac{D(\mathbf{q}\|\mathbf{p})}{\|\mathbf{p} - \mathbf{q}\|},
\]

\[
\mathbb{R}_+ := \{ x \in \mathbb{R} \mid x \geq 0 \},
\]

and \( p(\rho) \) denotes the probability \( (p_1, p_2, \ldots, p_d) \) in the case that \( p_i \) is eigenvectors of \( \rho \) and \( p_1 \geq p_2 \geq \ldots \geq p_d \). Note that the RHS of (7) is independent of \( p \). Letting \( \delta := n^{-1/3} \) and \( \delta_4 := n^{-1/3} - n^{-1/2} \), we can check that the RHS of (7) tends to 0, and that the RHS of (8) tends to the RHS of (3).

Next, deforming the code \( (\mathbf{E}^{\delta,n}, \mathbf{D}^{\delta,n}) \), we construct a universal quantum variable-length source code that achieves the optimal rate in the general case with no trivial redundancy. Define a sets \( \mathcal{S} \) and \( Y_{\delta,\delta_4,n}(\mathcal{S}) \) as

\[
\mathcal{S} := \{ \mathcal{T}_p \mid p \in \mathcal{P} \}
\]

\[
Y_{\delta,\delta_4,n}(\mathcal{S}) := \left\{ \mathbf{k} \in Y_{\delta,n} \mid \exists \rho \in \mathcal{S}, \left\| \mathbf{p}(\rho) - \frac{k}{n} \right\| \leq \delta_4 \right\}.
\]

When the data \( \mathbf{k} \) belongs to \( Y_{\delta,\delta_4,n}(\mathcal{S}) \), we send the state \( \mathbf{E}^{\delta,n}_{\mathbf{k}}(\rho_{\mathbf{k}}) \). Otherwise, we send only the classical information 0, except for \( Y_{\delta,\delta_4,n}(\mathcal{S}) \). Then, the encoder can be regarded as an instrument whose data set is \( Y_{\delta,\delta_4,n,+}(\mathcal{S}) := Y_{\delta,\delta_4,n}(\mathcal{S}) \cup \{0\} \). The decoder is defined as

\[
\mathbf{D}^{\delta,\delta_4,n}_{\mathbf{k}} := \mathbf{D}^{\delta,n}_{\mathbf{k}}, \quad \forall \mathbf{k} \in Y_{\delta,\delta_4,n}(\mathcal{S}).
\]

As is proven later, the quantum variable-length source code \( (\mathbf{E}^{\delta,\delta_4,n,S}, \mathbf{D}^{\delta,\delta_4,n,S}) \) on \( \mathcal{H}^\otimes n \) satisfies

\[
\epsilon_{n,p}(\mathbf{E}^{\delta,\delta_4,n,S}, \mathbf{D}^{\delta,\delta_4,n,S}) \leq 1 - \frac{C_{1,d}(n\delta_1)}{C_{1,d}(\delta)} \left( 1 - (n + d + 1)^{3d} \exp \left( -nC_{2,d}(\delta - \delta_4) \right) \right) \]

\[
\frac{-1}{n} \min_{\mathbf{q} \in \mathbb{R}^d : H(\mathbf{q}) \geq R - \frac{d}{2} \log(n + d) \exists \rho \in \mathcal{S}, \|\mathbf{q} - \mathbf{p}(\rho)\| \leq \delta_4 \mathbf{q}' \in \mathbb{R}^d : \|\mathbf{q}' - \mathbf{q}\| \leq 2\delta} D(\mathbf{q}'\|\mathbf{p}(\mathcal{T}_p)),
\]

for \( \forall \mathcal{T}_p \in \mathcal{P} \). Letting \( \delta := n^{-1/3} \) and \( \delta_4 := n^{-1/3} - n^{-1/2} \), we can show that the RHS of (10) tends to 0, and that the RHS of (11) tends to the RHS of (3).

### 6 Discussion

In our code, \( \delta \) does not equal 0. We can expect that the quantum variable-length source code \( (\mathbf{E}^{\delta,n}, \mathbf{D}^{0,n}) \) is universal. However, this code destroys the input state by a quantum measurement as follows.
Lemma 1 Assume that $d = 2$ and $\{|e_1\rangle, |e_2\rangle\}$ is a CONS of $\mathbb{C}^2$. If the support of $p$ is pure states $\{|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|\}$, the average error $\epsilon_{n,p}(E^{0,n}, D^{0,n})$ does not tends to 0.

As is understood from our proof of Theorem 2, bound (3) cannot be achieved unless $\delta$ tends to 0. It seems essential to approximate the nonzero number $\delta > 0$ to 0.

Next, we discuss how rapidly the average error $\epsilon_{n,p}$ tends to 0 in our code. Assume that $d = 2$ and $\{|e_1\rangle, |e_2\rangle\}$ is a CONS of $\mathbb{C}^2$. In this case, unless the delta satisfies $|\delta_n| \leq 1$, the coding length always equals $2n$. Then, we can assume that $|\delta_n| < 1$.

Lemma 2 If the support of $p$ is pure states $\{|e_1\rangle\langle e_1|, |e_2\rangle\langle e_2|\}$, the relation

$$
\lim_{n} \frac{-1}{n} \log \epsilon_{n,p}(E^{\delta_n,n}, D^{\delta_n,n}) = 0,
$$

holds for any sequence $\{\delta_n\}$ satisfying $|\delta_n| < 1$.

Therefore, it seems impossible to construct a universal code whose average error $\epsilon_{n,p}$ exponentially tends to 0.

7 Conclusion

We construct a quantum variable-length code satisfying equation (3). This is optimal in the sense of Theorem 1 when the family $P$ consists of probabilities on $\mathcal{S}(\mathcal{H})$ with no trivial redundancies. However, in our code the average error does not exponentially vanish. The construction of such a code seem to be difficult.

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Appendix

A Representation theoretical type method

For our proof, we need the following two lemmas.

Lemma 3 The relation

$$
\dim V_n \leq C(n)(n + d)^{2d} \leq (n + d)^{2d} \exp \left( n H \left( \frac{n}{n} \right) \right)
$$

holds, where $C(n)$ is defined as

$$
C(n) := \frac{n!}{n_1!n_2! \ldots n_d!}.
$$
Proof: According to Weyl [14], Iwahori [16], the following equation holds, and it is evaluated as
\[
\dim V_n = \frac{n!}{(n_1 + d - 1)! (n_2 + d - 2)! \ldots n_d!} \prod_{j > i} (n_i - n_j - i + j)
\]
\[
\leq \frac{n!}{n_1! n_2! \ldots n_d!} \prod_{j > i} (n_i - n_j - i + j) \leq C(n) (n + d)^{2d}
\]
\[
\leq (n + d)^{2d} \exp \left( n H \left( \frac{n}{n} \right) \right).
\]
The following is essentially equivalent to Keyl and Werner’s result [8]. For the reader’s convenience, we give a simpler proof.

Lemma 4 The following relations
\[
\sum_{\frac{n}{n} \notin U_{p, \delta}} \text{Tr} P_n \rho^\otimes n \leq (n + d)^{3d} \exp \left( -n D \left( \frac{n}{n} \| p \right) \right) \tag{14}
\]
\[
\sum_{\frac{n}{n} \notin U_{p, \delta}} \text{Tr} P_n \rho^\otimes n \leq (n + d)^{3d} \exp \left( -n \min_{q \notin U_{p, \delta}} D(q \| p) \right), \quad \forall \delta > 0 \tag{15}
\]
hold.

Proof: Let $U'_n$ be an irreducible representation of $SU(d)$ in $\mathcal{H}^\otimes n$, which is equivalent to $U_n$. We denote its projection by $P'_n$. We can calculate the operator norm by
\[
\| P'_n \rho^\otimes n P'_n \| = \prod_{i=1}^{d} p_i^{n_i}, \tag{16}
\]
where $\| X \| := \sup_{x \in \mathcal{T}} \| X x \|$. Since $\dim U'_n \leq (n + d + 1)^d$, from (13) and (16), the relations
\[
\text{Tr} P_n \rho^\otimes n \leq \dim V_n \times \text{Tr} P'_n \rho^\otimes n \leq (n + d + 1)^{3d} C(n) \prod_{i=1}^{d} p_i^{n_i} = (n + d + 1)^{3d} \text{Mul}(p, n)
\]
hold, where we denote the multinomial distribution of $p$ by $\text{Mul}(p, \bullet)$. Thus, we obtain (14). Sanov’s theorem (See p. 43 in [7]) guarantees that
\[
\sum_{\frac{n}{n} \notin U_{p, \delta}} \text{Tr} P_n \rho^\otimes n \leq (n + d + 1)^{3d} \exp \left( -n \inf_{q \notin U_{p, \delta}} \cap \mathbb{R}_+^d D(q \| p) \right).
\]

B Proof of (8) and (11)

First, we prove inequality (8). For a sufficiently large integer $n$, the relations
\[
|Y_{\delta,n}| \leq \# \{ k \in \mathbb{Z}^d | k_i \geq 0 \} \leq (n + 1)^d
\]
hold. Since \( \dim U_n \leq (n + d + 1)^d \), for any \( k \in Y_{\delta,n} \), we have
\[
\log |Y_{\delta,n}| + \log \dim \mathcal{H}_{k,n,\delta} \leq d \log (n + 1) + \max_{n \in Y_{\delta,n} \cap U_{k,n,\delta}} \log \dim U_n + \log \dim V_n
\]
\[
\leq 4d \log (n + d + 1) + \max_{n \in Y_{\delta,n} \cap U_{k,n,\delta}} nH \left( \frac{n}{n} \right).
\]
From (15), we have
\[
\text{Tr} M_{k,n,\delta}^{\otimes n} \leq \frac{|Y_{\delta,n}|}{C_{1,2}(n\delta)} (n + d)^{3d} \max_{n' \in Y_{\delta,n} \cap U_{k,n,\delta}} \exp \left( -nD \left( \frac{n'}{n} \right| p \right)
\]
\[
\leq (n + d)^{4d} \max_{n' \in Y_{\delta,n} \cap U_{k,n,\delta}} \exp \left( -nD \left( \frac{n'}{n} \right| p \right)
\]
\[
\leq (n + d)^{4d} \max_{q \in \mathbb{R}_+^{d+1}} \exp \left( -nD \left( q \right| p \right).
\]
Thus,
\[
P_{\mathcal{E}_n}^{E_{\delta,n}} \left\{ \frac{1}{n} \left( \log |Y_{\delta,n}| + \log \dim \mathcal{H}_{k,n,\delta} \right) \geq R \right\}
\]
\[
\leq \max_{k \in Y_{\delta,n}} \sum_{n \in Y_{\delta,n} \cap U_{k,n,\delta}} \text{Tr} M_{k,n,\delta}^{\otimes n}
\]
\[
\leq |Y_{\delta,n}| (n + d)^{4d} \max_{n' \in Y_{\delta,n} \cap U_{k,n,\delta}} \exp \left( -nD \left( \frac{n'}{n} \right| p \right)
\]
\[
\leq (n + d)^{5d} \max_{q \in \mathbb{R}_+^{d+1}} \exp \left( -nD \left( q \right| p \right).
\]
Then, we obtain (8).

Next, we proceed to (11). Since \( |Y_{\delta,\delta_4,n} + (S)| \leq |Y_{\delta,n}| \), we have
\[
P_{\mathcal{E}_n}^{E_{\delta,n}} \left\{ \frac{1}{n} \left( \log |Y_{\delta,\delta_4,n} + (S)| + \log \dim \mathcal{H}_{k,n,\delta} \right) \geq R \right\}
\]
\[
\leq \max_{k \in Y_{\delta,\delta_4,n} + (S)} \sum_{n \in Y_{\delta,\delta_4,n} \cap U_{k,n,\delta}} \text{Tr} \text{Tr} M_{k,n,\delta}^{\otimes n}
\]
\[
\leq |Y_{\delta,n}| (n + d)^{4d} \max_{n' \in Y_{\delta,n} \cap U_{k,n,\delta}} \exp \left( -nD \left( \frac{n'}{n} \right| p \right)
\]
\[
\leq (n + d)^{5d} \max_{q \in \mathbb{R}_+^{d+1}} \exp \left( -nD \left( q \right| p \right).
\]
Then, we obtain (11).

10
B.1 Proof of (7) and (10)

We can evaluate the average error as

$$
\epsilon_{n,p}(E^{\delta,n}, D^{\delta,n}) = \sum_{\vec{\rho}_n \in S(H^\otimes n)} p^n(\vec{\rho}_n) \sum_{k \in Y_{\delta,n}} \text{Tr} M_{k,n,\delta} \vec{\rho}_n \left( 1 - \text{Tr} \left( \sqrt{M_{k,n,\delta} \vec{\rho}_n \sqrt{M_{k,n,\delta} \vec{\rho}_n}} \right) \right)
$$

and

$$
= 1 - \sum_{\vec{\rho}_n \in S(H^\otimes n)} p^n(\vec{\rho}_n) \sum_{k \in Y_{\delta,n}} \sqrt{\text{Tr} M_{k,n,\delta} \vec{\rho}_n \text{Tr} \sqrt{M_{k,n,\delta} \vec{\rho}_n \sqrt{M_{k,n,\delta} \vec{\rho}_n}}} \vec{\rho}_n
$$

where inequality (17) follows from Jensen’s inequality concerning the convex function $x \mapsto x^{3/2}$.

The relations

$$
C_{1,d}(n\delta_4) = \#(Y_{\delta,n} \cap U_{np,n\delta_4}), \quad 0 < \delta_4 < \delta
$$

(19)

$$
P_{k,n,\delta} \geq \sum_{n \in Y_n \cap U_{np,n(\delta-\delta_4)}} P_n, \quad \forall k \in Y_{\delta,n} \cap U_{np,n\delta_4}
$$

(20)

hold. Using Lemma 4, and equations (20) and (9), we have

$$
\text{Tr} \, P_{k,n,\delta} \bar{P}_p^{\otimes n} \geq 1 - (n + d)^{3d} \exp \left( -n \min_{q \in C_{2,d}(\delta-\delta_4)} D(q\|p) \right)
$$

$$
\geq 1 - (n + d)^{3d} \exp \left( -nC_{2,d}(\delta - \delta_4)) \right).
$$

(21)

It follows from (19) and (21) that

$$
\sum_{k \in Y_{\delta,n}} \frac{1}{C_{1,d}(n\delta)} \left( \text{Tr} \, P_{k,n,\delta} \bar{P}_p^{\otimes n} \right)^{\frac{3}{2}} \geq \frac{1}{C_{1,d}(n\delta)} \left( \sum_{k \in Y_{\delta,n} \cap U_{np,n\delta_4}} \text{Tr} \, P_{k,n,\delta} \bar{P}_p^{\otimes n} \right)^{\frac{3}{2}} \geq \frac{C_{1,d}(n\delta_4)}{C_{1,d}(n\delta)} \left( 1 - (n + d + 1)^{3d} \exp \left( -nC_{2,d}(\delta - \delta_4)) \right) \right)^{\frac{3}{2}}.
$$

(22)

Inequality (7) follows from (18) and (22).
Similarly, we can prove that
\[ \epsilon_{n,p}(\mathbf{E}^{0,n}, \mathbf{D}^{0,n}) \leq 1 - \sum_{k \in \mathcal{Y}_{\delta,n}(S)} \frac{1}{C_{1,d}(n\delta)} \left( \text{Tr} \mathcal{P}_p^{\otimes n} P_{k,n,\delta} \right)^{\frac{3}{2}}. \]  

(23)

Since \( Y_{\delta,n,+}(S) \cap U_{n,p}(\mathcal{S}), n\delta_4 = Y_{\delta,n} \cap U_{n,p}(\mathcal{S}), n\delta_4 \), we can prove that
\[ \sum_{k \in \mathcal{Y}_{\delta,n,+}(S)} \frac{1}{C_{1,d}(n\delta)} \left( \text{Tr} \mathcal{P}_p^{\otimes n} P_{k,n,\delta} \right)^{\frac{3}{2}} \]
\[ \geq \frac{1}{C_{1,d}(n\delta)} \sum_{k \in \mathcal{Y}_{\delta,n} \cap U_{n,p}(\mathcal{S})} \left( \text{Tr} \mathcal{P}_p^{\otimes n} P_{k,n,\delta} \right)^{\frac{3}{2}} \]
\[ \geq \frac{C_{1,d}(n\delta_4)}{C_{1,d}(n\delta)} (1 - (n + d + 1)^{3d} \exp (-nC_{2,d}(\delta - \delta_4)))^{\frac{3}{2}}. \]  

(24)

Inequality (10) follows from (23) and (24).

C Proof of Lemma 1

In this case, the average error is calculated as
\[ \epsilon_{n,p}(\mathbf{E}^{0,n}, \mathbf{D}^{0,n}) = 1 - \sum_{n \in \mathcal{Y}_n} \sum_{\bar{e}_n} p(\bar{e}_n) \left( \langle \bar{e}_n | P_n | \bar{e}_n \rangle \right)^{\frac{3}{8}} \]
\[ = 1 - \sum_{\bar{e}_n} p(\bar{e}_n) \sum_{n \in \mathcal{Y}_n} \left( \langle \bar{e}_n | P_n | \bar{e}_n \rangle \right)^{\frac{3}{8}}, \]

where \( \bar{e}_n := e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \in \mathcal{H}^{0,n} \). We define \( \mathbf{n}(\bar{e}_n) := (n_1(\bar{e}_n), n_2(\bar{e}_n)) \) by
\[ n_i(\bar{e}_n) := \# \{ j \in [1, n] | e_{i,j} = e_i \}. \]  

(25)

Now, we focus on a typical element \( \bar{e}_n \), i.e. \( \frac{n(\bar{e}_n)}{n} \approx p_i \). The number satisfying (25) is \( n(\bar{e}_n)(n_2(\bar{e}_n)) \), and \( \dim \mathcal{V}_{n'} = (n)(n_2(\bar{e}_n) - n(n_2(\bar{e}_n) - 1) \), where \( \mathbf{n}(\bar{e}_n) = (n_1(\bar{e}_n), n_2(\bar{e}_n)) \in Y_n \). Then,
\[ \langle \bar{e}_n | P_{\mathbf{n}(\bar{e}_n)} | \bar{e}_n \rangle = n(n_2(\bar{e}_n))^{-1} (n(n_2(\bar{e}_n) - n(n_2(\bar{e}_n) - 1) \]
\[ = 1 - \frac{n_2(\bar{e}_n)}{n_1(\bar{e}_n) + 1} = \frac{n_1(\bar{e}_n) + 1 - n_2(\bar{e}_n)}{n_1(\bar{e}_n) + 1} \approx \frac{p_1 - p_2}{p_1}. \]

Since \( x^{\frac{3}{8}} + y^{\frac{3}{8}} \leq (x + y)^{\frac{3}{8}} \) for \( 0 < x, y < 1 \), we can evaluate
\[ \sum_{n \in \mathcal{Y}_n} \left( \langle \bar{e}_n | P_n | \bar{e}_n \rangle \right)^{\frac{3}{8}} \leq \left( \sum_{n \in \mathcal{Y}_n \setminus \{n'\}} \langle \bar{e}_n | P_n | \bar{e}_n \rangle \right)^{\frac{3}{8}} + \left( \langle \bar{e}_n | P_{n'} | \bar{e}_n \rangle \right)^{\frac{3}{8}} \]
\[ \leq \left( 1 - \frac{p_1 - p_2}{p_1} \right)^{\frac{3}{8}} + \left( \frac{p_1 - p_2}{p_1} \right)^{\frac{3}{8}} \]
\[ = \left( \frac{p_2}{p_1} \right)^{\frac{3}{8}} + \left( \frac{p_1 - p_2}{p_1} \right)^{\frac{3}{8}} < 1. \]
Therefore,
\[
\lim \epsilon_{n,p}(E^{0,n}, D^{0,n}) \geq 1 - \left( \left( \frac{p_2}{p_1} \right)^{\frac{3}{2}} + \left( \frac{p_1 - p_2}{p_1} \right)^{\frac{3}{2}} \right) > 0.
\]

**D Proof of Lemma 2**

For any \( n \in Y_n, \delta_n > 0 \), we denote \( ([n_1 - \frac{1}{\sqrt{2}}\delta_n], n - [n_1 - \frac{1}{\sqrt{2}}\delta_n]) \in Y_{\delta,n} \) as \( k(n, \delta_n) \), where \([x]\) is defined as the maximum integer \( n \) satisfying \( n \leq x \). The element \( k(n, \delta_n) \) satisfies
\[
n = (n_1, n_2) \in U_{k(n, \delta_n), \delta_n}
(n_1 + 1, n_2 - 1) \notin U_{k(n, \delta_n), \delta_n}.
\]
Therefore
\[
\epsilon_{n,p}(E^{\delta,n}, D^{\delta,n})
= \sum_{\tilde{e}_n} p^n(\tilde{e}_n) \left( 1 - \sum_{k \in Y_{\delta,n}} \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n)} - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n), \delta_n})) \right)
\geq \sum_{\tilde{e}_n} p^n(\tilde{e}_n) \left( 1 - \sum_{k \in Y_{\delta,n}} \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n) - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n), \delta_n})) \right)
\geq \sum_{\tilde{e}_n} p^n(\tilde{e}_n) \left( 1 - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n) - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n), \delta_n})) \right)
\geq \sum_{\tilde{e}_n: \tilde{e}_n \geq p_i} p^n(\tilde{e}_n) \left( 1 - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n) - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n), \delta_n})) \right)
\geq \sum_{\tilde{e}_n: \tilde{e}_n \geq p_i} p^n(\tilde{e}_n) \left( 1 - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n) - \frac{1}{C_{1,d}(n\delta)}(\text{Tr} P_{k(n, \delta_n), \delta_n})) \right)
\leq \frac{1}{C_{1,d}(n\delta)} \left( \frac{p_1 - p_2}{p_1} - \left( \frac{p_1 - p_2}{p_1} \right)^{\frac{3}{2}} \right).
\]

However,
\[
-1 n \log \frac{1}{C_{1,d}(n\delta)} \leq -1 n \log \frac{1}{2Y_n} \leq \frac{1}{n} \log 2(n+1)^2 \to 0.
\]
Therefore, we obtain (12).

**References**


[16] N. Iwahori, Taishougun to Ippansenkeigun no Hyougenron, (Iwanami, Tokyo, 1978).(In Japanese)