The Casimir energy of a dilute homogeneous nonmagnetic dielectric ball at zero temperature is derived analytically within a \textit{microscopic} realistic model of dielectrics for an arbitrary physically possible frequency dispersion of dielectric permittivity $\varepsilon(i\omega)$. Divergences are absent in calculations, a minimum interatomic distance $\lambda$ is a \textit{physical} cut-off. Casimir surface force is proved to be attractive. A physical definition of the Casimir energy is discussed.

1. Microscopic Approach

Consider a dielectric nonmagnetic ball of the radius $a$ and permittivity $\varepsilon$ at zero temperature, surrounded by a vacuum. The ball is dilute, i.e. all final expressions are obtained under the assumption $\varepsilon - 1 \ll 1$ in the order $(\varepsilon(i\omega) - 1)^2$, the lowest order that yields the energy of interaction between atoms of the ball.

The Casimir energy of a disjoint macroscopic system (two dispersive dielectric parallel plates is a classic example by Lifshitz) depends only on the distance between macroscopic bodies and dispersion of dielectrics\textsuperscript{1}. On the other hand, it was first argued in Refs.\textsuperscript{2,3} that for a dilute connected dielectric the Casimir energy at zero temperature is equal to the energy of dipole-dipole pairwise interactions of all atoms constituting the dielectric and thus should also depend on an average minimum distance between atoms of a dielectric $\lambda$. For a dilute dispersive dielectric ball with an arbitrary frequency dependent dielectric permittivity the Casimir energy at zero temperature was first derived in Ref.\textsuperscript{4}.

A dipole-dipole interaction of two neutral atoms with atomic polarizabilities $\alpha_1(i\omega)$ and $\alpha_2(i\omega)$ is described by a potential \textsuperscript{5}

$$U(r) = -\frac{1}{\pi r^2} \int_0^{+\infty} \omega^4 \alpha_1(i\omega) \alpha_2(i\omega) e^{-2\omega r} \left[ 1 + \frac{2}{\omega r} + \frac{5}{(\omega r)^2} + \frac{6}{(\omega r)^3} + \frac{3}{(\omega r)^4} \right] d\omega,$$

(1)

where $r$ is a distance between two atoms. The energy calculation for the ball is illustrated by Fig.1. Suppose that an atom with an atomic polarizability $\alpha(i\omega)$ is located at the point $B$. One has to integrate interaction of an atom at the point $B$ via a potential (1) with the atoms separated by distances greater than interatomic distances $\lambda$ from the point $B$, integrate over all atom locations $B$ inside the ball and multiply by a factor $1/2$ to calculate the energy.

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Assuming homogeneity of the ball (it results in $\alpha_1(i\omega) = \alpha_2(i\omega) = \alpha(i\omega)$ and the condition that the number density of atoms $\rho$ doesn’t depend on the point inside the ball), the Casimir energy is equal to

$$E = \frac{\rho^2}{2} \left( \int_{0}^{a-\lambda} dp \frac{4\pi p^2}{4\pi} \int_{\lambda}^{a-p} dr \, 4\pi r^2 U(r) + \int_{0}^{a-\lambda} dp \frac{4\pi p^2}{4\pi} \int_{a-p}^{a+p} dr \, 2\pi r^2 \left( 1 - \frac{r}{2p} - \frac{p^2 - a^2}{2pr} \right) U(r) \right)$$

Performing calculations, it is possible to derive an analytic result:

$$E = -\rho^2 \frac{\pi}{48} \int_{0}^{+\infty} d\omega \alpha^2(i\omega) \left( \frac{\alpha^3}{\lambda^3} e^{-2\omega\lambda}(128 + 256\omega\lambda + 128\omega^2\lambda^2 + 64\omega^3\lambda^3) - \frac{\alpha^2}{\lambda^2} (e^{-2\omega\lambda}(144 + 288\omega\lambda + 120\omega^2\lambda^2 + 48\omega^3\lambda^3) - 96\omega^2\lambda^2 E_1(2\omega\lambda)) + (e^{-2\omega\lambda}(41 + 34\omega\lambda + 14\omega^2\lambda^2 + 4\omega^3\lambda^3) + 24 E_1(2\omega\lambda)) + (e^{-4\omega a}(-21 + 12\omega a) - E_1(4\omega a)(24 + 96\omega^2 a^2)) \right),$$

where $E_1(x) = \int_{1}^{+\infty} e^{-tx} / t \, dt$.

Eq.(3) finally solves the problem of the Casimir energy for a dilute dielectric ball at zero temperature. This energy is finite and physical only when a finite minimum separation between atoms $\lambda$ is taken into account.
Some words need to be said about the change of a viewpoint on the physical definition of the Casimir energy. When an interatomic distance $\lambda$ is taken into account, the Casimir energy can be calculated without divergences by use of Eq.(2), and so calculated energy doesn’t require further renormalization. No further renormalization is needed since it is obvious from the method of calculations that the Casimir energy (3) coincides with a potential (binding) energy of the ball when $\lambda$ is large enough, so that effects of short-range interatomic repulsive forces and other interactions can be neglected. In this case there is no classical part of the energy in the beginning of calculations, there exists only a set of atoms with specified interactions between them due to quantum fluctuations. The sum of all these interactions between atoms should give the Casimir contribution to the binding energy of a dielectric ball – a macroscopic classical system.

The leading contribution from the last line in (3) is equal to

$$\rho^2 a^2(0) \frac{\pi}{48} \int_0^{+\infty} d\omega \left( e^{-4\omega a} (-21 + 12\omega a) - E_1(4\omega a)(24 + 96\omega^2 a^2) \right)$$

$$= \rho^2 a^2(0) \frac{23\pi}{96a} = \frac{23}{1536\pi a} (\varepsilon - 1)^2 = E_{ld}. \quad (4)$$

The term (4) can be called a contribution from large distances to the Casimir energy of the ball. Usually all terms different from (4) were simply discarded or added up to macroscopic quantities of the ball (e.g., volume, surface energies) during the renormalization procedure, as it often happens in field theory. So only the term (4) was usually considered as the Casimir energy term. In fact, this was an erroneous definition of the Casimir energy since no physical quantities can be calculated by use of Eq.(4). All physical quantities that can be measurable at least in principle for a dilute ball (e.g., surface force or trajectories of the particles near the ball surface during the ball collapse in the adiabatic approximation) have to be calculated by use of a total potential energy of the ball (3). The term (4) has little influence on physics. The reason is quite simple - the term (4) is much less in magnitude than the terms in the first, second and third lines of (3). For example, ratio of terms in the first line of (3) to (4) is $\sim (\omega_0 a)(a/\lambda)^3 \gg 1$, where $\omega_0$ is a characteristic atomic absorption frequency. However, the term (4) was really important for development of the theory of Casimir effect in connected dielectrics since this term has been derived via microscopic $^6$ and macroscopic $^7$ techniques - so the equivalence of large distance parts of the Casimir energy for a dilute dielectric ball derived by microscopic and macroscopic approaches was proved.

It is important to stress that so far macroscopic methods did not yield satisfactorily short distance contributions to the Casimir energy of connected dielectrics$^8$. The reason is simple: these methods were developed for disjoint, not connected dielectrics.

From (2) it follows that Casimir surface force on a dilute dielectric ball is attractive. It is convenient to define $N \equiv a/\lambda, p \equiv \omega \lambda$. Then Eq.(3) can be rewritten
in a general form

\[ E = - \frac{\rho^2}{\lambda} \int_0^{+\infty} dp \alpha^2 \left( i \frac{p}{\lambda} \right) f(N,p). \]  

(5)

The function \( f(N,p) > 0 \) for \( N > 1/2, p > 0 \). Conservation of atoms inside the ball and homogeneity impose the condition

\[ \rho \frac{4\pi a^3}{3} = \text{const.} \]  

(6)

From the condition of an atomic conservation (6) it follows that during the ball collapse or expansion

\[ N = \text{const.} \]  

(7)

It is convenient to use Kramers–Kronig relations in the form

\[ \alpha(i\omega) = \int_0^{+\infty} dx \frac{xg(x)}{x^2 + \omega^2}, \]  

(8)

where the condition \( g(x) > 0 \) always holds. Using (5), (6), (7), (8), Casimir force on a unit surface is equal to

\[ F = - \frac{1}{4\pi a^2} \frac{\partial E}{\partial a} = - \rho^2 \frac{4\pi a^3}{3} \int_0^{+\infty} d\omega \int_0^{+\infty} dx \frac{x(7x^2 + 3\omega^2)g(x)}{(x^2 + \omega^2)^2} \alpha(i\omega)f(N,\omega\lambda) < 0. \]  

(9)

\( F < 0 \) because all functions inside integrals are positive. Casimir surface force is attractive for every model of atomic polarizability consistent with general causal requirements.

References