An Improved Quantum Scheduling Algorithm

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Abstract
The scheduling problem consists of finding a common 1 in two remotely located \( N \) bit strings. Denote the number of 1s in the string with the fewer 1s by \( \epsilon N \). Classically, it needs at least \( O(\epsilon N \log_2 N) \) bits of communication to find the common 1. The best known quantum algorithm would require \( O(\sqrt{N \log_2 N}) \) qubits of communication. This paper gives a quantum algorithm to find the common 1 with only \( O(\sqrt{\epsilon N \log_2 N}) \) qubits of communication.

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1 Introduction

Alice & Bob each have a calendar of their appointments for $N$ slots of time. They need to find a common slot for a meeting when each of them is available. How many bits of information do they need to exchange? This problem is also known as the intersection problem, i.e. the problem is to find a common 1 in two remotely located strings.

It has been shown that if Alice & Bob were exchanging classical bits, they would need to exchange $O(N)$ bits of information [1] (the intuition is that if they were to try to find the answer while exchanging fewer bits, they might leave out information about the common slot).

It was a surprising result when Buhrman, Cleve & Wigderson (BCW) [2] discovered a quantum mechanical technique through which Alice & Bob could identify the common slot while exchanging only $O(\sqrt{N \log N})$ qubits. This was surprising because there is a well known result called Holevo’s Theorem which proves that a single qubit cannot carry more than one bit of classical information [3]. It takes some thought to realize that the BCW result does not violate Holevo’s Theorem since Alice & Bob could not use it to transmit $N$ bits of information. What the BCW algorithm accomplishes is to reduce the communication complexity of the scheduling problem from $O(N)$ classical bits to only $O(\sqrt{N \log N})$ quantum mechanical qubits. This was the first significant reduction in communication complexity achieved through quantum communication.

The basis of the algorithm of BCW was to carry out a quantum search on the $N$ possible slots for the common slot. Since Alice & Bob each have part of the information, this requires a distributed search with the qubits being transferred back and forth as described in section 3. This requires $O(\sqrt{N \log N})$ qubits of communication which is considerably fewer than the $O(N)$ qubits of classical communication that would be required.

As described above, the number of qubits of communication required is approximately the square-root of the number of bits of classical communication required in the most general case. However, in the special case when the string of either Alice or Bob has few 1s, it is possible to design a classical algorithm requiring much less communication. For example, if Alice’s string has $\epsilon N$ 1s, she could encode the position of these using $\epsilon N \log_2 N$ bits and send these to Bob who could then find the common slot. This paper gives a modification to the BCW algorithm so that it gives a square-root advantage over the best classical algorithm even when the strings of Alice or Bob have few 1s.
2 The Quantum Search Algorithm

In order to describe the algorithm of BCW, we first need to describe the quantum search algorithm [4]. There are only three operations required by the search algorithm: \( W, I_0, I_t \). These are described below.

\( I_t \) is a selective inversion of the target state. This can be achieved provided we have a quantum mechanical black box that can evaluate whether or not a given state is the target state - note that this does not need any a priori knowledge of which the target state is. See ?? for a quantum circuit that accomplishes a selective phase inversion of \( t \) using such a black box.

\( I_0 \) is the selective inversion of the \( 0 \) state (i.e. the state in which all qubits are 0). \( W \) is the Walsh-Hadamard Transformation.

The quantum search algorithm showed that, if we start from \( \overline{t} \) and carry out \( \pi \sqrt{N} \) repetitions of the sequence of operations \( I_0 W I_t W \), followed by \( W \), we reach the \( t \) state with certainty. Equivalently:

\[
W(I_0 W I_t W)\ldots(I_0 W I_t W)(I_0 W I_t W)(I_0 W I_t W)|0\rangle = |t\rangle
\]

A measurement after this will reveal which the \( t \) state is. Note that this requires only \( O(\sqrt{N}) \) operations.

3 Distributed Searching

In the scheduling problem, part of the information is with Alice & part of it with Bob. The insight of BCW was to observe that the quantum search algorithm could still be carried out by transferring the qubits back & forth. There are only three operations required by the search algorithm: \( W, I_0, I_t \). The first two are independent of the \( t \) state and can be carried out anywhere (\( I_0 \) requires all qubits to be available together).

\( I_t \) clearly depends on the solution the information about which is with Alice & Bob. It requires that both Alice’s and Bob’s schedules be satisfied. A modification of the circuit shown in the figure in the previous section accomplishes this. It requires part of the function to be evaluated by Alice & the resulting qubits to be passed to Bob who evaluates the rest of the function. This ensures that both Alice’s & Bob’s functions evaluate to 1. Please see [2] for details.
4 Amplitude Amplification

A few years after the invention of the quantum search algorithm, it was generalized to a much larger class of applications known as the amplitude amplification algorithms [5] (similar results are independently proved in [6]). In these algorithms, the amplitude produced in a particular state by a unitary operation $U$, can be amplified by successively repeating the sequence of operations: $Q = I_s U^\dagger I_t U$. It was proved that if we start from the $s$ state and repeat the operation sequence $I_s U^\dagger I_t U$, $\eta$ times followed by a single repetition of $U$, then the amplitude in the $t$ state becomes approximately $\eta U_{ts}$ (provided $\eta U_{ts} \ll 1$). Also, if we start from $s$ and carry out $\pi 4 |U_{ts}|$ repetitions of $Q$ followed by a single repetition of $U$, we reach $t$ with certainty.

The quantum search algorithm was a particular case of amplitude amplification with the Walsh-Hadamard Transformation being the $U$ operation and $s$ being the $\overline{0}$ state. For any $t$, $|U_{ts}| = 1/\sqrt{N}$. It follows from the amplitude amplification principle that if we start from $\overline{t}$ and carry out $\pi \sqrt{N} 4$ repetitions of the sequence of operations $I_{\overline{t}} W I_t W$, followed by $W$, we reach the $t$ state with certainty.

In this paper we use the amplitude amplification principle for developing a new scheduling algorithm. This is achieved by designing a sequence of transformations that produce an amplitude of $1 \sqrt{\epsilon N}$ in the $t$ state and much smaller amplitudes in other states while requiring $\log_2 N$ qubits of communication. Therefore by the amplitude amplification principle, in $\pi 4 \sqrt{\epsilon N}$ repetitions of this transformation, we can concentrate most of the amplitude in the $t$ state.

5 Improved Scheduling Algorithm

The following algorithm assumes a single common 1 in the two strings. The algorithm is easily extended to the situation with multiple common 1s using standard approaches, e.g. [8]. Alice & Bob each count the number of 1’s in their strings (or equivalently the number of slots they are each available for). They exchange this information using $\log_2 N$ bits of communication. The person with the fewer 1s (say Alice) starts.

Assume that Alice has $\epsilon N$ 1’s in her $N$ bit string. She starts with a register of $\log_2 N$ qubits that encode the $N$ slots. She starts with these in the $\overline{0}$ state. Next consider the transformation consisting of $\pi 4 \sqrt{\epsilon N}$ applications of the quantum search operator $I_{\overline{t}} W I_t W$, followed by $W$ (here $I_A$ inverts the amplitude of each of the states when Alice is available). This produces a superposition
concentrated in states in which she is available. Denote this composite transformation that transforms \( |\rangle \) into the superposition corresponding to her available slots, by \( U \), i.e.

\[
U \equiv W(I_{\pi}WI_AW) \cdots (I_{\pi}WI_AW)(I_{\pi}WI_AW)(I_{\pi}WI_AW)
\]

\[
\pi \sqrt{\epsilon N}\text{repetitions}
\]

Note that \( U^\dagger \) consists of the application of the adjoints of the operations that constitute \( U \) but in the opposite order, i.e.

\[
U^\dagger \equiv W(I_AWI_{\pi}W) \cdots (I_AWI_{\pi}W)(I_AWI_{\pi}W)(I_AWI_{\pi}W)
\]

\[
\pi \sqrt{\epsilon N}\text{repetitions}
\]

Next apply the following sequence of transformations:

\[
U \left( I_{\pi}U^\dagger I_BU \right) \cdots \left( I_{\pi}U^\dagger I_BU \right) \left( I_{\pi}U^\dagger I_BU \right) |0\rangle
\]

\[
\pi \sqrt{\epsilon N}\text{repetitions}
\]

Note that this needs only \( \pi \sqrt{\epsilon N}2\log_2 N \) qubits of communication since Alice can carry out all but the \( I_B \) operations for which the register needs to be sent to Bob and returned. The number of times the register needs to be sent to Bob is equal to the number of \( I_B \) operations.

It follows by the amplitude amplification principle that if we start from \( |0\rangle \), then after \( \pi 4 \sum_B |U_{B\pi}\|^2 \) repetitions of the \( I_BU_I_{\pi}U^\dagger \) followed by a single application of \( U \), all the amplitude is concentrated in the states inverted by the \( I_B \) operation (i.e. the set of states that satisfy Bob’s schedule) with amplitude in each \( B \) state proportional to \( U_{B\pi} \). The amplitude \( U_{B\pi} \), is 0 except in the states that satisfy Alice’s schedule. Therefore the superposition is entirely concentrated in states that satisfy both Alice’s & Bob’s schedule. For convenience we have assumed a single common slot, the method of [8] easily generalizes it to multiple common slots.

6 Other Tradeoffs & Applications

The algorithm of this paper requires \( O(\sqrt{\epsilon N}) \) cycles, each cycle requires \( O\left( \sqrt{N\epsilon N} \right) \) steps of computation (assuming the busier person has \( \epsilon N \) openings and each
selective inversion takes $O(1)$ steps - which gives $O(\sqrt{N})$ total steps of computation while requiring only $O(\sqrt{\epsilon N})$ communication (ignoring log factors). This is in comparison to the BCW [2] algorithm that requires $O(\sqrt{N})$ steps of computation and $O(\sqrt{N})$ communication. As described above, this gives our algorithm an advantage when the communication cost is significant. It is interesting to speculate about the situation when Alice’s & Bob’s selective inversions take different numbers of steps - it might be possible to design algorithms that are more efficient even in terms of computation, or in terms of some combination of computation and communication.

Alternatively, one can extend this method to solve problems where the problem is easiest posed as an intersection of two different queries, one of which is more selective than the other.

References


