On the $SO(2,1)$ symmetry in General Relativity

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Abstract

The role of the $SO(2,1)$ symmetry in General Relativity is analyzed. Cosmological solutions of Einstein field equations invariant with respect to a space-like Lie algebra $\mathcal{G}_r$, with $3 \leq r \leq 6$ and containing $so(2,1)$ as a subalgebra, are also classified.

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Gravitational models are usually classified in terms of their group of isometries, namely the number of Killing vectors, the values of the structure constants and the transitivity regions, i.e. the regions on which the isometry group acts transitively. The classification of all non-isomorphic isometry groups is a well known and solved issue in the context of Group Theory; for instance, there exist 9 non-isomorphic 3 -dimensional groups, $G_3$, which yield the so-called Bianchi models.

A simpler, but important, example is provided by the $G_2$ groups. In this case the corresponding Lie algebra $\mathcal{G}_2$ is described in terms of two Killing vectors $X, Y$ satisfying the commutation relation $[X,Y] = sY$ with $s = 0, 1$, this corresponding respectively to the Abelian and non-Abelian case.

Interesting gravitational fields are represented by metrics possessing a higher degree of symmetry (for an exhaustive review on the various classifications of such models available in the literature see for example [11]). Among them particular attention have attracted homogeneous or hypersurface–homogeneous models like the Friedmann-Robertson-Walker metric (FRW) which is $G_6$-symmetric,
the Gödel metric which is $G_5$-symmetric or the Kantowski-Sachs metric, $G_4$-symmetric.

To better understand the role of the symmetries in the classification let us give a few definitions.

The Lie algebra of all Killing vector fields of a given metric $g$ will be denoted by $\mathfrak{Kil}(g)$, while Killing algebra will denote any subalgebra $\mathcal{G}$ of $\mathfrak{Kil}(g)$. The group corresponding to a subalgebra $\mathcal{G}$ of $\mathfrak{Kil}(g)$ is called group of motions or group of isometries and denoted by $G_r$ where $r$, the order of the group, is the number of generators. If $\mathcal{G} = \mathfrak{Kil}(g)$, then the corresponding group is called complete group of motions.

A metric manifold is said to be homogeneous if its group of motions, $G_r$, acts transitively on it, that is the whole manifold is an orbit of $G_r$.

A metric manifold is said to be maximally symmetric if it has the maximum number of Killing vector fields, i.e. if it admits a complete group $G_{(n(n+1)/2)}$ of motions, where $n$ denotes the dimension of the manifold. Of course, a maximally symmetric manifold is homogeneous.

The isotropy or stability group of a point $p$ is a subgroup $H_p$ of $G_r$ leaving $p$ fixed. The manifold is said to be isotropic about $p$ if its isotropy group is $n$-dimensional. It can be easily shown that if the manifold is spacelike, then $H_p = SO(n)$.

A manifold isotropic about every point $p$ is said to be isotropic.

It is easy to see that:

- A maximally symmetric metric manifold has constant curvature. The converse is also true.

- A metric manifold of constant curvature is isotropic. The converse is also true.

Thus, a maximally symmetric manifold is isotropic and has a constant curvature.

A cosmological model or shortly a cosmology is a Lorentzian 4-dimensional differential manifold foliated by 3-dimensional submanifolds $S$ on which the restriction $g |_S$ of the metric $g$ is positive definite. These submanifolds $S$ will be also called leaves. A cosmology is said to be homogeneous and/or isotropic if the leaves $S$ are homogeneous and/or isotropic.

The present letter moves from a previous series of articles [12] where vacuum gravitational fields, invariant for a non-Abelian 2-dimensional Killing algebra $\mathcal{G}_2$, are exhaustively classified as described below.

If $g$ is a metric on the space-time and $\mathcal{G}_2 = span\{X, Y\}$ one of its Killing algebras

$$X, Y \in \mathcal{G}_2 \quad [X, Y] = sY, \quad s = 0, 1,$$  \hspace{1cm} (1)

then the integrable involutive distribution $\mathcal{D}$ generated by $X$ and $Y$ is $2$-dimensional$^{*}$.

$^{*}$The fields $X$ and $Y$, leaving invariant a metric, i.e. a symmetric not degenerate $(0, 2)$ tensor field, cannot be parallel.
Moreover, if the orthogonal distribution is integrable and transversal \(^\dagger\), then all the \(G_2\) invariant solutions of the vacuum Einstein field equations are characterized either in terms of solutions of an algebraic equation (the tortoise equation) or in terms of solutions of a linear partial differential equation in the plane, depending on whether \(g(Y,Y) \neq 0\) or \(g(Y,Y) = 0\), respectively. It was also shown that in the case \(g(Y,Y) \neq 0\) there exists a third Killing field such that \(\mathcal{Kil}(g)\), the complete Killing algebra, is isomorphic to \(so(2,1)\) and the Killing leaves, i.e., the transitivity regions, are 2-dimensional Riemann surfaces of constant curvature. Moreover, all the invariant vacuum gravitational fields with Lorentzian signature, are locally diffeomorphic to

\[
g = \frac{r - A}{r} dr^2 - \frac{r}{r - A} d\tau^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) \quad (2)
\]

where \(r \in [0, \infty[, \vartheta \in R, \varphi \in [0, 2\pi]\) are pseudo-spherical coordinates and \(A\) an arbitrary constant. This solution is static and \(so(2,1)\) invariant; it will be called the pseudo-Schwarzschild metric because of its resemblance to the Schwarzschild one which is static and \(so(3)\) invariant.

These results suggest to clarify the role of the \(SO(2,1)\) symmetry in General Relativity and in particular to analyze the metric

\[
g = dt^2 - a^2(t) \left[ -\frac{dr^2}{kr^2 + 1} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) \right] \quad (3)
\]

which is similar in form to the FRW solution and invariant with respect to the \(so(2,1)\) Lie algebra

\[
[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2,
\]

which, in the pseudo-spherical coordinates, is spanned by

\[
\begin{align*}
X_1 &= \sin \varphi \partial_\vartheta + \cos \varphi \coth \vartheta \partial_\varphi \\
X_2 &= -\cos \varphi \partial_\vartheta + \sin \varphi \coth \vartheta \partial_\varphi \\
X_3 &= \partial_\varphi.
\end{align*}
\]

For \(-kr^2 > 1\) the metric has Lorentzian signature and the 2-dimensional surfaces defined by \((r, t = \text{const})\) may be identified with one of the sheets of the two-sheeted space-like hyperboloid. They are also known as pseudo-spheres.

The pseudo-sphere \([1]\) is a surface with constant negative Gaussian curvature \(\mathcal{R} = -1/r^2\). It can be globally embedded in a 3-dimensional Minkowskian space. Let \(y_1, y_2, y_3\) denote the coordinates in the Minkowskian space, where the separation from the origin is given by \(y^2 = -y_1^2 + y_2^2 + y_3^2\). These coordinates are connected to the pseudo-spherical coordinates \((r, \vartheta, \varphi)\) by \(y_1 = \]

\(^\dagger\)The class of metrics, so characterized, encompasses a wide variety of gravitational models. It suffices to mention that this class includes the Robinson-Bondi plane-waves, the cylindrical-wave solutions, the homogeneous cosmological models of Bianchi types I through VIII, the pseudo-Schwarzschild \([10, 12]\) and Kerr solutions, the Belinskii-Khalaktinov \([2]\) general cosmological solution with a physical singularity on portions of the so-called long eras.
The equation \( y_2^2 = -r^2 \), i.e. the locus of points equidistant from the origin, specifies a hyperboloid of two sheets intersecting the \( y_1 \) axis at the points \( \pm r \) called poles in analogy with the sphere. Either sheet (say the upper sheet) models an infinite spacelike surface without a boundary; hence, the Minkowski metric becomes positive definite (Riemannian) upon it. This surface has constant Gaussian curvature \( (R = -1/r^2) \), and it is the only simply connected surface with this property. Other embeddings of the pseudo-sphere in the 3-dimensional Euclidean space are also available, for example it can be regarded as the 2-dimensional surface generated by the tractrix [13], but they are not global.

The form of the metric (3) is conjectured on the basis of the well known FRW solution by essentially replacing the \( SO(3) \) 2-dimensional orbits with those of \( SO(2,1) \). Let us briefly recall the main features of the FRW metric. In spherical coordinates \( (r \in [0, \infty[ , \vartheta \in [0, \pi[ , \varphi \in [0, 2\pi[ ) \) it has the form

\[
g = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right].
\]

(5)

It is a homogeneous and isotropic cosmology and the 3-dimensional leaves \( S \), defined by \( t = \text{const} \), have constant scalar curvature \( R \) proportional to \( k \) \( (R = 6k/a^2) \). Indeed, \( \dim\text{Kil}(g|_S) = 6 \) and the Killing vector fields \( X_i, P_i \{i = 1,..3\} \) of \( g \)

\[
X_i = -\epsilon_{ijk}x_j \partial_k, \quad P_i = (1 - \frac{kr^2}{4})\partial_i + \frac{1}{2}kx_i(r\partial_r)
\]

close the Lie algebra

\[
[X_i, X_j] = \sum_k \epsilon_{ijk}X_k, \quad [P_i, P_j] = k \sum_k \epsilon_{ijk}X_k, \quad [X_i, P_j] = \sum_k \epsilon_{ijk}Y_j.
\]

Here \( \epsilon_{ijk} \) is the Levi-Civita tensor density and the standard relation between spherical and Cartesian coordinates, \( x_i \), is understood. For positive \( k \) the six vectors span the Lie algebra of \( SO(4) \), for \( k = 0 \) they span the Lie algebra of the semidirect product \( SO(3) \times' \mathbb{R}^3 \), whereas \( k \) negative corresponds to the proper Lorentz group \( SO(3,1) \). Since the maximally symmetric and isotropic leaves \( S \) are spacelike, the isotropy group is for all the three cases \( SO(3) \). In order to evidentiate the special role of \( SO(3) \) and because of some parallelism we may trace for \( so(2,1) \) invariant situations, it is useful to rewrite the three Lie algebras respectively as \( so(3) \oplus so(3) \), \( so(3) \oplus' \mathbb{R}^3 \), \( so(3) \oplus sb(2, C) \), where \( \oplus' \) denotes the semidirect sum and \( \oplus \) denotes a fully non-commutative sum of two Lie algebras which act nontrivially on each other by adjoint action (more details on this structure are given later on); \( sb(2, C) \) is the Lie algebra of type \( V \) in the Bianchi classification.

Going back to the metric (3), the question arises whether it admits other Killing fields and, if this is the case, what is the complete Killing algebra. Solving the Killing equations

\[
P^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu P^\alpha g_{\alpha\nu} + \partial_\nu P^\alpha g_{\mu\alpha} = 0
\]

(6)
we find three more Killing fields,

\[
\begin{align*}
P_1 &= b(r) \left[ \sinh \vartheta \cos \varphi \partial_r - \frac{\cos \varphi \cosh \vartheta}{r} \partial_\vartheta + \frac{\sin \varphi}{r \sinh \vartheta} \partial_\varphi \right] \\
P_2 &= b(r) \left[ \sin \vartheta \sin \varphi \partial_r - \frac{\sin \varphi \cosh \vartheta}{r} \partial_\vartheta - \frac{\cos \varphi}{r \sinh \vartheta} \partial_\varphi \right] \\
P_3 &= b(r) \left[ \cosh \vartheta \partial_r - \frac{\sinh \vartheta}{r} \partial_\vartheta \right]
\end{align*}
\]

(7)

with \(b(r) = \sqrt{1 - kr^2}\). Together with the vector fields \(X_i\) of Eqs. (4) they span the Lie algebra

\[
[X_i, X_j] = c_{ijk} X_k, \quad [X_i, P_j] = c_{ijk} P_k, \quad [P_i, P_j] = kc_{ijk} X_k.
\]

(8)

According to the value of \(k\) three different situations occur. It can be easily checked that for \(k = 0\), \(\text{Kil}(g)\) is the semidirect sum \(\mathfrak{so}(2,1) \oplus' \mathbb{R}^3\) while for positive \(k\), \(\text{Kil}(g)\) is the direct sum \(\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)\), that is to say \(\mathfrak{so}(2,2)\). Both the choices yield a metric of indefinite signature. The case \(k\) negative and \(-kr^2 > 1\) is the interesting one from a cosmological point of view. Here the signature of the metric is Lorentzian and, performing the change of basis

\[
Y_1 = \frac{P_1}{\sqrt{-k}}, \quad Y_2 = \frac{P_2}{\sqrt{-k}}, \quad Y_3 = -X_3, \\
Q_1 = X_1, \quad Q_2 = X_2, \quad Q_3 = \frac{P_3}{\sqrt{-k}}
\]

to reach a more conventional form, we discover that the Killing algebra (8) is nothing but the Lorentz algebra \(\mathfrak{so}(3,1)\). The scalar curvature of the spacelike submanifolds \(S\) is constant and negative

\[
\mathcal{R}(S) = 6 \frac{k}{a^2}.
\]

(9)

Therefore, we conclude that the metric (3) with \(k\) negative is diffeomorphic to the \(FRW\) metric of negative curvature.

To complete the analysis let us investigate the other cosmological solutions of the Einstein field equations, invariant with respect to a spacelike algebra \(\mathcal{G}_r\), with \(3 \leq r \leq 6\), containing \(\mathfrak{so}(2,1)\) as a subalgebra.

The most general \(\mathfrak{so}(2,1)\) invariant metric can be locally written in pseudospherical coordinates as

\[
g = A(r, t) dt^2 - B(r, t) dr^2 - F(r, t) r^2 (d\vartheta^2 + H(\vartheta) d\varphi^2)
\]

(10)

where \(H(\vartheta)\) is one of the functions \(\sinh^2 \vartheta\) or \(-\cosh^2 \vartheta\). With positive definite functions \(A, B, F\) and \(H(\vartheta) = \sin^2 \vartheta\) the metric is Lorentzian and the 2-dimensional surfaces defined by \((r, t = \text{const})\) are pseudo-spheres, that is, spacelike surfaces of constant negative curvature. In the other case, corresponding to
− cosh^2 \vartheta, the 2-dimensional surfaces defined by \((r, t = \text{const})\) are one-sheeted time-like hyperboloids and will not be considered here. As already mentioned the pseudo-Schwarzschild metric (2) is locally the only \(so(2,1)\)-invariant solution of the Einstein field equations in the vacuum \(^1\). This is the static solution found in [12] and also, in the context of warped solutions, in [10].

The more physically interesting gravitational field, generated by a distribution of matter described by an energy-momentum tensor \(T_{\mu\nu}\) and reducing in the vacuum to the previous one, is given by

\[
g = f(r)dt^2 - h(r)dr^2 - r^2(d\vartheta^2 + \sinh^2 \vartheta d\varphi^2),
\]

(11)

where the \(so(2,1)\)-invariant positive functions \(f(r)\) and \(h(r)\) satisfy the equations

\[
8\pi T_{00} = h'(rh^2)^{-1} + \frac{1}{r^2}(1 - h^{-1})
\]

\[
8\pi T_{11} = f'(rhf)^{-1} - \frac{1}{r^2}(1 - h^{-1})
\]

\[
8\pi T_{22} = \frac{1}{2}f'(rh)^{-1} - \frac{1}{2}h'(rh^2)^{-1} + \frac{1}{2}(fh)^{-1/2}[(fh)^{-1/2}f]'',
\]

the apex denoting the derivation with respect to \(r\).

For a perfect fluid, with energy momentum tensor field of the type \(T_{\mu\nu} = \rho u_{\mu}u_{\nu} + P(g_{\mu\nu} + u_{\mu}u_{\nu})\), the above equations give

\[
\frac{d\psi}{dr} = \frac{4\pi Pr^3 + m}{r(r - 2m)},
\]

\[
\frac{dP}{dr} = -(P + \rho)\frac{4\pi Pr^3 + m}{r(r - 2m)}
\]

where \(\psi = \ln \sqrt{f}\) and \(m(r) = 4\pi \int_1^r \rho(r)r^2dr + C\), the constant \(C = h(1)^{-1} - 1\) being determined by the boundary conditions.

The quantity \(m(r)\), when integrated over the whole volume of the source, represents the total mass. Unlike the \(so(3)\) invariant compact case, the hyperbolic symmetry gives to the source an infinite extension. Then, a constant density is not allowed whereas each function of \(r\) decreasing faster than \(1/r^3\) will do.

• \( \mathcal{G}_3 = so(2,1) \) invariant cosmologies

The most general \(so(2,1)\) invariant cosmology is given by the metric (10) with \(A(r, t) = 1\):

\[
g = dt^2 - B(r, t)dr^2 - F(r, t)r^2(d\vartheta^2 + \sinh^2 \vartheta d\varphi^2)
\]

(12)

Cosmologies with maximal symmetry \(SO(2,1)\) are not homogeneous since the orbits of \(SO(2,1)\) are 2-dimensional. They belong to the class of

\(^1\)This yields an extension of the Birkhoff theorem to the \(so(2,1)\) invariant case.
Bianchi cosmologies [6]. An interesting point of view which deserves more investigation is to regard them as limiting cases of models with higher symmetries in the presence of sources which retain only the $SO(2,1)$ symmetry.

- $G_4$ invariant cosmologies
  
  An additional space-like Killing field for the above metric has to commute with the $so(2,1)$ generators thus yielding a central extension of $so(2,1)$ and will have the general form
  
  $$P = P(r, t) \frac{\partial}{\partial r}.$$  

  The Killing equations (6) are satisfied by
  
  $$P = P(r) \frac{\partial}{\partial r}, \quad g = -dt^2 + \frac{B(t)}{P(r)^2} dr^2 + F(t)(d\vartheta^2 + \sinh^2 \varphi d\varphi^2),$$  

  where $P(r)$ and $B(t)$ are arbitrary non vanishing positive functions and the time coordinate $t$ has been rescaled by $\sqrt{A(t)}$. By defining a new radial coordinate, this gravitational field may be reexpressed as
  
  $$g = -dt^2 + b(t)dr^2 + f(t)(d\vartheta^2 + \sinh^2 \varphi d\varphi^2).$$  

  In this form it is immediately recognized as one of the Kantowski–Sachs cosmological solutions§ [9]. The metric (15) is spatially homogeneous as the leaves $S$ are orbits of $G_4$, but not isotropic, $SO(2,1)$ being only a global symmetry [9, 11].

- $G_5$ invariant cosmologies

  There are no $G_5$ invariant cosmologies because a 3-dimensional metric manifold cannot admit a complete group $G_5$ of motions [7].

  The Gödel model [13]

  $$g = dx^2 + e^{2x} dy^2/2 + dz^2 - (e^x dy + cd t)^2,$$  

  with $\dim \mathbb{K}il(g) = 5$ and Killing fields

  $$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial x} + (e^{-2x} - \frac{1}{2} y^2) \frac{\partial}{\partial y} - 2e^{-x} \frac{\partial}{\partial t},$$

  $$Y_1 = \frac{\partial}{\partial z}, \quad Y_2 = \frac{\partial}{\partial t},$$

  is a static solution whose isometry group acts transitively on the whole space-time which is then a homogeneous manifold. The Killing fields $Y_i$ commute between them and the vector fields $X_i$ close the $so(2,1)$ Lie algebra, but do not have space-like orbits.

§Among them the most known is the one invariant with respect to $G_4 = so(3) \oplus R$ obtained from the previous one by replacing the $SO(2,1)$ orbits with those of $SO(3)$.
G\textsubscript{6} invariant cosmologies

In this case, the leaves S are maximally symmetric and then isotropic and of constant curvature. Hence we search for three additional Killing fields \(P_i\) which, together with the \(so(2,1)\) generators, span the Lie algebra of any possible \(G_6\) with space-like 3-dimensional orbits. Apart from \(SO(2,1)\), because of the isotropy, it must contain \(SO(3)\) as a subgroup. On the other hand, maximally symmetric manifolds are uniquely specified by the numbers of eigenvalues of the metric that are positive or negative and by the sign of the scalar curvature [5, 14]; it follows that the \(G_6\) group we are looking for is the proper Lorentz group \(SO(3,1)\) and the corresponding invariant cosmology has to be diffeomorphic to the FRW cosmology (5) with negative spatial curvature, no matter what cumbersome coordinate system is adopted. Then, the only solution is represented by the metric in Eq. (3). Let us see how it can be derived in a more systematic manner.

The most general \(G_6\) including \(so(2,1)\) as a subalgebra may be given the Lie algebra structure [8]

\[
\begin{align*}
[X_i, X_j] &= c_{ijk} X_k \quad (17) \\
[X_i, P_j] &= \epsilon_1 c_{ijk} P_k + \epsilon_2 f_{ijk} X_k \quad (18) \\
[P_i, P_j] &= \epsilon_3 c_{ijk} X_k + \epsilon_4 f_{ijk} P_k \quad (19)
\end{align*}
\]

where \(c_{ijk}\) are the structure constants of \(so(2,1)\), \(f_{ijk}\) are some unknown structure constants and \(\epsilon_i = 0, \pm 1\), all of them constrained by the Jacobi identity. Up to isomorphisms we have the following cases.

i) \(\epsilon_1 = 1, \ \epsilon_2 = \epsilon_4 = 0, \ \epsilon_3 = 0, \pm 1\). This corresponds to search \(G_6\) in the form of a principal fibre bundle having \(SO(2,1)\) as structure group;

ii) \(\epsilon_1 = \epsilon_2 = \epsilon_3 = 0, \ \epsilon_4 = 1\). This is the direct sum, \(so(2,1) \oplus G_3\), which can be considered for any of the 9 Bianchi types. Imposing the Killing equations to the generators of \(G_3\) we find no solutions of this form, if \(so(2,1)\) is realized as in (4);

iii) \(\epsilon_1 = \epsilon_4 = 1, \ \epsilon_3 = 0\). This is the sum of two 3-dimensional Lie algebras not commuting between them, one of them being \(so(2,1)\).

The case \(\epsilon_2 = 0\) corresponds to a semidirect sum, whereas \(\epsilon_2 = 1\) yields a fully non-commutative sum of Lie algebras. In this case \(so(2,1)\) is said to be a Lie bialgebra [4]. The compatibility condition for two Lie algebras to be given such a structure is

\[
c_{ijk} f_{krs} + c_{irk} f_{jks} - f_{irk} c_{kjs} - c_{jrk} f_{iks} + f_{jrk} c_{kis} = 0 \quad (20)
\]

There are only two solutions for this equation, one with \(f_{ijk}\) all vanishing which yields the semidirect sum \(so(2,1) \oplus R^3\) (this is the only semidirect sum compatible with the \(so(2,1)\) structure), the other one with \(f_{ijk}\) given by the structure constants of \(sb(2,C)\), the Lie algebra
Let us analyze the case (i) in more detail. From the condition (18), with $X_i \in \mathfrak{so}(2,1)$ the fields $P_i$ must be of the form (7) with $b(r)$ an arbitrary function of $r$. The condition (19) fixes the $r$ dependence to be $b(r) = (-\epsilon_3 r^2 + C)^{1/2}$, with $C$ an arbitrary constant. Moreover, the Killing equations for the metric (12) restricted to the leaves $\mathcal{S}$, give $B(r) = \frac{B}{(-\epsilon_3 r^2 + C)^{1/2}}$, $F(r) = -\frac{B}{r}$, with $B$ an arbitrary constant. Thus, the restricted metric takes the form
\begin{equation}
g|_{\mathcal{S}} = -\frac{dr^2}{1 + \epsilon_3 r^2 / B} + r^2 (d\vartheta^2 + \sinh^2 \varphi \, d\varphi^2),
\end{equation}
where the coordinate $r$ has been rescaled by the factor of $\sqrt{-B/C}$. The space-time metric is then
\begin{equation}
g = dt^2 - a^2(t)\left[\frac{dr^2}{1 + \epsilon_3 r^2 / B} + r^2 (d\vartheta^2 + \sinh^2 \varphi \, d\varphi^2)\right].
\end{equation}
This can be seen to coincide with the solution (3) with $\epsilon_3 / B = k$. Hence, according to the value of $\epsilon_3 / B$ the three situations already described occur.

To summarize, we have found yet an alternative form of the FRW metric, for the case of negative scalar curvature, in terms of the symmetry subgroup $SO(2,1)$. This corresponds to the fact that both $SO(3)$ and $SO(2,1)$ are subgroups of the Lorentz group, so that we can choose to adapt our system of coordinates to either one or the other. With our choice the leaves $\mathcal{S}$ are foliated by pseudospheres, i.e space-like orbits of $SO(2,1)$ which are non-compact and of constant negative curvature. The conclusion is that the $SO(2,1)$ invariance is much more stringent than the $SO(3)$ invariance, because there is only one possible $G_6$-invariant cosmology admitting $SO(2,1)$ as a subgroup and this is the well known Lorentz invariant, negative curvature, FRW metric.

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