We show that a system of $2n$ identical two-level atoms interacting with $n$ cavity photons manifests entanglement and that the set of entangled states coincides with the so-called $SU(2)$ phase states. In particular, violation of classical realism in terms of the GHZ and GHSH conditions is proved. We discuss a new property of entanglement expressed in terms of local measurements. We also show that generation of entangled states in the atom-photon systems under consideration strongly depends on the choice of initial conditions and that the parasitic influence of cavity detuning can be compensated through the use of Kerr medium.
I. INTRODUCTION

It has been recognized that entanglement phenomenon touches on the conceptual problems of reality and locality in quantum physics as well as the more technological aspects of quantum communications, cryptography, and computing. In particular, the methods of quantum key distribution in communication channels secured from eavesdropping are based on the use of entangled states [1–5] (for recent review, see Refs. [6,7]). In turn, the realization of quantum computer [8] is dependent on the ability to form entangled states of initially uncorrelated single-particle states [9].

In recent years, many successful experiments have been performed to verify the violation of Bell’s inequalities and Greenberger-Horne-Zeilinger (GHZ) equality [10,11] and to develop the methods of engineered entanglement for quantum cryptography and quantum key distribution. In particular, the recent advances in the field of cavity QED and techniques of atom manipulation, trapping, and cooling enable a number of experiments which investigates the entanglement in the atomic systems (see [11–17] and references therein).

It has been shown recently [18] that a pure entangled state of two atoms can be obtained in an optical resonator through the exchange by a single photon. The main idea in Ref. [18] is that a single excitation of the system is either carried by a photon or shared between the atoms. If the photon can leak out from the resonator, the absence of photon counts in the process of continuous monitoring of the cavity decay can be associated with the presence of the pure entangled atomic state. The importance of this scheme is caused by the fact that its realization seems to be easy available with present experimental technique.

There are the two main objectives of this paper. On the one hand, we will show that the entangled states in the “atoms plus photons” systems of the type discussed in [18] are represented by the so-called $SU(2)$ phase states corresponding to the $SU(2)$ algebra of the odd “spin”

$$j = \frac{1}{2} \left( \left( \frac{2n}{n} \right) - 1 \right),$$

where $2n$ is the even number of atoms and $n = 1, 2, \cdots$ is the number of cavity photons. In particular, the system considered in Ref. [18] corresponds to the phase states of ”spin” $j = 1/2$. The $SU(2)$ phase states were introduced in [19] for an arbitrary spin and then generalized in [20,21] to the case of the $SU(2)$ subalgebra in the Weyl-Heisenberg algebra of photon operators (for recent review, see [22]).

From the mathematical point of view, this is the $N = 2j + 1$ qubit system defined in the Hilbert space $H_N = (C^2)^\otimes N$ with componentwise action of $SU(2)^N$.

On the other hand, we will discuss a new condition of entanglement has been proposed recently by one of us [23]. Let us note in this connection that, in the usual treatment of entanglement, the entangled states of a two-component (in general, multi-component) system are considered as the nonseparable states with respect to the subsystems (e.g., see [24]). For example, if the individual components of a two-component system are described by the states $|\xi_i\rangle$ and $|\chi_i\rangle$, respectively, the state

$$|\psi_{\text{ent}}\rangle = \sum_i b_i |\xi_i\rangle \otimes |\chi_i\rangle,$$

is entangled if $b_i \neq 0$ for at least two distinct values of the subscript $i$. From the mathematical point of view, the entanglement is caused by the combination of the superposition principle in quantum mechanics with the tensor product structure of the space of state of the two- or multi-component system [25].

Very often, the existence of entanglement is verified in terms of violation of Bell’s inequalities and their generalizations [26–31]. Another way is based on the use of GHZ theorem [10]. A possibility to introduce more general inequalities is also discussed [32].

It should be noted that the use of Bell’s inequalities and their numerous generalizations demonstrate nothing but the nonexistence of hidden variables. Moreover, it is possible to say that the unique, general, and mathematically correct definition of entanglement still does not exist (e.g., see Ref. [32]).

An interesting observation has been done recently [23]. The states usually considered in the context of entanglement have a common property which can be expressed in terms of the local measurements. Viz, the averages over such a states of all operators, representing local measurements, are equal to zero. This statement can be illustrated by the atoms-plus-photons systems under consideration. Consider first the set of two identical two-level atoms. Let $|e\rangle$ and $|g\rangle$ denote the excited and ground atomic states of the $\ell$th atom, respectively. Then, the entangled, maximum excited atomic states in the system "2 atoms plus 1 photon" considered in [18] are

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1g_2\rangle \pm |g_1e_2\rangle).$$

The local measurement can be described by the Pauli matrices

$$\sigma_1^{(\ell)} = |e\rangle\langle g| + |g\rangle\langle e|,$$

$$\sigma_2^{(\ell)} = -i|e\rangle\langle g| + i|g\rangle\langle e|,$$

$$\sigma_3^{(\ell)} = |e\rangle\langle e| - |g\rangle\langle g|,$$

i.e., by the infinitesimal generators of the algebra $SL(2)$. It is now a straightforward matter to check that

$$\forall i, \ell \quad \langle \psi_{\pm}| \sigma_{i}^{(\ell)} |\psi_{\pm}\rangle = 0,$$

where averaging is taken over the states (4). Another example is provided by the GHZ states [10]
corresponding to the maximum atomic excitation in the 3 + 3-system. It is easily seen that the averaging of the local operators (4) over (6) gives the same result as (5).

We will show that the SU(2) phase states of spin (1) in a 2n + n-type atom-photon system obey the nonseparability conditions of the type of (2), have the property (5), and manifest the violation of classical realism expressed in terms of the GHZ [10] and CHSH (Clauser-Horne-Shimony-Holt) [33] conditions.

The paper is organized as follows. In Sec. II, we consider the representation of the SU(2) phase states. As a particular example, we examine the system of two identical two-level atoms, interacting with a single cavity photon and show that the maximum entangled atomic states of the Ref. [18] belong to the class of the SU(2) phase states of spin j = 1/2. Then, we generalize this result on the case of 2n + n system. As a nontrivial example we consider in Sec. III the system of four identical two-level atoms interacting with the two cavity photons. In this case, the set of entangled, maximum excited atomic states is provided by the six orthogonal SU(2) phase states of spin j = 5/2. For these states, we prove violation of classical realism through the use of GHZ and CHSH conditions. In Sec. IV, we discuss how the entangled atomic states can be achieved in the process of steady-state evolution. In particular, we show that the maximum entanglement can be achieved if the initial state of the system contains the photons and does not contain the atomic excitations. We also show that the presence of the cavity detuning hampers the creation of pure entangled states and that the parasitic influence of detuning can be compensated through the use of the Kerr medium inside the cavity. Finally, in Sec. V, we briefly discuss the obtained results.

II. REPRESENTATION OF THE SU(2) PHASE STATES

An arbitrary spin j can be described by the generators J+, J−, Jz of the SU(2) algebra such that

\[ [J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm, \]

\[ J^2 = J^2 + \frac{1}{2}(J_+ J_- + J_- J_+) = j(j + 1) \times 1, \]

where 1 is the unit operator in the 2j + 1 dimensional Hilbert space. Since

\[ J_\pm = J_z \pm iJ_y, \]

it is possible to say that the generators J+, J−, Jz in (7) correspond to the Cartesian representation of the SU(2) algebra. Following [19], one can introduce the representation in spherical coordinates via the polar decomposition of (7) of the form

\[ J_+ = J_r e^\epsilon, \quad J_- = J_r^* e^{-\epsilon}, \quad \epsilon^+ = 1, \]

where the Hermitian operator \( J_r \) corresponds to the radial contribution, while \( \epsilon \) gives the exponential of the azimuthal phase operator. It is a straightforward matter to show that \( \epsilon \) can be represented by the following \((2j + 1) \times (2j + 1)\) matrix

\[ \epsilon = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \]

in the \(2j + 1\)-dimensional Hilbert space. Here \( \psi \) is an arbitrary real parameter (reference phase). The eigenstates of the operator (9)

\[ \epsilon |\phi_n^{(j)}\rangle = e^{i\psi} |\phi_n^{(j)}\rangle, \quad n = 1, \ldots, (2j + 1), \]

form the basis of the so-called phase states

\[ |\phi_n^{(j)}\rangle = \frac{1}{\sqrt{2j + 1}} \sum_{k=0}^{2j} e^{ik\psi} |\psi_k\rangle \]

dual with respect to the basis of individual states \( |\psi_k\rangle \) of the Hilbert space.

As a physical example of some considerable interest, consider now the system of the two identical two-level atom interacting with the single cavity photon (see [18]). If the cavity photon is absorbed by either atom, the atomic subsystem can be observed in the following states

\[ |\psi_1\rangle = |e_1 g_2\rangle, \quad |\psi_2\rangle = |g_1 e_2\rangle, \]

where \( |e_1 g_2\rangle = |e_1\rangle \otimes |g_2\rangle \) and \( |e\rangle \) and \( |g\rangle \) denote the excited and ground atomic states, respectively. The subscript marks the atom. Using the atomic basis (12), we can construct the following representation of the SU(2) algebra:

\[ J_+ = |e_1 g_2\rangle \langle g_1 e_2|, \quad J_- = |g_1 e_2\rangle \langle e_1 g_2|, \]

\[ J_3 = \frac{1}{2}(|e_1 g_2\rangle \langle e_1 g_2| - |g_1 e_2\rangle \langle g_1 e_2|). \]

This representation formally corresponds to (7) at the spin \( j = 1/2 \). Then, the corresponding exponential of the phase operator (9) takes the form

\[ \epsilon = |e_1 g_2\rangle \langle g_1 e_2| + e^{i\psi} |g_2 e_1\rangle \langle g_1 e_2|. \]

In turn, the phase states (10) and (11) are

\[ |\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|e_1 g_2\rangle \pm e^{i\psi} |g_2 e_1\rangle), \]

\[ \phi_\pm = \psi/2 + (1 \mp 1)\pi/2. \]

It is easily seen that the phase states (14) form the set of entangled atomic states in the two-atom system under
either excited or de-excited atoms: on the states (16) leads to the change of the number of atoms only for
\( \sigma_{i}^{(5)} \). In fact, the action of the flip-operators can be used to construct a representation of the algebra (7) of spin (1). Here
\( i \)

is the total number of such a states. In the basis (15), we can construct the polar decomposition of the \( SU(2) \) algebra of spin (1) and the corresponding exponential of the phase operator (9) and the phase states (11). Let us rename the states (15) as follows
\[
|\psi_{k}\rangle \rightarrow |\psi_{k'}\rangle, \quad k' \equiv k - 1 = 0, \ldots, N - 1.
\]
Then, the \( SU(2) \) phase states (11) take the form
\[
|\phi_{k}\rangle = \frac{1}{\sqrt{N}} \sum_{k'=0}^{N-1} e^{ik'\phi_{k}} |\psi_{k'}\rangle,
\]
where
\[
\phi_{k} = (\psi + 2k\pi)/N.
\]
These states (16) form a basis dual with respect to (15) and spanning the Hilbert space of the maximum excited atomic states in the \( 2n + n \) system under consideration. By construction, the phase states (16) are nonseparable with respect to contributions of individual atoms and thus entangled [24]. Let us stress that the choice of the phase factors in (16) is irrelevant to entanglement, which holds for arbitrary phase factors. This choice is caused by the aspiration for getting the dual with respect to (15) basis of entangled states.

We now note that the coordinates of a general \( N \)-qubit state may be naturally placed at vertices of the \( N \)-dimensional cube. Then, the state is maximally entangled if the opposite faces of the cube are orthogonal and have the same norm equal to \( 1/\sqrt{2} \). This condition holds for the superposition of all excited states of \( N \) atoms only for \( N = 2n \) as in the case under consideration.

It is easily seen that the states (16) obey the condition (5). In fact, the action of the flip-operators \( \sigma_{i}^{(5)} \) in (4) on the states (16) leads to the change of the number of either excited or de-excited atoms:
\[
\sigma_{i}^{(5)} \psi_{k} \rightarrow \left\{ \begin{array}{ll}
|\psi_{n-1} \rangle, & (\ell, \{g\}) \\
|\psi_{n+1} \rangle, & (\ell, \{e\})
\end{array} \right.
\]
and therefore \( \langle \sigma_{i}^{(5)} \rangle_{k} = 0 \) in the case of averaging over the states (16). Since each state (15) contains equal number of excited and de-excited atoms, the action of the parity operator in (4) on the phase states (16) should lead to the state which differ from (16) by the multiplication of a certain \( n \) terms by the factor of \(-1\). Hence
\[
\langle \sigma_{i}^{(5)} \rangle_{k} = \frac{1}{N} \left( \sum_{i=1}^{N/2} 1 - \sum_{i=N/2+1}^{N} 1 \right) = 0.
\]
By construction, \( N \) is always an even number.

Thus, the \( SU(2) \) phase states (16), corresponding to the maximum excited atomic states in the \( 2n + n \) system, are entangled because they are nonseparable and, at the same time, obey the condition (5) for the local measurements. In the next Section, we show that the states (16) manifest violation of classical realism as well.

Before we begin to discuss this subject, let us note that the \( SU(2) \) phase states of the atomic system under consideration with integer spin do not provide the entanglement. Consider as an example the system of three identical two-level atoms, interacting with a single cavity photon. There are the three excited atomic states
\[
|\psi_{122}\rangle, \quad |g_{1}e_{22}\rangle, \quad |g_{1}g_{2}e_{3}\rangle
\]
and the three dual phase states of the type of (16)
\[
|\psi_{k}\rangle = \frac{1}{\sqrt{3}} (|\psi_{122}\rangle + e^{i\phi_{k}} |g_{1}e_{22}\rangle + e^{2i\phi_{k}} |g_{1}g_{2}e_{3}\rangle).
\]
It is clear that the states (18) are the phase states of spin \( j = 1 \). Here
\[
\phi_{k} = (\psi + 2k\pi)/3, \quad k = 0, 1, 2.
\]
It is easily seen that the phase states (18) cannot be factorized with respect to atoms. At the same time, the average of the parity operator \( \sigma_{3}^{(5)} \) in (4) over the states (18) is
\[
\forall k, \ell, \quad \langle \sigma_{3}^{(5)} \rangle_{k} = -\frac{1}{3},
\]
although the averages of the flip-operators are
\[
\forall k, \ell, \quad \langle \sigma_{i}^{(5)} \rangle_{k} = 0.
\]
Thus, the nonseparable states (18) do not obey the condition (5). At the same time, these states do not manifest the maximum entanglement as well. Let us stress that the nonseparability is not a sufficient condition of maximum entanglement [24]. For example, from the measurement of the state of the first atom we can only learn that either the atoms 2 and 3 are both in the ground state with reliability or they are in the two-atom entangled state of the type discussed in [18]. Similar result can be obtained for the system of three atoms, interacting with two cavity photons. The only maximum entangled state of the system of three atoms is provided by the superposition of GHZ states (6).
Then, the six phase states (16) have the form (16) with
generators (7) has the form
the form
the following way

As well as (16), these states are nonseparable and hence
photons. The maximum excited atomic states at


\[ e = |e_1e_2g_34\rangle|e_1g_2e_34\rangle|e_1g_2g_3e_4\rangle \\
+|g_1e_2g_34\rangle|g_1e_2g_3e_4\rangle + |g_1e_2g_3e_4\rangle|g_1g_2e_3e_4\rangle + e^{i\phi}|g_1g_2e_3e_4\rangle|e_1e_2g_34\rangle. \]

Then, the six phase states (10) have the form (16) with \( N = 6 \) and

\[ \phi_k = \frac{\psi + k\pi}{3}, \quad k = 0, 1, \ldots, 5. \]

As well as (16), these states are nonseparable and hence
entangled and obey the condition (5) for local variables.

To show that these phase states violate the classical
realism, let us first represent the states (16) at \( N = 6 \) in the following way

\[ |\phi_k\rangle = \frac{1}{2\sqrt{3}}(|\chi_{1k}\rangle + e^{i\phi_k}|\chi_{2k}\rangle + e^{2i\phi_k}|\chi_{3k}\rangle), \]

where

\[ |\chi_{1k}\rangle = \frac{1}{2\sqrt{2}}(|e_1e_2g_34\rangle + e^{5i\phi_k}|g_1g_2e_3e_4\rangle), \]
\[ |\chi_{2k}\rangle = \frac{1}{2\sqrt{2}}(|g_1e_2g_34\rangle + e^{3i\phi_k}|e_1g_2g_3e_4\rangle), \]
\[ |\chi_{3k}\rangle = \frac{1}{2\sqrt{2}}(|g_1e_2g_3e_4\rangle + e^{i\phi_k}|e_1g_2e_3g_4\rangle). \]

It is easily seen that each set of six states \( |\chi_{\phi k}\rangle \)
with \( p = 1, 2, 3 \) and \( k = 0, \ldots, 5 \) consists of the nonseparable
and hence entangled states. Consider, for example, the
states \( |\chi_{1k}\rangle \) in (22). Because of the definition of the phase angle \( \phi_k \) at \( N = 6 \), they consist of the three sets of the pairwise orthogonal states

\[ \{|\chi_{10}\rangle, |\chi_{13}\rangle\}, \{|\chi_{11}\rangle, |\chi_{14}\rangle\}, \{|\chi_{12}\rangle, |\chi_{15}\rangle\}. \]

It is also seen that the second and third sets here are
obtained from the first set by the successive rotations of the reference frame.

Now the violation of classical realism can be proved
through the use of the GHZ theorem [10]. Consider first
the state \( |\chi_{10}\rangle \) in (22). It is easy to verify that this state
obey the conditions

\[ \forall \ell, \sigma \prod_{\ell=1}^{4} \sigma^{(\ell)}_i |\chi_{10}\rangle = |\chi_{10}\rangle \] (23)

and

\[ \sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = -|\chi_{10}\rangle, \]
\[ \sigma_2^{(1)} \sigma_1^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = -|\chi_{10}\rangle, \]
\[ \sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = |\chi_{10}\rangle, \]
\[ \sigma_2^{(1)} \sigma_1^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = |\chi_{10}\rangle. \] (24)

It is possible to say that these equalities (23) and (24) express
a kind of EPR “action at distance” in the maximum
excited states of the system of four atoms interacting
with two photons. In other words, the correlations represented
by (23) and (24) permit us determine in a unique way the state
of the fourth atom via measurement of the states
of other three atoms.

The operator equalities (23) and (24) can be used to
obtain the relations similar to those in the GHZ theorem.
Following [10], we have to assign the classical quantities
\( m_i^{(\ell)} \) to the local operators. Here

\[ m_1^{(\ell)} m_2^{(\ell)} = \pm 1. \]

Then, it follows from (23) that

\[ \prod_{\ell=1}^{4} m_i^{(\ell)} = 1. \] (25)

At the same time, it follows from (24) that

\[ \sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = -|\chi_{10}\rangle. \]

Employing the classical variables instead of the local operators allows this to be cast into the form

\[ (m_1^{(1)} m_1^{(2)} m_2^{(3)} m_2^{(4)} m_1^{(3)} m_2^{(4)} m_1^{(4)} m_2^{(4)})^2 = -1. \]
Since \((m_1^{(1)})^2 = (m_2^{(1)})^2 = 1\), we get an equivalent equality
\[
m_1^{(1)} m_1^{(2)} m_1^{(3)} m_1^{(4)} = -1,
\]
which contradicts (25). Hence, the state \(|\chi_{10}\rangle\) in (22) obey the GHZ theorem. Similar result can be obtained for all other states in (22) and hence, for the phase states (21).

Our consideration so far have applied to the local measurements touching on a single atom. We now note that the phase states (21) allow another kind of entanglement in the case of pairwise measurement. Consider again the state \(|\chi_{10}\rangle\) in (22) and assume that the measurements \(a\) and \(b\) corresponds to a pair of atoms:
\[
\begin{align*}
a &= \cos \theta_a |e_1 e_2\rangle \langle e_1 e_2| + \sin \theta_a |e_1 e_2\rangle \langle e_1 e_2| g_1 g_2| g_1 g_2\rangle, \\
b &= \cos \theta_b |e_3 e_4\rangle \langle e_3 e_4| + \sin \theta_b |e_3 e_4\rangle \langle e_3 e_4| g_3 g_4| g_3 g_4\rangle.
\end{align*}
\]
Assume now that we make the two measurements \(a\) and \(a'\) with the angles \(\theta_1 = \pi\) and \(\theta'_a = \pi/2\) and the two more measurements \(b\) and \(b'\) with the angles \(\theta'_b = -\theta_b\), respectively. Then, the averaging over the state \(|\chi_{10}\rangle\) gives
\[
\langle ab \rangle = \langle a'b' \rangle = \cos \theta_b, \quad \langle a'b \rangle = \sin \theta_b = -\langle a'b' \rangle.
\]
Employing the CHSH inequality [33]
\[
|\langle ab \rangle + \langle a'b \rangle + \langle a'b' \rangle - \langle a'b' \rangle| \leq 2
\]
then gives
\[
|\cos \theta_b - \sin \theta_b| \leq 1.
\]
Violation of this inequality and hence, of the classical realism occurs at small negative \(\theta_b\), when we can put
\[
|\cos \theta_b - \sin \theta_b| \sim 1 + |\theta_b| > 1.
\]
Similar consideration can be done for all over states in (22) through the use of proper pairwise measurements. At the same time, the phase states (21) do not manifest entanglement with respect to the pairwise measurements.

The phase states (16) for the \(6 + 3, 8 + 4, \cdots\) systems, corresponding to the spin (1) equal to 19/2, 69/2, \cdots, respectively, can be considered as above.

IV. INITIAL CONDITIONS AND ATOMIC ENTANGLEMENT

It is clear that the evolution of the \(2n + n\) system strongly depends on the choice of initial conditions. To trace the proper choice leading to the atomic entanglement, let us ignore the relaxation processes. Then, the steady-state evolution of the \(2n + n\) system under consideration is governed by the Hamiltonian
\[
H = \Delta a^+ a + \omega_0 n + \gamma \sum_{\ell} R_{\ell} a + a^+ R_{\ell}.
\]
Here \(\Delta\) is the cavity detuning, \(\omega_0\) is the atomic transition frequency, \(\gamma\) is the atom-field coupling constant, operators \(a\) and \(a^+\) describe the cavity photons,
\[
\mathcal{N} = a^+ a + \sum_{\ell, \ell' \neq \ell} |e_{\ell}\rangle \langle e_{\ell}| \otimes |1^{(\ell)}\rangle,
\]
and the atomic operators are defined as follows
\[
R_{\ell}^\pm = \frac{\sigma_{\ell}^{(\ell)} \pm i \sigma_{\ell}^{(\ell)}}{2}.
\]

Consider first the case of two atoms and single cavity photon when \(\ell = 1, 2\) and the Hamiltonian (28) coincides with that of Ref. [18]. For simplicity, we use here the same coupling constant \(\gamma\) for both atoms. Our consideration can easily be generalized on the case of coupling constant depending on the atomic position. Let us note that, in the case of only two atoms, the Hamiltonian (28) can be represented as follows
\[
H \to H_\phi = \Delta a^+ a + \omega_0 \mathcal{N}_\phi + \gamma \sqrt{2}(\mathcal{R}^+ a + a^+ \mathcal{R}),
\]
where
\[
\mathcal{N}_\phi = a^+ a + \sum_{k = \pm 1} |\phi_k\rangle \langle \phi_k|,
\]
and
\[
\mathcal{R}^+ = |\phi_+\rangle \langle g_1 g_2|.
\]
Here \(|\phi_{\pm}\rangle\) denote the phase states (14).

Using the Hamiltonian (29) as the generator of evolution, for the time dependent wave function we get
\[
|\Psi(t)\rangle = e^{-iH_\phi t} |\Psi(0)\rangle = |C_-(t)|\phi_-\rangle + C_+(t)|\phi_+\rangle \otimes |0\rangle_{ph} + C(t)|g_1 g_2\rangle \otimes |1\rangle_{ph},
\]
where \(|\cdots\rangle_{ph}\) denotes the states of the cavity field. The coefficients \(C_\pm(t)\) and \(C(t)\) in (30) are completely determined by the initial conditions and normalization condition.

It is easily seen that the state \(|\phi_-\rangle \otimes |0\rangle_{ph}\) is the eigenstate of the Hamiltonian (29). Hence, at
the atomic phase state $|\phi_-\rangle$ in (14) provides the stationary, maximum entangled atomic state in the system under consideration [18]. At the same time, it is not very clear how to prepare such a state.

Therefore we consider a more realistic initial state provided by excitation of either atom, while the cavity field is in the vacuum state. To realize such a state, we can assume, for example, that one of the atoms (initially de-excited) is trapped in the cavity, while the second atom (initially excited) slowly passes through the cavity like in the experiments discussed in [14,15]. Assume for definiteness that

$$|\Psi(0)\rangle = |e_1 g_2\rangle \otimes |0\rangle_{ph}. \quad (31)$$

Then, the coefficients of the wave function (30) take the form

$$C_- (t) = \frac{1}{\sqrt{2}} e^{-i\omega_0 t},$$

$$C_+ (t) = \frac{1}{\sqrt{2}} \left( \cos \Omega t + \frac{i\Delta}{2\Omega} \sin \Omega t \right) e^{-i(\omega_0 + \Delta/2)t},$$

$$C(t) = -\frac{i\gamma}{\Omega} e^{-i(\omega_0 + \Delta/2)t} \sin \Omega t,$$

where $\Omega = [2\gamma^2 + (\Delta/2)^2]^{1/2}$. At first site, the probabilities

$$P_{\pm} (t) = \langle |0\rangle_{ph} \otimes \langle \phi_{pm} | \Psi(t) \rangle \rangle^2 = |C_{\pm} (t)|^2$$

to observe the states (14) corresponding to the maximum atomic entanglement, are

$$P_- (t) = \frac{1}{2},$$

$$P_+ (t) = \frac{\Delta^2}{8\Omega^2} + \frac{\gamma^2}{\Omega^2} \cos^2 \Omega t \leq \frac{1}{2},$$

respectively. At the same time, the absence of photon counts, which is considered in [18] as a sign of the atomic entanglement, corresponds here to the case when both probabilities $P_{\pm} (t_k) = 1/2$ at a certain time $t_k$. In other words, the mutually orthogonal entangled states (14) have the same probability to be observed at $t = t_k$. This means that there is no atomic entanglement at all but we definitely know which atom is in the excited state.

Consider one more realistic initial state when both atoms are trapped in the cavity in de-excited state, while the cavity field contains a photon:

$$|\Psi(0)\rangle = |g_1 g_2\rangle \otimes |1\rangle_{ph}. \quad (32)$$

Then, for all times we get $C_- (t) = 0$ and

$$C_+ (t) = \frac{i\gamma \sqrt{2}}{\Omega} e^{-i(\omega_0 + \Delta/2)t} \sin \Omega t,$$

$$C(t) = \left( \cos \Omega t - \frac{i\Delta}{2\Omega} \sin \Omega t \right) e^{-i(\omega_0 + \Delta/2)t}.$$  

Hence, under this initial condition, the entangled state $|\phi_-\rangle$ cannot be achieved at all, while the second entangled state $|\phi_+\rangle$ in (14) can be achieved. It is seen that, in the case of initial state (32), the probability to detect the photon is

$$P_{ph} (t) = |C(t)|^2 = \cos^2 \Omega t + \frac{\Delta^2}{4\Omega^2} \sin^2 \Omega t.$$

This expression takes the minimum value

$$\min P_{ph} = P_{ph} (t_m) = \frac{\Delta^2}{4\Omega^2}$$

at $t = t_m = \pi (2m + 1)/\Omega$, $m = 0, 1, \cdots$. At the same time, the absence of photon in (14) can be achieved. It is seen that, the pure atomic entanglement with $P_+ (t_m) = 1$ is realized at $t = t_m$ only in the absence of the cavity detuning when $\Delta \to 0$.

The parasitic influence of the cavity detuning can be compensated through the use of Kerr medium filling the cavity. In this case, the Hamiltonian (28) should be supplemented by the term

$$H_k = \kappa (a^+ a)^2,$$

which leads to the following renormalization of the Rabi frequency

$$\Omega \to \Omega_k = \sqrt{2\gamma^2 + (\Delta + \kappa)^2/4}.$$

Then, the proper choice of the Kerr parameter $\kappa = -\Delta$ should lead to the pure entangled atomic state $|\phi_+\rangle$ at a certain times.

Consider now the case of four atoms and two photons. In contrast to the previous case, neither phase state in (21) is an eigenstate of the Hamiltonian (28). Then, the choice of the initial state either as a state with two excited atoms or as a state with one excited atom plus cavity photon does not lead to a pure atomic entanglement. As in the case of two atoms, the pure atomic entanglement can be reached under the choice of the state with the absence of the atomic excitations in the initial state. The influence of the cavity detuning can be compensated by the presence of Kerr medium as well as in the case of two atoms.

V. CONCLUSION

Let us briefly discuss the obtained results. For the model of two identical two-level atoms interacting with a single photon has been proposed in [18], it is shown that the maximum entangled atomic states are represented...
by the SU(2) phase states of spin 1/2. This result is
generalized on the case of a system with an arbitrary
even number 2n of the two-level atoms, interacting with
n photons (n ≥ 1), when the entangled states correspond
to the SU(2) phase states of the half-integer spin (1). In
particular, it is proved that the SU(2) phase states of
spin (1) violate the classical realism.

It should be noted that the SU(2) phase states can
easily be constructed because they are the eigenstates
of a well-defined non-Hermitian exponential of the phase
operator (9). Let us stress that the phase states can be
represented as the eigenstates of the Hermitian operators
of cosine and sine
\[ C = \frac{1}{2}(\epsilon + \epsilon^+), \quad S = \frac{1}{2i}(\epsilon - \epsilon^+) \]
as well.

It is also shown that the realization of a pure atomic
entanglement in the 2n + n-type systems strongly depends
on the choice of initial state. Viz, the entangled states
can be reached in the process of steady-state evolution
only if all 2n atoms are initially in the de-excited states,
while the cavity contains just n photons. This condition
has an intuitively clear explanation: the excitations of
different atoms have the same probability and therefore
each photon in the 2n + n-system is shared with a couple
of atoms.

It is shown that the presence of the cavity detuning
hampers the creation of a pure entangled atomic state.
This negative effect can be compensated through the use
of Kerr medium in the cavity.

Since we are primarily interested in the algebraic struc-
ture of the entangled states, we neglected the dissipation
processes in our consideration. The obtained results can
be generalized on the cavity with leakage in the same way
as in Ref. [18].

It seems to be interesting that the condition (5) for
the local measurements always accompanies the conven-
tional entangled states such as the Bell and GHZ states.
This condition is also valid in the case of the SU(2) phase
states in the 2n + n-systems. At the same time, this con-
dition is invalid in the case of 3 + 1 system, in which the
phase states, corresponding to the absence of free photon,
do not manifest pure entanglement although they obey
the nonseparability condition. Let us note in this context
that the nonseparability of the states is not a sufficient
condition of the maximum entanglement [24].

It is clear that the condition (5) has a nontrivial mean-
ing. In fact, all one can expect is that the total spin of
the system is equal to zero because the Bell states are
usually obtained by decay of a spinless system (e.g., see
[34]). Instead the condition (5) assumes that all compo-
nents of the local spin operators give zero averages over
the entangled states. Since the unique and general math-
ematical definition of entanglement is something hitherto
undiscovered, it seems to be tempting to examine the
condition (5) as a possible candidate [23].

It is also interesting that, in some cases, the entangled
states coincide with the SU(2) phase states. The latter
have direct connection with the polarization of photons
[21,22,35]. It seems to be possible to discuss the polar-
ization entanglement in terms of the phase states [36]. In
particular, if we consider a more realistic electric-dipole
transition in the two-level atoms, the atomic entangle-
ment may arise in the system of two atoms interacting
with two photons of different helicity.

We also note that, in spite of the fact that the SU(2)
phase states of the type of (18) do not manifest the entan-
glement in the case of the (3+1) and (3+2) atom-photon
systems, this may occur in some other ”spin”-1 systems.
An example is provided by the ”biphoton” states, namely,
photon pairs in symmetric Fock states [0, 2], [2, 0], and
[1, 1] [37]. The SU(2) phase states (18) constructed from
the basis of the biphoton states have been discussed re-
cently in the context of quantum cryptography with the
three-state systems [38].

Another example is provided by photons emitted by a
single two-level atom with the cascade transition
\[ |J = 2, m = 0\rangle \rightarrow |J' = 0, m' = 0\rangle. \]

Here J, J’ and m, m’ denote the angular momentum and
projection of the angular momentum of the excited and
ground states, respectively. This transition gives rise to
e first photon twins propagating in the opposite directions
(e.g., see [39]). Each photon carries spin 1, but because
of the conservation of angular momentum in the process
of radiation, the sum of projections for the two photons
should be equal to zero. Therefore, the possible states of
the photon subsystem, specified by the projection of the
angular momentum, are:
\[ |+ 1; −1\rangle, \quad |0 ; 0\rangle, \quad |− 1; +1\rangle. \]

These three ”individual” states can be used to construct the
dual basis of the SU(2) phase states
\[ |\phi_k\rangle = \frac{1}{\sqrt{3}}(|+ 1; −1\rangle + e^{i\phi_k}|0 ; 0\rangle + e^{2i\phi_k}|− 1; +1\rangle, \]
\[ \psi_k = \frac{2k\pi}{3}, \quad k = 0, 1, 2, \]
similar to (18) and to the biphoton states of Ref. [37].
The states of the same structure have been discussed in the
context of the quantum phase of the angular momen-
tum of photons [21,35,36] and in [39] in connection with
the quantum cryptography. It is easily seen that these
states are nonseparable with respect to the contributions
of individual photons and, at the same time, obey the
condition (5).

The more detailed consideration of the above examples
and of the condition (5) as well as of the connection be-
tween the entangled and phase states definitely deserves
a further discussion.

The authors would like to thank Prof. P.L. Knight and
Prof. A. Vourdas for useful discussions.

8
[23] A.A. Klyachko, to be published.
[34] M.O. Scully and M.S. Zubairy, Quantum Optics (Cambridge University Press, New York, 1997).
[36] A.S. Shumovsky, in Ref. [7].