Here is the propagator of the composite field, \( Z(u) \), where

\[
Z(u) = \left( \frac{\alpha}{\alpha - u} \right)^{\frac{\alpha - 1}{(\alpha - 1)^2}} \sum_{n=0}^{\infty} \frac{Z_n(0)}{u^n} \int_{\mathcal{E}} \frac{d^4 \mathbf{p}}{(2\pi)^4} \left( \frac{\mathbf{p}}{\mathbf{p} + \mathbf{q}} \right)^2 = \sqrt{\mu \alpha} e^{-\mu \alpha Z(u)}
\]

The propagator satisfies the equation of motion

\[
\left( -\frac{d^2}{dx^2} - m^2 + \alpha \right) Z(u) = \delta(u) - \frac{1}{\alpha - u}
\]

with boundary conditions

\[
Z(u) = \begin{cases} 1 & \text{for } u = 0 \\ \frac{1}{\alpha - u} & \text{for } u > 0 \end{cases}
\]

The effective action is given by

\[
S_{\text{eff}} = \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \left( \frac{\mathbf{p}}{\mathbf{p} + \mathbf{q}} \right)^2 = \sqrt{\mu \alpha} e^{-\mu \alpha Z(u)}
\]

Evaluation of the non-resonant reaction rate is very important for non-spherical cases.

We derived analytical formulas of the effective astrophysical factor, \( S_{\text{eff}} \), for non-spherical cases.
Fig. 1. Approximate values of $S_{\text{eff}}$ generated by $S_{\text{eff-std}}$ (closed squares) and $S_{\text{eff-Gam}}$ (open squares) together with the exact value of $S_{\text{eff}}$ (the solid line) at (a) $T = 1.3 \times 10^7$ K and (b) $T = 1.7 \times 10^7$ K. The astrophysical $S$ factor was Taylor-expanded up to $n_M$-th order.

$$P_0(n) = 1728 n^6 + 4320 n^5 - 4320 n^4 - 13320 n^3 - 288 n^2 + 6210 n + 665.$$ 

The values of $n_M$ and $k_M$ correspond to the numbers of terms in the Taylor expansion of $S(E)$ and the asymptotic expansion of $S_{\text{eff}}$ (expansion parameter $1/\tau$), respectively. The analytic formulas are expected to be useful for high temperature environments and reactions which have strong $E$-dependence around the corresponding Gamow peak energies. Note that the formulas are valid if $E_0 + \Delta/2$ is within the radius of convergence concerning the Taylor expansion of $S(E)$.

We applied our formulas to the $^7\text{Be}(p, \gamma)^8\text{B}$ reaction in the stellar interior of the sun, which is reported to have strong $E$-dependence around the corresponding Gamow peak energy $E_0 \approx 20$ keV$^5)$. Here we employed the function form of $S(E)$ obtained in Ref. 5). Figs. 1-(a) and 1-(b) show the approximate values of $S_{\text{eff}}$ for the reaction at $T = 1.3 \times 10^7$ K and $T = 1.7 \times 10^7$ K, respectively. In each figure closed and open squares represent the approximate values of $S_{\text{eff}}$ generated by Eqs. (0-2) and (0-3), respectively. The exact values of $S_{\text{eff}}$ is denoted by the solid line. Note that the two approximate values for a given $n_M$ are not modified by inclusion of terms of higher order than $k_M > 1$. Therefore, only the approximate values with $k_M = 1$ are shown in the figures. At both temperatures $S_{\text{eff-std}}$ converges to the exact result with $n_M = 4$ and $k_M = 1$ while $S_{\text{eff-Gam}}$ does with $n_M = 2$ and $k_M = 1$. The result can be understood by considering that $S_{\text{eff}}$ is evaluated even quantitatively by $S(E)$ in the energy region $E_0 - \Delta \leq E \leq E_0 + \Delta$. However, it is necessary to confirm more systematically whether the uniform approximation works well in approximating $S_{\text{eff}}$.

References

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