Introduction

The D-brane Probe of $G$-Holographic Manifolds
from deforming $\mathcal{N} = 4$ mirror pairs using both field theory and string theory techniques. Of course with such little supersymmetry ($\mathcal{N} = 1$ means 2 supercharges) we have little control over the strong coupling dynamics and must be wary of any conjectured duality. Our only savior is parity symmetry which may be used to prohibit the lifting of certain vacuum moduli spaces [13, 14]. We hope that the success of our mirror pairs in describing manifolds of $G_2$ holonomy goes some way towards convincing the reader of their utility.

The mirror pairs preserve only $\mathcal{N} = 1$ supersymmetry, and are given by

**Theory A:** $U(1)^r$ with $k$ scalars and $N$ hypers

**Theory B:** $U(1)^{N-r}$ with $(3N - k)$ scalars and $N$ hypermultiplets.

The abelian vector multiplets contain only a photon and a Majorana spinor, while the scalar multiplets, which we shall denote as $\Phi$, contain a single real scalar and a Majorana fermion. In contrast, the hypermultiplets fill out representations of the $\mathcal{N} = 4$ algebra: they each contain four Majorana fermions and two complex scalars, $q$ and $\bar{q}$. We write the superfield as a doublet, $W = (Q, Q^\dagger)^T$. The chiral multiplets $Q$ and $\bar{Q}$ carry conjugate charges under the gauge group. For Theory A, we denote the charge of the hypermultiplets as $R^q_i$, while for Theory B it is $R^q_i; i = 1, \ldots, N; a = 1, \ldots, r; p = 1, \ldots, N - r$. Each of these matrices is assumed to be of maximal rank, and mirror symmetry requires

$$\sum_{i=1}^{N} R^2_i R^2_p = 0 \quad \forall \, a, p$$  \hspace{1cm} (1)

In $\mathcal{N} = 1$ theories, there are no holomorphic luxuries and interactions are determined in terms of a real superpotential, $f$. For Theory A, this superpotential has the cubic form associated with the $\mathcal{N} = 4$ theories, and is determined by a triplet of $k \times N$ matrices $T_{c, \alpha} = 1, 2, 3$

$$f = \sum_{i=1,c=1}^{N} \sum_{\alpha=1}^{k} W_{i}^\alpha \tau_c W_{i}\cdot T^{\alpha}_{c,\alpha} \Phi_{\alpha}$$  \hspace{1cm} (2)

where $\alpha = 1, \ldots, k$ and $\tau^c$ are the three Pauli matrices. A similar coupling exists for Theory B, now with the triplet of $(3N - k) \times N$ coupling matrices $T_{c, \alpha}$ satisfying

$$\sum_{i=1}^{N} \sum_{c=1}^{3} T^{\alpha}_{c,\alpha} T^{\beta}_{c,\beta} = 0 \quad \forall \, \alpha, \beta$$  \hspace{1cm} (3)

Further details of these theories, together with the methods used to derive them, can be found in [1]. Here let us restrict ourselves to a few comments. The Coulomb branch of Theory A has dimension $(N + r)$, which coincides with the dimension of the Higgs branch of Theory B. (The converse also holds). Mass and FI parameters may be added to both theories, partially lifting some branches of vacua, and the mirror map for these deformations is known.

**The $G_2$ Quotient Construction**

Let us now apply the mirror pairs described above to a D2-brane probe of a D6-brane background. We take $i = 1, \ldots, N$, flat D6-branes, each of which has spatial world-volume direction.

$$D_{ci} = 123[47][58]_6[69]_7$$

The D6-branes lie on a special lagrangian locus if each rotation is contained in $SU(3)$ [15] or, more simply, if

$$\theta^i + \theta^j + \theta^k = 0 \quad \text{mod} \, 2\pi \, \forall \, i$$  \hspace{1cm} (4)

ensuring that $\mathcal{N} = 1$ supersymmetry (4 supercharges) is preserved on their common world-volume. (For non-generic angles, more supersymmetry may be preserved. We will assume this is not the case).

As described in the introduction, we probe this configuration with a D2-brane lying in the $x^1 - x^2$ plane. This breaks supersymmetry by a further half, resulting in a $d = 2 + 1$ dimensional world-volume theory with $\mathcal{N} = 1$ supersymmetry (2 supercharges). For the singular case of intersecting, flat D6-branes, the theory on the D2-brane probe is simple to write down. The 2-2 strings give rise to the usual gauge field and seven scalars. Of these, there is one free $\mathcal{N} = 1$ scalar multiplet parameterizing motion in the $x^3$ direction common to all D6-branes. Further fields arise from the 2-6 strings. These give rise to $N$ hypermultiplets. Thus, we have the interacting $\mathcal{N} = 1$ supersymmetric theory on the probe,

**Theory A:** $U(1)$ with 6 scalar multiplets and $N$ hypermultiplets

where each hypermultiplet has charge +1 under the gauge field. The couplings of the hypermultiplets to the scalar multiplets are determined by the geometry of the D6-branes: each hypermultiplet couples minimally to the three scalar fields orthogonal to the corresponding D6-brane. If we define the scalar fields $\phi_{\alpha} = x^{\alpha + 3}$, $\alpha = 1, \ldots, 6$, then the superpotential is of the form (2) with the couplings determined by the D6-brane orientations,

$$T^{\alpha}_{\delta_{\alpha},\alpha} = -\sin \theta^\delta_{\alpha} \phi_{\delta_{\alpha}} \phi_{\alpha} + \cos \theta^\delta_{\alpha} \phi_{\delta_{\alpha}-3} \quad \alpha = 1, 2, 3$$  \hspace{1cm} (5)

From the IIA space-time picture, we are lead to the natural conjecture that the Coulomb branch of this theory, parameterized by the six real scalars $\phi_{\alpha}$, together with the dual photon $\sigma$, is a seven dimensional manifold $X$ that admits a metric of $G_2$ holonomy. However, this description of $X$ in terms of Coulomb branch variables is not overly useful. In particular, the isometries of $X$ are lost in the reduction to IIA, and are only expected to be recovered as isometries of the Coulomb branch in the strong coupling limit. It would be desirable to have an algebraic description of $X$, in which the isometries are manifest. This is exactly what the mirror theory provides for us.
Since Theory A is in the class of theories discussed above, we may simply write down the mirror theory whose Higgs branch is conjectured to give the $G_2$ manifold $X$.

**Theory B:** $U(1)^{N-1}$ with $3(N-2)$ scalar and $N$ hypermultiplets

The gauge couplings are determined by the $A_{N-1}$ quiver diagram: i.e. the $r^i$ gauge group acts on the $r^i$ hypermultiplet with charge $+1$, and the $(1,1)^{ab}$ hypermultiplet with charge $-1$. All other hypermultiplets are neutral. The Yukawa terms are of the form (2), with the triplet of coupling matrices $T$ determined by (3). The Higgs branch of this theory is parameterized by $w_i = (q_i, \tilde{q}_i)^T$, the $N$ doublets of complex scalars in the hypermultiplets. These are constrained by the $3(N-2)$ $D$-terms, modulo $(N-1)\ U(1)$ gauge quotients,

$$\sum_{i,j} T_{ij} w_i \sigma^j w_i = 0 \quad \rho = 1, \ldots, 3(N-2) \quad (6)$$

This quotient construction yields a conical manifold, which is expected to admit a metric of $G_2$ holonomy. To avoid the resolution of the right-hand side of (6). This blows up two-cycles, and in the IIA picture, corresponds to translating the D6-branes in the $\bar{x}^4 - \bar{x}^3$ directions. Note that when the Yukawa matrices $T$ fill into suitable $SU(2)$ triplets, the above method coincides with the toric hyperKähler quotient construction, supplemented by a further quotient by a tri-holomophic isometry to yield a manifold of dimension seven. This is the construction discussed by Acharya and Witten [5]. However, in general, the charges in (6) differ.

The above theory may also be considered as a $N = (1,1)$ supersymmetric linear sigma model in $d = 1+1$ dimensions, cf. [10]. However, in the absence of something akin to Yau’s theorem, we cannot be sure that the Ricci flat metric to which the theory flows has $G_2$ holonomy.

**An Example**

Let us now examine the $G_2$-quotient construction applied to a specific example. Our choice for consideration is the $G_2$ manifold $X$ given by the cohomology of the flag manifold $SU(3)/U(1)^2$ [17, 18]. This example was also discussed in detail by Aliyah and Witten [4]. They show that, with a suitable choice of $M$-theory circle, $X$ can be reduced to three, flat, intersecting D6-branes in type IIA string theory. The symmetry of $X$ (to be discussed below) together with the special lagrangian condition (4) determines the angles of these three branes to be $\theta_x = 0$, $\theta_y = 2\pi/3$ and $\theta_z = 4\pi/3$ for each $c$. The configuration is drawn in Figure 1.

In order to make the symmetries of the configuration manifest, we define two triplets of scalars, $\phi_1 = (x^7, x^8, x^9)^T$ and $\phi_2 = (x^5, x^6, x^3)^T$ in terms of which

![FIG. 1: Intersection of special Lagrangian D6-branes dual to M-theory on $G_2$ holonomy core over $SU(3)/U(1)^2$ (a), and its non-singular deformation (b).](image)

the orientation of the $i$th D6-brane can be described by the set of linear equations:

$$D6_1 : \quad \vec{\phi}_1 = 0
$$

$$D6_2 : \quad \frac{1}{2} \vec{\phi}_1 + \sqrt{3} \vec{\phi}_2 = 0
$$

$$D6_3 : \quad - \vec{\phi}_1 + \sqrt{3} \vec{\phi}_2 = 0
$$

The original $G_2$ holonomy manifold $X$ enjoyed an $SU(3)$ continuous isometry. It is not surprising that, upon taking the quotient to IIA string theory, this isometry is partially lost. In fact, the D6-brane background has only a $SU(2)$ symmetry, under which each $\vec{\phi}_i$ transforms as a triplet. The M-theory circle itself provides one further, hidden, $U(1)$ action. We therefore conclude that the reduction to IIA string theory has broken the isometry group to $SU(3) \to SU(2) \times U(1)$. Now, let us introduce a probe D2-brane in this background, and look at the $N = 1$ gauge theory on its world-volume.

**Theory A:** $U(1)$ with 6 scalars and 3 hypermultiplets

As described above, the 6 scalar multiplets combine into two triplets whose interactions with the hypermultiplets are of the form (2), where the interaction matrices are determined by (5). The Coulomb branch of this theory is parameterized by the six scalars, together with the dual photon. It has the $SU(2) \times U(1)$ isometry group, which is expected to be enhanced to the full $SU(3)$ only in the strong coupling, infrared limit.

Using the results described earlier, the mirror theory is the $N = 1$ gauge theory with matter content.

**Theory B:** $U(1)^2$ with 3 scalar and 3 hypermultiplets

The charges of the three hypermultiplets under the $U(1)^2$ gauge group are $+1, -1, -(0,0,0,0)$. The three scalars couple through the usual superpotential (2), with interactions determined by (3) and (5) to be $T_{ij}^c = \delta_{ij}^c$ for all $i$. Let us examine the Higgs branch of this theory. The superpotential provides 3 real constraints on the 12 real scalar fields contained in the hypermultiplets. After dividing by the gauge group, we are left with a Higgs branch of dimension 7, as required. The constraints are,

$$\sum_{i=1}^{3} \bar{q}_i \bar{q}_i = 0, \quad \sum_{i=1}^{3} \bar{q}_i q_i = 0 \quad (7)$$
Firstly notice that this space has a manifest $SU(3)$ symmetry, thus recovering the full isometry group of $X$. It is not difficult to further show that the space is indeed isomorphic to the cone over $SU(3)/U(1)^3$, ensuring that the full Higgs branch is the flag manifold $SU(3)/U(1)^3$.

There is a single normalizable deformation of this space, which yields a smooth $G_2$ manifold:

$$X \cong \mathbb{R}^3 \times \mathbb{C}P^2$$

(8)

In the D6-brane picture, the singularity is resolved by deforming the singular locus of flat, intersecting D6-branes into a smooth special Lagrangian curve $L \subset C^3$:

$$L \equiv \mathbb{R} \times S^2 \cup \mathbb{R}^3$$

(9)

In the present case this deformation involves only two out of the three D6-branes. To see this more explicitly, let us choose the first and second D6-branes, which deform to lie on the special Lagrangian curve:

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = -|\vec{\sigma}_1||\vec{\sigma}_2| \quad \quad (\vec{\sigma}_1 \cdot (3|\vec{\sigma}_1| - |\vec{\sigma}_2|) = \rho)$$

This curve has a remarkable property: it creates a hole through which the remaining D6-brane can pass, see Figure 1. Therefore, it suffices to deform only two of the three D6-branes in order to completely remove the conical singularity. Of course, one has three different ways to pick a pair of D6-branes, leading to three different resolutions of the space, meeting at a singular point. This is precisely the picture suggested in [4].

It is natural to ask how the probe theory responds to such a deformation. From the perspective of Theory A, one can show that there is essentially a unique deformation consistent with all the symmetries of the model; it is a Yukawa term coupling a pair of hypermultiplets. Moreover, the locus of zeroes of the fermion mass matrix has the same topology as the locus (10). For more details, see [1].

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