Anti-de Sitter Black Holes, Thermal Phase Transition and Holography in Higher Curvature Gravity

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Abstract

We study anti-de Sitter black holes and evaluate different thermodynamic quantities in the Einstein-Gauss-Bonnet and the general $R^2$ gravity theories. We examine the possibility of Hawking-Page type thermal phase transitions between AdS black hole and thermal anti-de Sitter space in such theories. In Einstein theory with a possible cosmological term, one observes a Hawking-Page phase transition only if the event horizon is a hypersurface of positive constant curvature ($k = 1$). But in Einstein-Gauss-Bonnet gravity there can occur a similar transition even for a horizon of negative constant curvature ($k = -1$), which may allow one to study the boundary conformal theory with different background geometries. For the Gauss-Bonnet black holes, one can relate the entropy of the black hole as measured at horizon to a variation of the geometric property of the horizon based on first law and Noether charge. With (Riemann)$^2$ terms, however, we can do this only approximately, and the two results agree in the limit $r_H \gg L$, the size of the horizon is much bigger than the AdS curvature. In (Riemann)$^2$ gravity, we establish certain relations between bulk data associated with the AdS black hole in five dimensions and boundary data defined on the horizon of the AdS geometry, in which case we do not expect a sensible holographic dual. We also give a heuristic approach to estimate the difference between Hubble entropy and Bakenstein-Hawking entropy with (Riemann)$^2$ term.

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1 Introduction

Anti-de Sitter black hole thermodynamics, which produces an aggregate of ideas from thermodynamics, quantum field theory and general relativity, is certainly one of the most remarkable tools to study quantum gravity in space-time containing a horizon [1]. In recent years there has been a great deal of attention in such black holes. Among different reasons, the one which has dominated others is the spirit of AdS/CFT correspondence [2], in particular, due to Witten’s interpretation [3, 4] of the Hawking-Page phase transition between thermal AdS and AdS black hole [5] as the confinement-deconfinement phases of the Yang-Mills (dual gauge) theory defined on the asymptotic boundaries of the AdS geometry. Many results in the literature are based on the Einstein theory with a negative cosmological constant, where one enjoys the well known Bakenstein-Hawking area-entropy law

\[ S = \frac{k_B c^3}{\hbar} \frac{A}{4G}, \]  

where \( A \) is volume of the horizon corresponding to the surface at \( r = r_H \). In the following, we adopt the standard convention of setting \( c = \hbar = k_B = 1 \). One of the impressive features of (1) is its universality to all kinds of black holes [6, 7] irrespectively of their charges, shapes and rotation. Nonetheless, it has been known that (1) is no longer applied to the higher curvature (HC) theories in \( D > 4 \) dimensions (see Ref. [8] for review).

On general grounds, any effective gravity action will involve, besides the usual Einstein term (and a possible cosmological term), the higher curvatures and also derivative terms corresponding to the low energy matter fields (see for example [9]). These high powers of curvature tensors have certain roles within the modern paradigm of effective field theories. When the effect of gravitational fluctuations are small compared to the large number of matter fluctuations, one can neglect graviton loops, and look for a stationary point of the combined gravitational action, and the effective action for the matter fields. This is implied by solving

\[ R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi G \langle T_{ab} \rangle, \]  

where the source being the expectation value of the matter energy momentum tensor, which may include the contribution from higher curvature terms. If we allow non-conformally invariant matter fields, one must take into account the non conformally invariant local terms [10], which in four dimensions read [11, 12]

\[ \langle T \rangle = \alpha_1 F - \alpha_2 G + \alpha_3 \nabla^2 R, \]  

where the Gauss-Bonnet invariant \( G \) and the square of Weyl tensor are

\[ G = R^2 - 4 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}, \]

\[ F = \frac{1}{3} R^2 - 2 R_{ij} R^{ij} + R_{ijkl} R^{ijkl}. \]  

Here \( \alpha_i \) are defined by certain combinations of the number of real scalars, Dirac fermions and vectors in that particular theory, \( \nabla^2 R \) is the variation of the local term \( \int d^4 x \sqrt{|g|} R^2 \) in effective action which
generally do not carry dynamical information. One can derive (3) for $d + 1 = 5$ in conformal anomaly from $R^2$ gravity using the AdS/CFT correspondence [13, 14]. The effective supergravity action also contain certain combination of higher curvature terms as corrections of large N expansion of boundary CFTs in the strong coupling limit [2, 12, 13].

Any effective stringy gravity action may include higher curvature terms of different order as loop corrections to string amplitudes. The suggestive combination of the higher derivative curvature terms is perhaps the Gauss-Bonnet(GB) invariant, which is attributed to the low energy effective string action [15, 16]. In this case, the resulting field equations contain no more than second derivatives of the metric tensor, thus the theory is free of ghost when expanding about the flat space. This is also true [17] also in the recently proposed Randall-Sundrum type warped geometry [18] (see [19] for discussion with GB term). It may be incorrect to assume that the higher derivative correction terms with “small” coefficients will just produce small modifications of the solution of the unperturbed (Einstein) theory [20]. Any higher curvature theory actually contains whole classes of new solutions as flat space and (anti-)de Sitter vacua (see [21] for a discussion in $D = 4$). The Einstein-Gauss-Bonnet theory in $D \geq 5$ clearly exhibits various new black hole solutions, which are unavailable to the classical Einstein theory (see Ref. [22]) for a recent work).

In this paper, we study the AdS black hole thermodynamics in $R^2$ gravity. If the event horizon of AdS black hole is a hypersurface with a zero ($k = 0$) or negative ($k = -1$) constant curvature, the black hole is always stable and the corresponding boundary field theory defined at finite temperature is always dominated by the black hole [4]. While for a constant positive curvature hypersurface ($k = 1$), one sees a Hawking-Page phase transition between AdS black hole and thermal AdS [5]. This is precisely the result one has in Einstein gravity, but in Einstein-Gauss-Bonnet theory we clearly see a possibility of similar phase transition for AdS black holes with hyperbolic horizons. For $\alpha \neq 0$, the thermal phase structures actually depend on the spatial dimensions $d$ and horizon geometry $k$.

We also study the perturbative AdS black hole solutions with general $R^2$ terms and discuss their thermodynamic behavior. Wald [23] has shown that one can relate the variations in properties of the black hole as measured at horizon to the variations of the geometric property of the horizon based on the first law and evaluation of the Noether charge (see Refs. [24, 25, 26] for a clear generalization). This can be realized for Gauss-Bonnet black hole, in which case we also have the exact solutions. But in (Riemann)$^2$ gravity, the above prediction is only a good approximation. In particular, the two entropies we find with (Riemann)$^2$ term would be in a close agreement in the limit $r_H >> L$, while, they completely agree for $k = 0$.

A version of the AdS/CFT correspondence asserts that physics in the bulk of AdS spacetime is fully described by a CFT on the boundary, an intuitive notion holography [27]. As a result, in Einstein theory, with a possible negative cosmological constant, the thermodynamic quantities of the holographic dual CFT theory defined on $S^3 \times S^1$ at high temperature can be identified with those of the
bulk AdS Schwarzschild black holes with spherical horizons \( k = 1 \) [4]. In fact, the higher derivative terms should correspond to finite coupling effects in the CFT, which then generally allow terms squares in Ricci scalar, Ricci tensor and Riemann tensor in the effective action. But, a boundary field theory defined with \((\text{Riemann})^2\) term could be non-conformal, and besides that, such term is known to give spin-two ghost in the bulk theory, thus one cannot expect a sensible holographic dual in the latter case. Nonetheless, we consider this term to study AdS black hole thermodynamics, and also to justify this as an example of non-CFT/AdS dual by evaluating different thermodynamic quantities.

The paper is organized as follows. In next section we shall begin with our effective action and present some curvature quantities for a general metric ansatz. Section 3 deals in detail with the Gauss-Bonnet black hole thermodynamics in AdS space, including thermal phase structures. In section 4, we study AdS black holes with \( \gamma = 0 \) (trivial \((\text{Riemann})^2\) interaction) and evaluate different thermodynamic quantities. In section 5, we shall begin with our discussion of the black hole thermodynamics in \((\text{Riemann})^2\) gravity, where we present formulas for free energy, entropy and the energy. In section 6, we will present certain realizations of the FRW-type brane equations. A comparison between Bakenstein-Hawking entropy of the black hole and the Hubble (or holographic) entropy will be made. Section 7 contains conclusions.

2 Action, metric ansatz and curvature quantities

Perhaps, a natural tool to explore the AdS/CFT as well as AdS/non-CFT correspondences is to implement the general higher derivative terms to the effective effect, and to study the thermodynamics of the anti-de Sitter black holes. To begin with, we consider the following \((d + 1)\) dimensional gravitational action containing terms up to quadratic in the curvatures

\[
I = \int d^{d+1}x \sqrt{-g_{d+1}} \left[ \frac{R}{\kappa_{d+1}} - 2\Lambda + \alpha R^2 + \beta R_{ab}R^{ab} + \gamma R_{abcd}R^{abcd} + \cdots \right] + \frac{2}{\kappa} \int_{\partial\mathcal{B}} d^dx \sqrt{|g(d)|} K + \cdots ,
\]

(5)

where \(\kappa_{d+1} = 16\pi G_{d+1}\), \(K = K_a^a\) is the trace of the extrinsic curvature of the boundary, \(K^{ab} = \nabla^a n^b\), where \(n^a\) is the unit normal vector on the boundary. The second action above is attributed to the Gibbon-Hawking boundary action. When working in \((d+1)\)-dimensional anti-de Sitter space \((\Lambda < 0)\), one may drop the surface terms including the Gibbon-Hawking action. However, these terms, including the higher order, might be essential to evaluate the conserved quantities [28] when the solutions are extended to de Sitter (dS) spaces.

Let us define the metric ansatz in the following form

\[
ds^2 = -e^{2\phi(r)} dt^2 + e^{-2\phi(r)} dr^2 + r^2 \sum_{i,j} h_{ij} dx^i dx^j ,
\]

(6)

where \(h_{ij}\) is the horizon metric for a manifold \(\mathcal{M}^{d-1}\) with the volume \(V_{d-1} = \int d^{d-1}x \sqrt{h}\). For (6),
the non-vanishing components of the Riemann tensor are
\[ R_{trtr} = e^{2\phi(r)}(\phi'' + 2\phi' r) , \quad R_{titj} = e^{4\phi(r)} r \phi' h_{ij} = -e^{4\phi(r)} R_{rirj} , \]
\[ R_{ijkl} = r^2 R_{ijkl}(h) - r^2 e^{2\phi(r)} (h_{ik} h_{jl} - h_{il} h_{jk}) . \] (7)

We readily obtain the following non-trivial components of the Ricci tensor
\[ R_{tt} = -e^{4\phi(r)} R_{rr} = e^{4\phi(r)} \left( \phi'' + 2\phi' r + \frac{(d-1)\phi'}{r} \right) \]
\[ R_{ij} = \mathcal{R}_{ij}(h) - e^{2\phi(r)} \left( (d-2) + 2r\phi' \right) h_{ij} , \] (8)

where \( \mathcal{R}_{ij} = (d-2)k h_{ij} \) with \( k \) being the curvature constant, whose value determines the geometry of the horizon. The boundary topology of the Einstein space \( (\mathcal{M}^{d-1}) \) looks like
\[ k = 1 \rightarrow S^{d-1} : \text{Euclidean de Sitter space (sphere)} \]
\[ k = 0 \rightarrow \mathbb{R}^{d-1} : \text{flat space} \]
\[ k = -1 \rightarrow H^{d-1} : \text{anti-de Sitter space (hyperbolic)} . \] (9)

This means that the event horizon of the black hole can be a hypersurface with positive, zero or negative curvature. Note, for the spherically symmetric black holes, the event horizon is generally a sphere surface with \( k = 1 \). While, if the horizon is zero or negative constant hypersurface, the black holes are referred as topological black holes. The thermodynamics of the topological and asymptotically anti-de Sitter black holes in Einstein’s theory were investigated in Refs. [29, 30, 31, 32, 33]. It would be essential to include the higher curvature terms in order to better understand the thermodynamic behavior, including thermal phase structure.

Before going forward, let us assume that the \((d+1)\)-dimensional spacetime is an Einstein space
\[ R_{abcd} = -\frac{1}{\ell^2} (g_{ac}g_{bd} - g_{ad}g_{bc}) , \quad R_{ab} = -\frac{d}{\ell^2} g_{ab} , \] (10)

where \( \ell^2 > 0 \) \((< 0)\) if \( \Lambda < 0 \) \((> 0)\). This is always possible provided that the horizon geometry is also an Einstein space [32]
\[ \mathcal{R}_{ijkl}(h) = k (h_{ik} h_{jl} - h_{il} h_{jk}) , \quad \mathcal{R}_{ij}(h) = (d-2)k h_{ij} . \] (11)

Then one easily computes the Ricci scalar in \((d+1)\) spacetime dimensions
\[ R = \frac{(d-1)(d-2)k}{\ell^2} - e^{2\phi(r)} \left( 2\phi'' + 4\phi' r + \frac{4(d-1)\phi'}{r} + \frac{(d-1)(d-2)}{r^2} \right) \]
\[ = \frac{(d-1)(d-2)k}{\ell^2} - \frac{1}{r^{d-1}} \left( r^{d-1} e^{2\phi(r)} \right)'' . \] (12)
Let us set at first $\alpha = -\beta/4 = \gamma$ in (5), and also drop the Hawking-Gibbon term. Then the equations of motion following from (5) simply read

$$\kappa_{d+1}^{-1} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) + \Lambda g_{ab} - \frac{\alpha}{2} g_{ab} R_{GB}^2 + 2\alpha \left( R R_{ab} - 2 R_{abcd} R^{cd} + R_{acde} R^{cde} - 2 R_a c R_{bc} \right) = 0$$

(13)

with the Gauss-Bonnet invariant $R_{GB}^2 = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}$. The explicit form of the metric solution following from (13) is (see also the Refs. [15, 34, 35, 22])

$$e^{2\phi} = k + \frac{r^2}{2\hat{\alpha}} + \epsilon \frac{r^2}{2\alpha} \left[ 1 - \frac{4\hat{\alpha}}{\ell^2} \left( 1 - \hat{\alpha} \right) + \frac{4\hat{\alpha} m}{r^d} \right]^{1/2},$$

(14)

where $\epsilon = \mp 1$, $\hat{\alpha} = (d-2)(d-3)\alpha \kappa_{d+1}$ and $m$ is an integration constant with dimensions of $(\text{length})^{d-2}$, which is related to the ADM mass $M$ of the black hole via

$$M = \frac{(d-1) V_{d-1}}{\kappa_{d+1}} m.$$

(15)

Here $V_{d-1} = \int d^{d-1} x \sqrt{h}$ is the volume of the spatial $(d-1)$ dimensional constant curvature manifold $\mathcal{M}^{d-1}$. We should note that the AdS curvature squared term $\ell^2 (\equiv -\ell_{dS}^2)$ is related to the cosmological constant $\Lambda$ via

$$\Lambda = -\frac{d(d-1)}{2 \kappa_{d+1} \ell^2}, \quad \text{where} \quad \frac{1}{\ell^2} = \frac{1}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right),$$

(16)

so that $\ell^2 > 0$ for $\Lambda < 0$ (anti-de Sitter), while $\ell^2 < 0$ for $\Lambda > 0$ (de Sitter). Notice that there are two branches in the solution (14), because $e^{2\phi(r)}$ is determined by solving a quadratic equation, we denote by $r_+$ the $\epsilon = -1$ branch, and by $r_-$ the $\epsilon = +1$ branch. For large $r$

$$-g_{00}(r_+) \sim k + \frac{m}{r^{d-2}} + \frac{r^2}{\ell^2} + \mathcal{O}(r^{4-2d}),$$

(17)

$$-g_{00}(r_-) \sim k + \frac{m}{r^{d-2}} - \frac{r^2}{\ell^2} + \frac{r^2}{\hat{\alpha}} + \mathcal{O}(r^{4-2d}).$$

(18)

For $\ell^2 > 0$, in the limit $r \rightarrow \infty$, the $\epsilon = -1$ branch gives the $(d+1)$-dimensional Schwarzschild anti-de Sitter (SAdS) solution, while the $\epsilon = +1$ branch gives the Schwarzschild de Sitter (SdS) solution with a negative gravitational mass if $m < 0$. However, the lower branch ($\epsilon = +1$) solution corresponds to an unstable region for certain parameter values of $\hat{\alpha}$ and $m$ [15]. Besides that, in order to reconcile the above solution with that of $\hat{\alpha} = 0$, one finds the upper branch ($\epsilon = -1$) as the physical one in flat or anti-de Sitter spaces. At any rate, we consider in this paper only the exact solution (14), rather than its perturbative cousins (17, 18).
3.1 Thermodynamic quantities

From (14), the mass of the black hole $M$ can be expressed in terms of the horizon radius $r_+ = r_H$

$$M = \frac{(d-1) V_{d-1} r_+^{d-2}}{\kappa_{d+1}} \left[ k + \frac{r_+^2}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) + \frac{\hat{\alpha} k^2}{r_+^2} \right], \quad (19)$$

where, as defined earlier, $\hat{\alpha} = (d-2)(d-3)\alpha\kappa_{d+1}$. The positions of the horizons may be determined as the real roots of the polynomial $q(r = r_H) = 0$, where

$$q(r) = kr^{d-2} + \hat{\alpha} k^2 r^{d-4} + \frac{r^d}{\ell^2} - m. \quad (20)$$

Of course, in the limit $\ell^2 \to \infty$ (i.e. $\Lambda \to 0$) and $k = 1$, one recovers the Eq. (5) of Ref. [34]. One derives $q(r) = 0$ directly from Eq. (14) or from the original field equations by setting $e^{2\phi(r)} = 0$ at the horizon $r = r_H$, but we always satisfy $e^{2\phi(r)} > 0$ for $r > r_H$. When $d + 1 = 5$, the black hole horizon is at

$$r_H^2 = \frac{l^2}{2} \left[ -k + \sqrt{k^2 + \frac{4(m - \hat{\alpha} k^2)}{l^2}} \right]. \quad (21)$$

As noted in [22], there is a mass gap at $m = \hat{\alpha}k^2$, so all black holes have a mass $M \geq 3V_{d-1}k^2/\kappa_{d+1} \equiv M_0$, the requirement $M > M_0$ is needed to have a black hole interpretation.

From Eq. (14) and Eq. (20), we easily see that the roots of $q(r)$ must also satisfy

$$1 + \frac{4\hat{\alpha}^2 k^2}{r^4} + \frac{4\hat{\alpha} k}{r^2} \geq 0. \quad (22)$$

When $k \geq 0$, this is trivially satisfied in any spaces ($\ell^2 > 0$ (AdS), $\ell^2 < 0$ (dS), and $\ell^2 = 0$ (flat)), since $\hat{\alpha} > 0$. If the horizon of the black hole is a hypersurface with a negative curvature ($k = -1$), one has

$$1 - \frac{4\hat{\alpha}}{r^2} \left( 1 - \frac{\hat{\alpha}}{r^2} \right) \geq 0. \quad (23)$$

This implies that $r^2 \geq 2\hat{\alpha}$ at the horizon $r = r_H$, which indeed defines the minimum size of the black hole horizon in an asymptotically anti-de Sitter space. For $k = +1$, Eq. (22) would rise to give $r_H^2 \geq -2\hat{\alpha}$, which was first noticed by Myers and Simon [34]. Note, for $k = 0$, the horizon size of black hole is not constrained in terms of $\hat{\alpha}$.

To study the black-hole thermodynamics, it is customary to find first the Euclideanized action by analytic continuation. That is, after Wick-rotating the time variable $t \to i\tau$, one regularizes Euclidean section $\mathcal{E}$ by identifying the Killing time coordinate with a period $\tau = \beta_0$. One subtracts energy of the reference geometry [5, 36], which is simply anti-de Sitter space produced by setting $M = 0$ in (14). The Euclideanized action $\hat{I}$ therefore reads

$$\hat{I} = -\frac{V_{d-1} r_H^{d-4}}{\kappa_{d+1} (d-3)} \left[ (d-1)\beta_0 (kr_H^2 - \hat{\alpha} k^2) - 8\pi r_H^3 + 3(d-1)\beta_0 \frac{r_H^4}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right]. \quad (24)$$
where $\beta_0 = 1/T$ is the periodicity in Euclidean time. With $\ell^2 = \infty$ (i.e. $\Lambda = 0$) and $k = 1$, we correctly reproduce the result in [34]. Since the temperature of the black hole horizon has been identified by the periodicity in imaginary time of the metric, $T$ is identified by the Hawking temperature ($T_H$) of the black hole defined by $(e^{2\phi(r)})^T |_{r=r_+} = 4\pi T_H$, where $r_+ = r_H$. Hence

$$\frac{1}{\beta_0} = T_H = \frac{(d-2)}{4\pi r_+} \frac{1}{(r_+^2 + 2\hat{\alpha} k)} \left[ k r_+^2 + \frac{d-4}{d-2} \hat{\alpha} k^2 + \frac{d}{d-2} \frac{r_+^4}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right].$$

(25)

We plot the inverse temperature of the black hole for the case $\hat{\alpha} = 0$ in Fig. (1), and also for the $\hat{\alpha} \neq 0$ case with $k = 1$ in $d = 3, 4, 5$, and $d = 9$ in Fig. (2). In term of temperature, we clearly see that only in $d + 1 = 5$ dimensions there occurs a new phase of locally stable small black hole with $k = 1$. In Einstein gravity, one simply discards the region $\beta_0 \to 0$ as $r \to 0$, because thermodynamically it is an unstable region [5, 4]. In EGB gravity, however, in $d + 1 = 5$ one finds both conditions as physical ones, i.e., $\beta_0 \to \infty$ as $r \to 0$, and $\beta_0 \to 0$ as $r \to \infty$.

As in [5], we identify the Euclidean action with the free energy times $1/T$. Hence

$$F = -\frac{V_{d-1} r_+^{d-4}}{\kappa_{d+1} (d-3)} \left[ (d-1) \left( k r_+^2 - \hat{\alpha} k^2 + 8\pi r_+^3 T_H + 3(d-1) \frac{r_+^4}{\ell^2} \right) \right]
= \frac{V_{d-1} r_+^{d-4}}{\kappa_{d+1} (d-3)} \frac{1}{(r_+^2 + 2\hat{\alpha} k)} \left[ (d-3) r_+^4 \left( k - \frac{r_+^2}{\ell^2} \right) - \frac{6(d-1) \hat{\alpha} k r_+^4}{\ell^2} + (d-7) \hat{\alpha} k^2 r_+^2 + 2(d-1) \hat{\alpha}^2 k^3 \right],$$

(26)

where $\ell^2 = \ell^2 (1 - \hat{\alpha}/\ell^2)^{-1}$. In the second line above we have substituted the value of $T_H$ from (25). Interestingly, the free energy (26) was obtained in Ref. [22] by using the thermodynamic relation $F = M - TS$, where entropy $S$ was evaluated there using $S = \int_0^{r_+} T^{-1} dM$. So these two apparently different prescriptions for free energy ($F$ read from the Euclideanized action and $F$ derived from the first law) give the same results. One therefore computes the energy

$$E = \frac{\partial \widehat{I}}{\partial \beta_0} = M,$$

(27)

where $M$ is still given by (19), and the entropy

$$S = \beta_0 E - \widehat{I} = \frac{4\pi V_{d-1} r_+^{d-1}}{\kappa_{d+1}} \left[ 1 + \frac{(d-1)}{(d-3)} \frac{2\hat{\alpha} k}{r_+^2} \right].$$

(28)

The minimum of the Hawking temperature is given by solving $\partial T_H/\partial r_+ = 0$, where

$$\frac{\partial T_H}{\partial r_+} = \frac{1}{4\pi} \frac{1}{(r_+^2 + 2\hat{\alpha} k)^2} \left[ -(d-2) k r_+^2 + \frac{d r_+^4}{\ell^2} + \frac{6d \hat{\alpha} k r_+^2}{\ell^2} - (d-8) \hat{\alpha} k^2 - \frac{2(d-4) \hat{\alpha}^2 k^3}{r_+^2} \right].$$

(29)

If $\hat{\alpha} = 0$, for $k = 0$ and $k = -1$, one easily sees that there is no minimum of temperature, thus $k = 0$ and $k = -1$ black holes exist for all temperatures. But the situation is different for $\hat{\alpha} \neq 0$. Below we implement these results to investigate the thermal phase transition between AdS black hole and thermal AdS space.
As noticed by Hawking and Page [5], black hole at high temperature is stable, while it is unstable at low temperature, and there may occur a phase transition between thermal AdS and AdS black hole at some critical temperature \( T_c \). Witten [4] interpreted this behavior as the confinement-deconfinement transition in dual gauge theory from AdS/CFT vantage point. The most radical point in Witten’s interpretation is that the thermodynamics of the black hole corresponds to the thermodynamics of the strongly coupled super-Yang-Mills (SYM) theory in the unconfined phase, while the thermal anti-de Sitter corresponds to the confined phase of the gauge theory. In a dual field theory description [4] black hole dominates the path integral when the horizon \( r_H \) is large compared to \( \ell \) (i.e. \( F<0 \)), while thermal AdS geometry dominates for sufficiently low temperature, where \( F>0 \).

At first we briefly review some known results in Einstein’s theory, which are obtained by setting \( \alpha = 0 \) in the above expressions of the Hawking temperature \( T_H \) and the free energy \( F \), and read

\[
T_H = \frac{(d-2)}{4\pi r_H} \left( k + \frac{d}{d-2} \frac{r_H^2}{\ell^2} \right),
\]

\[
F = \frac{V_3 r_H^{d-2}}{\kappa_5} \left( k - \frac{r_H^2}{\ell^2} \right).
\]

When \((d+1) = 5\), the black hole horizon is at

\[
r_H^2 = r_1^2 = \frac{\ell^2}{2} \left( -k + \sqrt{k^2 + 4m \ell^2} \right), \quad m = r_H^2 \left( k + \frac{r_H^2}{\ell^2} \right).
\]

The minimum of \( T_H \) occurs at \( r_H = \ell \sqrt{k/2} \), which implies that \( T_{min} = \sqrt{2k/(\pi \ell)} \). Note that BH cannot exist when \( T < T_{min} \), and global AdS space is preferred below \( T_{min} \). By definition, free energy of the global AdS vacuum is zero, but from (31), \( F = 0 \) only at \( r_H = r_2 = \ell \sqrt{k} \). The critical Hawking temperature at \( r_H = r_2 \) is therefore \( T_H = T_c = 3\sqrt{k} / (2\pi \ell) \). Thus, in Einstein gravity, the \( k = 0 \) and \( k = -1 \) cases are of no interest to explain the phase transition. When \( k = 1 \), the above temperature \( T_c \) defines a possible Hawking-Page phase transition. Specifically, when \( T_H > T_c \), the BH free energy is negative, thereby implies a stable AdS black hole.

Hawking-Page Transition in Einstein-Gauss-Bonnet Gravity

In this subsection, we allow a non-trivial \( \alpha \). It is suggestive to consider the \( d+1 \geq 6 \) case separately, since the properties of the solutions might differ from the the special case \( d+1 = 5 \), for example, when \( d = 4 \), the second term in (25) drops out (see also Fig. (2). In this paper we consider only the \( d+1 = 5 \) case for two reasons: from \( AdS_5/CFT_4 \) perspective, and other than the \( d+1 = 5 \) case, there is no new phase transition of locally stable small black hole. With \( d = 4 \), one has

\[
F = -\frac{V_3}{\kappa_5} \left[ 3k (r_H^2 - \hat{\alpha} k) - 8\pi r_H^3 T_H + \frac{9r_H^4}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right],
\]
where $\hat{\alpha} = 2\alpha\kappa_5$. Thus, free energy can be zero at the critical Hawking temperature

$$T_H = T_c = \frac{1}{8\pi r_H} \left[ 3k \left( 1 - \frac{\hat{\alpha} k}{r_H^2} \right) + \frac{9r_H^2}{\ell^2} \left( 1 - \frac{\hat{\alpha}}{\ell^2} \right) \right].$$

(34)

For $d = 4$, the minimum of $T_H$ is given by solving

$$r_+^2 \left( k - \frac{2r_+^2}{\ell^2} \right) + 2\hat{\alpha} k \left( k - \frac{6r_+^2}{\ell^2} \right) = 0,$$

(35)

provided $(r_+^2 + 2\hat{\alpha} k) \neq 0$. In the following, it would be suggestive to consider the $k = 0, 1$ and $-1$ cases separately. Regarding the thermal phase transitions, we will be brief in this paper, leaving behind the details on confining-deconfining phases in $(d + 1) \geq 5$ dimensions and corresponding dual gauge theory description for a forthcoming publication. We simply note that the $k = 1$ case is still the most interesting one, since the boundaries of the bulk manifold has an intrinsic geometry that of that background of the boundary field theory at finite temperature.

**$k = 0$ case:** In Einstein’s gravity ($\hat{\alpha} = 0$), the free energy, Eq. (31), is always negative, thus one has a stable AdS black hole solution. For $\hat{\alpha} \neq 0$ case, free energy (33) is zero (from Eq. (25) with $(d + 1) = 5$ and Eq. (34), only if

$$1 - \frac{\hat{\alpha}}{\ell^2} = 0.$$  

(36)

At a critical value $\ell^2 = \hat{\alpha}$, one has $T_H = T_c = 0$. For $\ell^2 > \hat{\alpha}$, since $F < 0$ is always satisfied, the black hole with $k = 0$ is always stable. While, if $\ell^2 < \hat{\alpha}$, free energy (33) appears to be positive, but this limit is not allowed for AdS black hole, for with $\ell^2 < \hat{\alpha}$ the sign of $\Lambda$ in Eq. (16) becomes positive, and the black hole mass in Eq. (19) becomes negative. Thus $\ell^2 \geq \hat{\alpha}$ puts a limit for a flat ($k = 0$) AdS black hole. We see no evidence of a phase transition with $k = 0$ even for $\hat{\alpha} \neq 0$. This behavior is not changed even for $(d + 1) > 5$, unlike the case will be with $k \neq 0$.

**$k = +1$ case:** For $k = +1$, the size of the AdS black hole is constrained by $r_H^2 \geq -2\hat{\alpha}$. Thus to evaluate the behavior at the critical horizon $r_H^2 = -2\hat{\alpha}$, one requires $\hat{\alpha} < 0$, since $r_H$ has to be positive. At $r_H^2 = -2\hat{\alpha}$, free energy, Eq. (33), is zero when $r_H^2 = -\ell^2$. This is suggestive, and to understand it properly, one replaces $\ell^2$ by $-\ell_{dS}^2$, and also $r_H$ by the cosmological horizon $r_c$, so the solutions then correspond to de Sitter black hole, but we will not study such black holes here.

Now from the Eqs. (25) and (34), for $d = 4$, we have $F \geq 0$, if

$$\left( 1 - \frac{r_+^2}{\ell^2} \right) - \frac{3\hat{\alpha}}{r_+^2} \left( 1 - \frac{2\hat{\alpha}}{r_+^2} + \frac{6r_+^2}{\ell^2} \right) \geq 0.$$  

(37)

When $\hat{\alpha} = 0$, $F = 0$ at $r_H = r_+ = \ell$ (also note that with $\hat{\alpha} = 0$, $\ell^2 = \ell^2$), while $F < 0$ if $r_+ > \ell$, and $F > 0$ if $r_+ < \ell$. For $\hat{\alpha} \neq 0$, however, the sign for free energy $F$ is determined from the above inequality. Notably, $F > 0$ if $T < T_c$, while $F < 0$, if $T > T_c$. We should note that a requirement of $T > T_c$ to ensure $F < 0$ is a complementary condition for $k = 1$ or $k = 0$, but with $k = -1$, free
energy can be positive even for $T > T_c$ (see below or Figs. (7,8)). Like the case with $\hat{\alpha} = 0$, the $k = 1$ is still the most probable case for a possible Hawking-Page phase transition in Einstein-Gauss-Bonnet gravity (see Fig. (6)). As seen in Fig. (6), a small Gauss-Bonnet black hole with a spherical horizon, which has a small positive free energy at start, evolves to thermal AdS phase, attains a maximum positive free energy at some $r = r_+$, and eventually attains a stable black hole phase for large $r$.

$k = -1$ case: In this case, the size of the AdS black hole is constrained by $r_H^2 \geq 2\hat{\alpha}$, which defines the minimum size of the AdS BH. For $k = -1$, the $F \leq 0$ condition reads

$$
\left(1 + \frac{r_+^2}{l^2}\right) + \frac{3\hat{\alpha}}{r_+^2} \left(1 + \frac{2\hat{\alpha}}{r_+^2} - \frac{6r_+^2}{l^2}\right) \geq 0 .
$$

(38)

For $\hat{\alpha} = 0$, $F$ is always negative, thus a hyperbolic AdS black hole is always stable with $\hat{\alpha} = 0$. However, with $\hat{\alpha} \neq 0$, the situation is different. At the critical radius $r_+^2 = 2\hat{\alpha}$, which indeed defines the minimum size of the horizon, free energy (33) becomes zero at $r_+^2 = \ell^2$, but $F$ is always negative for $r_+^2 > 2\hat{\alpha}$ and $r_+^2 > \ell^2$. One sees that $F$ can be positive for $2\hat{\alpha} \leq r_+^2 < \ell^2$. Thus unlike in Einstein’s theory, there can occur a Hawking-Page transition even for hyperbolic ($k = -1$) anti-de Sitter black holes (see Figs. (8) and (9)).

Note that with $\hat{\alpha} = 0$ (since $\ell^2 = \ell^2$ for $\hat{\alpha} = 0$) and $k = -1$, there is a finite minimum radius $r_+ = \ell$ at which $g_{rr}$ diverges for an AdS metric. One therefore considers there $r_+^2 > \ell^2$. With $\hat{\alpha} \neq 0$ and $k = -1$, however, we have another singularity at $r = r_s$, where

$$
r_s^d \left(1 - \frac{4\hat{\alpha}}{l^2}\right) - 4\hat{\alpha} r_s^{d-2} \left(1 - \frac{r_s^2}{l^2}\right) = 0 ,
$$

(39)

other than at $r = 0$, and both are shielded by $r_+$. It is interesting that there exists a region of $F > 0$ before hitting the singularity at $r = r_s$. We also note from Fig. (9) that under a critical limit $0.48 < \hat{\alpha}/\ell^2 < 0.52$, free energy is always positive, which corresponds to thermal AdS phase. Hence, $k = -1$ allows one to study the boundary conformal theory with different background geometries. Due to a possible phase transition for black holes with a hyperbolic horizon, one may find it particularly amusing that when the hypersurface is $AdS_3 \times S^1$ ($AdS_3$ may be obtained by analytic continuation of $dS_3$) quantum gravity in $AdS_5$ can be dual to a boundary conformal field theory on $AdS_4$ background. This possibly reflects that the geometry on the boundary is not dynamical, since there are no gravitational degrees of freedom in the dual CFT [37].

4 Thermodynamic quantities with $\gamma = 0$

For $\gamma = 0$, the field equations following from (5), integrate to give the metric solution (6) with [38]

$$
e^{2\phi(r)} = k + \frac{r^2}{L^2} - \frac{\mu}{r^{d-2}} .
$$

(40)
The curvature squared $L^2$ in (40) is related to the cosmological term $\Lambda$ via

$$\Lambda = -\frac{d(d-1)}{2\kappa_{d+1} L^2} \left(1 - \frac{(d-3)\varepsilon}{2(d-1)}\right), \quad (41)$$

where

$$\varepsilon = \frac{2d(\beta + (d+1)\alpha)}{L^2}. \quad (42)$$

While, the integration constant $\mu$ is related to the black mass $M = (d-1)V_{d-1} \mu/\kappa_{d+1}$. When $\mu = 0$, the solution (40) locally corresponds to AdS metric, while $\mu \neq 0$ gives AdS black hole solutions. For $d + 1 = 5$, the horizon radius $r_H$ is given by (32), replacing $m$ by $\mu$, and $\ell$ by $L$.

For the background metric (40), the classical action takes the form

$$I = -\frac{V_3}{2\pi G_5 T} \int_{r_H}^{\infty} dr \, r^3 \left(1 - \varepsilon\right), \quad (43)$$

where $V_3$ is the volume of the manifold $M^3$ and $\varepsilon = 8\kappa_5 (\beta + 5\alpha)/L^2$. After a proper regularization, the Euclideanized action is identified with the free energy ($F$) times $1/T$. Following [6, 5], the free energy ($F$) and the Hawking temperature $T_H$ evaluated for $d = 4$ read

$$F = \frac{V_3 r_H^2}{16\pi G_5} \left(k - \frac{r_H^2}{L^2}\right) \left(1 - \varepsilon\right), \quad (44)$$

$$T_H = \frac{1}{4\pi} \left[e^{2\phi(r)}\right]'|_{r = r_H} = \frac{1}{\pi L^2} \left(r_H + \frac{k L^2}{2r_H}\right). \quad (45)$$

If one defines $\tilde{G}_5 \left(1 - \varepsilon\right) = G_5$ [38], entropy and energy take the usual form

$$S = -\frac{dF}{dT_H} = \frac{V_3 r_H^3}{4G_5}, \quad E = F + TS = \frac{3V_3 \mu}{16\pi G_5} \equiv M. \quad (46)$$

Thus energy of the AdS black hole can be identified simply by mass of the black hole, in terms of a renormalized Newton constant $\tilde{G}_5$. Moreover, these quantities may be identified, up to a conformal factor, except for entropy which is not subject to this rescaling, with the same quantities defined on the boundary of AdS$_5$. It is not unexpected that AdS gravity with $\gamma = 0$ is conformal, rather this may be related to AdS/CFT trace anomaly [13], since the leading supergravity contribution to the trace anomaly involves only the squares of the Ricci scalar and Ricci tensor of the boundary metric.

### 4.1 The role of boundary terms

For a definiteness, we work in $d + 1 = 5$, and define $e^{-\sigma(y)} = r$. Then we can bring the metric (6) in the following Randall-Sundrum type five-dimensional warped metric

$$ds^2 = e^{-2\sigma(y)} \left(-dt^2 + \sum_{i=1}^{3} \gamma_{ij} dx^i dx^j\right) + dy^2, \quad (47)$$

The related coordinate transformations are given in Ref. [38], which read: $e^{2\phi t^2} - e^{-2\phi e^{-2\phi}} = e^{-2\phi}; e^{2\phi \ell^2} - e^{-2\phi e^{-2\phi}} = 0$, and $e^{2\phi t^2} - e^{-2\phi e^{-2\phi}} = 1$, where $\sigma = \partial_t, \sigma' = \partial_y \sigma$. 

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where \( y \) denotes an extra (fifth) space transverse to the brane, which picks out a preferred family of hypersurfaces, \( y = \text{const} \). To make the role of the “brane” dynamic, one also adds to (5) the following boundary term, corresponding to the vacuum energy on the hypersurface,

\[
I_{\partial B} = \int_{\partial B} d^d x \sqrt{|g(4)|} \left( -T \right) + \cdots ,
\]

where, the ellipsoids represent the higher order surface terms, and \( \partial B \) denotes the hypersurface with a constant extrinsic curvature. To the leading order, one may drop the higher order surface terms by imposing certain restriction on their scalar invariants (see for example [39]), and this is the choice we adopt here. The extrinsic curvature \( K_{ab} \) can be easily calculated from (47). Then the variation \( \delta I + \delta I_{\partial B} \) at the 4d boundary would rise to give

\[
\delta I + \delta I_{\partial B} = \int d^4 x \sqrt{|g(4)|} \left[ \left( \frac{8(1 - \varepsilon)}{\kappa_5} - \frac{8}{\tilde{\kappa}} \right) \delta \sigma' + 4 \left( \frac{8}{\tilde{\kappa}} \sigma' - \frac{2(1 - \varepsilon)}{\kappa_5^2} \sigma' + \frac{T}{2} \right) \delta \sigma \right]
\]

As in [38], let us assume that the gravitational couplings \( \kappa \) and \( \tilde{\kappa} \) are related by

\[
\frac{1}{\tilde{\kappa}} - \frac{1 - \varepsilon}{\kappa_5} = 0 \Rightarrow \tilde{G} = \frac{G}{(1 - \varepsilon)},
\]

so that the first bracket in (49) vanishes. The dynamical equations on the brane then reduce to

\[
T = -\frac{12(1 - \varepsilon)}{\kappa_5} \sigma'|_{y=0^+} = \frac{12(1 - \varepsilon)}{\kappa_5 L} \quad \text{and} \quad \Lambda = -\frac{6}{\kappa_5 L^2} \left( 1 - \frac{\varepsilon}{6} \right),
\]

where \( \varepsilon \) is given by (42), and in the second step \( \sigma'|_{y=0^+} = -1/L \) has been used. The condition \( \sigma' = -1/L \) seems useful to recover the RS-type fine tunings.

5 Thermodynamic quantities with \( \gamma \neq 0 \)

In AdS/CFT, the higher curvature terms in the bulk have coefficients that are uniquely determined [12, 13], thus it is suggestive to specify the coefficient \( \gamma \). For example, one computes the trace anomaly of a \( D = 4, \mathcal{N} = 2 \) SCFT in \( AdS_5 \) supergravity or conformal anomaly on \( S^4 \) or \( H^4 \) (since \( H^4 \) can be obtained by analytic continuation of \( S^4 \)) in the \( \mathcal{N} = 4 \) SYM theory. One finds there the order \( N \) gravitational contribution to the anomaly from a (Riemann)\(^2 \) term. Moreover, the string theory dual to \( \mathcal{N} = 2, \ D = 4 \) SCFTs with the gauge group \( Sp(N) \) has been conjectured to be type IIB string theory on \( AdS_5 \times S^5/Z_2 \) [40, 13], whose low energy effective (bulk) action in five dimensions reads

\[
S = \int d^5 x \sqrt{-g} \left[ \frac{R}{\kappa_5} - 2\Lambda + \gamma R_{abcd} R^{abcd} \right] + \text{(boundary terms)},
\]

if we define

\[
\frac{L^3}{\kappa_5} = \frac{N^2}{4\pi^2}, \quad \Lambda = -\frac{6}{\kappa_5 L^2} = -\frac{6N^2}{4\pi^2} \frac{1}{L^5}, \quad \gamma = \frac{6N}{24 \cdot 16\pi^2} \frac{1}{L},
\]

hence \( \gamma > 0 \). Here we shall not elaborate more upon it, because our main interest in this section is to report correctly on the black hole parameters (free energy, entropy and energy) for the above theory.
For $\gamma \neq 0$, we can find only a perturbative metric solution, which in $(d + 1) = 5$ reads

$$e^{2\phi(r)} = k - \frac{\mu}{r^2} \left( 1 + \frac{2\tilde{\gamma}}{\ell^2} \right) + \frac{r^2}{\ell^2} \left( 1 + \frac{2\tilde{\gamma}}{3\ell^2} \right) + \frac{\tilde{\gamma} \mu^2}{r^6} + \mathcal{O}(\tilde{\gamma})^2,$$

where $\tilde{\gamma} = 2\gamma\kappa_5$. Since $\mu$ is an integration constant with dimension of (length)$^{d-2}$, we may rescale

$$\mu \left( 1 + \frac{2\tilde{\gamma}}{\ell^2} \right) \rightarrow \tilde{M}, \quad \frac{1}{\ell^2} \left( 1 + \frac{2\tilde{\gamma}}{3\ell^2} \right) \rightarrow \frac{1}{L^2},$$

in order to bring the solution (54) in the usual form

$$e^{2\phi(r)} = k - \frac{\tilde{M}}{r^2} + \frac{r^2}{L^2} + \frac{\tilde{\gamma} \tilde{M}^2}{r^6} + \mathcal{O}(\tilde{\gamma})^2.$$  \hspace{1cm} (56)

In large $r$ limit, this solution looks similar to the perturbative expansion of the Gauss-Bonnet black hole solution. In the latter case, however, there is no need to consider the perturbative solution, since there exist exact solutions as considered in section 3.

The integration constant $\tilde{M}$ at the singularity $e^{2\phi(r)} = 0$ reads, from Eq. (56), as

$$\tilde{M}_+ = r_+^2 \left[ k + \frac{r_+^2}{L^2} + \frac{\tilde{\gamma}}{r_+^2} \left( k^2 + \frac{2k}{L^2} + \frac{k^2}{L^2} \right) \right] + \mathcal{O}(\tilde{\gamma})^2,$$

where $r_+ = r_H$ is the black hole horizon. The corresponding Hawking temperature $T_H$ is

$$T_H = \frac{1}{4\pi} \left. \left[ e^{2\phi(r)} \right] \right|_{r=r_+} = \frac{1}{\pi L^2} \left[ r_+ + \frac{kL^2}{2r_+} - \frac{\tilde{\gamma}}{L^2} \left( r_+ + \frac{2kL^2}{r_+} + \frac{k^2L^4}{r_+^3} \right) \right] + \mathcal{O}(\tilde{\gamma})^2.$$ \hspace{1cm} (58)

We have plotted this temperature versus the horizon radius in Fig. (10) for $k = +1, 0$ and $-1$ at $\tilde{\gamma}/L^2 = 0.004$, and in Fig. (11) for $k = -1$ under a critical value of $\tilde{\gamma}/L^2 = 0.1/(0.325)^2$.

We follow the Euclidean prescription for regularizing the action [6, 5], and identify the Euclidean action with the free energy times $1/T$. For $(d + 1) = 5$, we find the following expression for free energy \footnote{This result differs in sign for the last term from that of [39]. Perturbative black hole solution with $R^2$ terms and its thermodynamic behavior was further discussed in [41].}

$$F = \frac{V_3}{\kappa_5} \left[ \left( 1 - \frac{2\tilde{\gamma}}{L^2} \right) \left( \tilde{M} - \frac{2r_+^4}{L^2} \right) - \frac{6\tilde{\gamma} \tilde{M}^2}{r_+^4} \right] + \mathcal{O}(\tilde{\gamma})^2$$

$$= \frac{V_3r_+^2}{\kappa_5} \left[ k - \frac{r_+^2}{L^2} - \frac{12\tilde{\gamma}k}{L^2} - \frac{3\tilde{\gamma}r_+^2}{L^4} - \frac{5\tilde{\gamma}k^2}{r_+^2} \right] + \mathcal{O}(\tilde{\gamma})^2$$ \hspace{1cm} (59)

where we have substituted the Eq. (57) in the second step above. In Fig. (12) the free energy of the black hole versus the horizon radius is plotted for $k = +1, 0$ and $-1$, where we see that the $k = 1$ case is still most viable one to explain the Hawking-Page type phase transition.

The entropy of the black hole can be determined by using the relation

$$S = -\frac{dF}{dT_H} = \frac{dF}{dr_+} \frac{dr_+}{dT_H},$$ \hspace{1cm} (60)
where the derivative terms $dF/dr_+$ and $dT_H/dr_+$ take the following forms

\[
\begin{align*}
\frac{dF}{dr_+} & = -\frac{4Vr_+^3}{\kappa_5 L^2} \left[ 1 - \frac{kL^2}{2r_+^2} + \frac{3\gamma}{L^2} \left( 1 + \frac{2kL^2}{r_+^4} \right) \right] + O(\gamma^2), \\
\frac{dT_H}{dr_+} & = \frac{1}{\pi L^2} \left[ 1 - \frac{kL^2}{2r_+^2} - \frac{\hat{\gamma}}{L^2} \left( 1 - \frac{2kL^2}{r_+^4} - \frac{3k^2L^4}{r_+^4} \right) \right] + O(\gamma^2).
\end{align*}
\]

(61)

Hence we obtain (since $\hat{\gamma} = 2\gamma \kappa_5$)

\[
S = \frac{4\pi V\gamma^3 r_+^3}{\kappa_5} \left[ 1 + \frac{8\gamma \kappa_5 L^2}{L^2} \left( 1 + \frac{kL^2}{2r_+^2} - \frac{3k^2L^4}{4r_+^4} \right) \left( 1 - \frac{kL^2}{2r_+^2} \right)^{-1} \right] + O(\gamma^2).
\]

(62)

For a large size black hole $2r_+^2 >> kL^2$, one may approximate the entropy

\[
S \simeq \frac{4\pi V\gamma^3 r_+^3}{\kappa_5} \left[ 1 + \frac{8\gamma \kappa_5 L^2}{L^2} \left( 1 + \frac{3kL^2}{2r_+^2} \right) \right] + O(\gamma^2).
\]

(63)

Thus entropy of the black hole with $(\text{Riemann})^2$ term does not satisfy the area formula $S = A/4G$. Rather, just as the Einstein term in the action is corrected by $(\text{Riemann})^2$ term, the Einstein contribution to black hole entropy ($S = A/4G$) receives $(\text{Riemann})^2$ corrections. This has been expected in the literature (see Ref. [8] for review), but here we further show that entropy (63) agrees with Wald’s formula for entropy [23] for $r_+^2 >> L^2$. Another new ingredient is that we implement these results in next section to reproduce boundary data on the horizon of the AdS geometry.

We can also find the black hole energy, defined as $E (\equiv TS + F)$,

\[
E = \frac{3V_3}{\kappa_5} \left[ \tilde{M}_0 + 2\gamma \kappa_5 r_H^2 \left( \frac{3r_H^4}{L^4} + \frac{5k r_H^2}{2L^2} - k^2 - \frac{k^3 L^2}{2r_H^2} \right) \left( 1 - \frac{kL^2}{2r_H^2} \right)^{-1} \right] + O(\gamma^2).
\]

(64)

Here $\tilde{M}_0 = r_H^2 (k + r_H^2/L^2)$ is the value of $\tilde{M}_+$ when $\gamma = 0$. For $k = 1$, one apparently sees a singularity at $2r_H^2 = L^2$, but actually there is no singularity in the formulas for energy Eq. (64) and entropy Eq. (62). One easily checks that at $r_H^2 = L^2/2$, for $k = 1$, the first round bracket in (64) also vanishes, thus the above formulas are applicable to all three choices: $k = 0, 1, -1$.

As we already mentioned above, entropy of the black hole can be expressed as a local geometric (curvature) density integrated over a space-like cross section of the horizon. Notably, an entropy formula valid to any effective gravitational action including higher curvature interactions, was first proposed in [23], and nicely generalized in Refs. [25, 26]. One can infer from [26] that the black hole entropy for the action (5) takes in five dimensions the following form

\[
\hat{S} = \frac{4\pi}{\kappa_5} \int_{\text{horizon}} d^3x \sqrt{h} \left\{ 1 + 2\alpha \kappa_5 R + \beta \kappa_5 \left( R - h^{ij} R_{ij} \right) + 2\gamma \kappa_5 \left( R - 2h^{ij} R_{ij} + h^{ij} h^{kl} R_{ijkl} \right) \right\},
\]

(65)

where $h$ is the induced metric on the horizon. For $d + 1 = 5$, since the curvatures are defined in the following form

\[
R = -\frac{20}{L^2}, \quad h^{ij} R_{ij} = -\frac{12}{L^2}, \quad h^{ij} h^{kl} R_{ijkl} = -\frac{6k}{r_H^2} + O(\gamma)
\]

(66)
from the Eq. (65), when \( \alpha = \beta = 0 \), we read
\[
\tilde{S} = \frac{V_3 r_H^3}{4 G_{(5)}} \left[ 1 + \frac{8 \gamma \kappa_5}{L^2} \left( 1 + \frac{3 k L^2}{2 r_H^2} \right) \right] + \mathcal{O}(\gamma^2). \tag{67}
\]
Hence the two expressions for entropies i.e., Eq. (65) and Eq. (62) are identical. Our results are suggestive, and clearly contradict with the observations made in Ref. [39] in this regard.

From the \((d+1)\) dimensional analogue of the formula (65), we may calculate entropy of the Gauss-Bonnet black hole, using the relation
\[
\alpha = -\beta/4 = \gamma,
\]
\[
\tilde{S} = \frac{4 \pi}{\kappa_{d+1}} \int d^{d-1}x \sqrt{h} \left[ 1 + 8 \gamma \kappa_5 L^2 \left( 1 + 3 k L^2 r_H^2 \right) \right] + \mathcal{O}(\gamma^2). \tag{68}
\]
At the horizon, one sets \( e^{2\phi(r_+)} = 0 \), and reads the value of \( \mathcal{R}(h) \) from (12). Hence
\[
S = \frac{V_{d-1} r_H^{d-1}}{4 G_{d+1}} \left[ 1 + \frac{(d-1)}{(d-3)} \frac{2\hat{\alpha} k}{r_H^2} \right], \tag{69}
\]
which coincides with Eq. (28). Myers and Simon [34] have derived the result (69) for entropy of the black hole in asymptotically flat backgrounds \((L^2 \to \infty \text{ or } \Lambda = 0)\) with \( k = 1 \), but here we see that this holds for arbitrary \( \Lambda \). This mimics that the geometry on the boundary or the cosmological term in the bulk is not dynamical. This is crucial and possibly gives some insights of the holography.

### 5.1 Quantities with non-trivial \( \alpha, \beta, \gamma \)

One may find a perturbative solution for arbitrary \( \alpha, \beta \) and \( \gamma \) at a time, but this calculation is complicated due to a perturbative expansion. Without loss of any generality, one can follow a different, but equivalent, prescription, in which one combines the results obtained for (i) \( \alpha, \beta \neq 0 \) and \( \gamma = 0 \), to the results obtained for (ii) \( \alpha = \beta = 0 \) and \( \gamma \neq 0 \). Thus, by combining the results (44,46) and (59,62,64), we obtain the following expressions for free energy \((F)\), entropy \((S)\) and energy \((E)\) of the 5d AdS black hole with quadratic curvature terms
\[
F = \frac{V_3}{\kappa_5} \left[ r_H^2 \left( k - \frac{r_H^2}{L^2} \right) (1 - \epsilon) - 10 \gamma \kappa_5 \left( k + \frac{r_H^2}{L^2} \right)^2 \right] + \mathcal{O}(\gamma^2), \tag{70}
\]
\[
S = 4 \pi \frac{V_3 r_H^3}{\kappa_5} \left[ (1 - \epsilon) + \frac{12 \gamma \kappa_5}{L^2} \left( 1 + \frac{k L^2}{2 r_H^2} - \frac{k^2 L^4}{2 r_H^4} \right) \left( 1 - \frac{k L^2}{2 r_H^2} \right)^{-1} \right] + \mathcal{O}(\gamma^2), \tag{71}
\]
\[
E = \frac{3 V_3}{\kappa} \left[ \tilde{M}_0 (1 - \epsilon) + 2 \gamma \kappa_5 \left( \frac{5 r_H^4}{L^4} + \frac{7 k r_H^2}{2 L^2} - 2 k^2 - \frac{k^3 L^2}{2 r_H^2} \right) \left( 1 - \frac{k L^2}{2 r_H^2} \right)^{-1} \right] + \mathcal{O}(\gamma^2), \tag{72}
\]
where \( \epsilon = 4 \kappa (\gamma + 2 \beta + 10 \alpha)/L^2 \), and \( \tilde{M}_0 \) is related to the mass parameter \( \tilde{M} \) when \( \alpha = \beta = \gamma = 0 \). The consistency of the above results is reflected from the regularities of the expressions for entropy (71) and energy (72) at \( k L^2 = 2 r_H^2 \).

In the large \( N \) limit, one has \( L^3 >> \kappa_5 \) (from the first expression of Eq. (53)), and in AdS/CFT, the coefficients of the higher curvature terms in the bulk, in general, satisfy the limit \( 1 > \epsilon > 0 \). In
Ref. [13] the subleading contribution to the $AdS_5/CFT_4$ trace anomaly was considered. In terms of boundary metric $g_{ij}^{(0)}$, supergravity prediction of $O(N)$ contribution to the trace anomaly was found to be [13]

$$\frac{6N}{24 \times 16\pi^2} \left[ R^{(0)}_{ijkl} R^{(0)ijkl} - \frac{13}{4} R^{(0)}_{ij} R^{(0)ij} + \frac{3}{4} R^{(0)^2} \right], \quad (73)$$

for which $1 < \varepsilon < 0$. Since $\gamma > 0$, the free energy (70) is always negative for $k = 0$ and $k = -1$, and such black holes are globally stable. This means that provided $(1 - \varepsilon) > 0$ and $\gamma > 0$, there may not occur a Hawking-Page phase transition for AdS black hole with a Ricci flat or hyperbolic horizon for the above theory. For the $k = 1$ case, $F$ can be negative for large black hole with $r_H^2 >> L^2$ since $\gamma > 0$. While, for a small size black hole $r_H^2 < L^2$, the free energy $F$ can be positive corresponding to a thermal AdS phase.

As a consistency check of the above formulas, we express the curvature squared terms in the Gauss-Bonnet form: $\alpha = -\beta/4 = \gamma$. In this case, since $\varepsilon = 12\alpha\kappa/L^2$, the free energy (70) reduces to

$$F = \frac{V_3}{\kappa_5} \left[ r_+^2 \left( k - \frac{r_+^2}{L^2} \right) + \hat{\alpha} \left( \frac{r_+^4}{L^4} - \frac{16 k r_+^2}{L^2} - 5 k^2 \right) \right] + O(\alpha^2), \quad (74)$$

where $\hat{\alpha} = 2\alpha\kappa_5$ and $r_+ = r_H$. This agrees with the exact expression of free energy (26), where one sets $d = 4$ and $(1 + 2\hat{\alpha} k/r_+^2)^{-1} \equiv (1 - 2\hat{\alpha} k/r_+^2)$, and $L \equiv \ell$. We may read entropy for the Gauss-Bonnet black hole from the formula Eq. (71), in the limit $r_+^2 >> L^2$,

$$S_{GB} = \frac{4\pi V_3 r_+^3}{\kappa} \left[ 1 + \frac{12\alpha\kappa_5 k}{r_+^2} \right]. \quad (75)$$

This very nicely agrees in five dimension ($d = 4$) with the expression (28) or (69) (obtained from two different methods). For $k = 0$, one has $S_{GB} = (4\pi V_3 r_H^3)/\kappa_5 \equiv A/4G$. This special connection is the reminiscent of the topological behavior of the GB invariant, but, as we have already seen, does not hold in the generic higher derivative theories.

6 Holography beyond AdS/CFT

Via holography [27], it has been known that thermodynamic quantities of a boundary CFT can be determined by those of the global AdS (supergravity) vacuum, the notion of the celebrated AdS/CFT correspondence [2]. Witten [4], further argued that such a correspondence might exist even if we give a finite temperature to the bulk AdS (so that pure AdS bulk is replaced by AdS-Schwarzschild black hole), and define the CFT on the boundary at finite temperature. One could therefore associate the mass (energy), temperature and entropy of the black hole with the corresponding quantities in the boundary CFT. With (Riemann)$^2$ term, however, the boundary field theory at finite temperature dual to AdS black hole is not conformal, which we exhibit below in the cosmological context.
6.1 Relating boundary and bulk parameters

To reproduce boundary data from bulk data and vice versa, one may consider the coordinate transformations given in the footnote (3), which are solved by \[38\]

\[\sigma^* = \sigma^2 - e^{2\phi(r)}e^{2\sigma(y)}, \tag{76}\]

where \(d\eta = e^{-\sigma(y)}d\tau\), with \(\eta\) being a new time parameter, \(\sigma^* = \partial_\eta \sigma\) and \(\sigma' = \partial_y \sigma\). One specifies the functions \(r = r(\eta), t = t(\eta)\), so that \(-\sigma^* \equiv \dot{r}/r\) defines the Hubble parameter \(H_0\). Eq. (76) ensures that the induced metric on the brane takes the standard Robertson-Walker form

\[ds^2_d = -d\eta^2 + r^2(\eta)d\Omega_{d-1}^2, \tag{77}\]

and the radial distance \(r\) measures the size of \(d\)-dimensional universe from the center of the black hole \[48\]. When applying the holography in the above context, one considers an \(d\)-dimensional brane with a constant tension in the background of an \((d+1)\)-dimensional AdS black hole. One then regards the brane as the boundary of the AdS geometry, and further assumes that \(\sigma'|_{y=0^+} = -L^{-1}\) at the horizon \(r = r_H\), where \(e^{2\phi(r)} = 0\). Then it is clear that Eq. (76) leads to \(H_0 = \pm 1/L\). Using Eq. (40), one finds the first Friedman equation in \((d+1)\)-dimensions \[39\]

\[H_0^2 + \frac{k}{r^2} = \frac{\dot{M}}{r^d} \equiv \frac{\kappa_d}{(d-1)(d-2)}\rho, \tag{78}\]

where \(\kappa_d = 16\pi G(d)\), \(\rho = \tilde{E}/V\) is the matter energy density on the boundary, and

\[\tilde{E} = \frac{(d-1)(d-2)\dot{M}V_{d-1}}{\kappa_d r}, \quad V = r^{d-1}V_{d-1}. \tag{79}\]

This energy \(\tilde{E}\) coincides with the gravitational energy \(E\) (Eq. (46)), up to a conformal factor. Differentiation of Eq. (78) gives the second Friedman equation

\[\dot{H}_0 - \frac{k}{r^2} = -\frac{\kappa_d}{2(d-2)}(\rho + p), \quad \text{where} \quad p = \frac{(d-2)\dot{M}}{\kappa_d r^d} \tag{80}\]

is the matter pressure on the hypersurface. Since \(-\rho + (d-1)p = 0\), the induced CFT matter is radiation-like \(T^\mu_\mu = 0\). For a non-zero \(\gamma\) (but with \(\alpha = \beta = 0\), the FRW equation reads

\[H_0^2 + \frac{k}{r^2} = \frac{\kappa_d}{6}\rho, \tag{81}\]

where \(\rho = \tilde{E}/V, V = V_3 r^3\) and

\[\tilde{E} = \frac{6V_3}{\kappa_d r^3} \left[\dot{M} - \frac{\gamma M^2}{r^4}\right]. \tag{82}\]

Differentiation of Eq. (81) gives the second FRW equation

\[\dot{H}_0 - \frac{k}{r^2} = -\frac{\kappa_d}{4} \left(\frac{\tilde{E}}{V} + p\right), \tag{83}\]
with
\[ p = \frac{2}{\kappa_4 r^4} \left[ \tilde{M} - \frac{5\tilde{\gamma} M^2}{r^4} \right]. \]  

(84)

From Eqs. (82) and (84), one obtains
\[ -\frac{\tilde{E}}{V} + 3p = -\frac{24\tilde{\gamma} M^2}{\kappa_4 r^8} + O(\tilde{\gamma}^2). \]  

(85)

Thus, one has \((T_\mu^\mu) \neq 0\), which reveals that a theory defined with \(\gamma \neq 0\) is not conformal. It is also desirable to define the relation between \(\kappa_4\) and \(\kappa_5\). They can be related by (see [42] for \(\varepsilon = 0\) case)
\[ \kappa_4 = \frac{2}{L} \frac{\kappa_5}{1 + \lambda \varepsilon}. \]  

(86)

If \(\alpha = \beta = 0\) and \(\gamma \neq 0\), one has \(\varepsilon = 4\kappa_5 \gamma / L^2\). The magnitude of \(\lambda\) can be fixed from the propagator analysis. If one regards the brane as the boundary of the AdS geometry, one has \(\lambda = 1\). We simply note, however, that for a gravity localized Randall-Sundrum type \(\delta\)-function brane, one actually finds \(\lambda = 1/3\) (see Ref. [17, 43] for the Gauss-Bonnet case). Assume that brane is the horizon of AdS geometry, then it is suggestive to consider a moment in the brane’s cosmological evolution at which the brane crosses the black hole horizon \(r = r_{brane} = r_H\), so that \(\tilde{M}(r_{brane}) = \tilde{M}(r_H)\). After implementing the relation (86), i.e.
\[ \kappa_4 = \frac{2}{L} \frac{\kappa_5}{1 + 2\tilde{\gamma}/L^2}, \]  

(87)

from Eq. (82) we find
\[ \tilde{E} = \frac{3V_3}{\kappa_5} \frac{L}{r} \left[ \tilde{M}_0 + \frac{4\gamma \kappa_5 r_{brane}^2}{L^2} \left( k + \frac{r_{brane}^2}{L^2} \right) \right] + O(\gamma^2). \]  

(88)

One may assume that the energy \(\tilde{E}\) is rescaled by
\[ \tilde{E} = \frac{L}{r} E_{AdS}. \]  

(89)

The origin of the factor \(L/r\) is entirely “holographic” in spirit. It has been argued in Ref. [44] that for non-critical branes (i.e. \(k \neq 0\)), the AdS length scale \(L\) may be replaced by \(1/T\), where \(T\) is the brane tension. At any rate, we see that the two expressions for energy, Eq.(64) and \(E_{AdS}\) (which one reads from Eqs. (88) and (89)) do not coincide, but for \(\gamma = 0\) they do coincide.

Finally, we comment upon the size and location of the brane for a flat and static brane. Instead of \(\sigma' = -1/L\), let us assume that \(\sigma' = -1/\ell\) holds. If the brane (hypersurface) is flat \((k = 0)\), the horizons defined by \(e^{2\phi(r)} = 0\) (Eq. (54)) read
\[ r_1^4 = \mu \ell^2 + \frac{\mu \tilde{\gamma}}{3} + O(\tilde{\gamma}^2), \quad r_1^4 = \mu \gamma \left( 1 - \frac{\tilde{\gamma}}{\ell^2} - \frac{16}{3} \frac{\tilde{\gamma}^2}{\ell^4} \right) + O(\tilde{\gamma}^3). \]  

(90)

If the brane is also static (i.e. \(H_0 = 0\)), the Friedmann equation (83) rise to give two horizons. However, the one which reduces to that of Einstein’s gravity for \(\gamma = 0\) and would be of more physical interest is
\[ r_{brane}^4 = \mu \gamma \left( 1 - \frac{2\tilde{\gamma}}{\ell^2} + \frac{2}{3} \frac{\tilde{\gamma}^2}{\ell^4} \right) + O(\tilde{\gamma}^3). \]  

(91)
Since $\gamma > 0$, there is a critical point at
\[ 1 = \frac{6\gamma}{l^2} \equiv \frac{12\gamma\kappa_5}{l^2}, \]
where the brane coincides with the black hole inner horizon $r_{brane}^4 = r_+^4$. The brane lies outside the (black hole) inner horizon if $l^2 > 6\gamma$, and it lies inside the black hole horizon if $l^2 < 6\gamma$.

6.2 Entropy bounds in holography

For completeness and comparison, we list some of the interesting proposals for entropy bounds in holography [45, 46]. Consider the Cardy formula [47] of two-dimensional conformal field theory
\[ S_H = 2\pi \sqrt{\frac{c}{6}\left( L_0 - \frac{c\kappa}{24} \right)}, \]
where the CFT generator (eigenvalue) $L_0$ represents the product $\tilde{E} r$ of the energy and radius, $k$ is the intrinsic curvature of the CFT boundary, $c/24$ is the shift in eigenvalues caused by the Casimir effect. In [48, 49], it was shown that the formula (93) can generalized to arbitrary $d$-dimensions, which then correspond to the FRW type brane equations of $d$ dimensions. One identifies there [49, 46]
\[ L_0 \equiv \frac{\tilde{E} r}{d-1}, \quad \frac{c}{6} = \frac{4(d-2)V_3 r^2}{\kappa_d}, \quad S_H \equiv \frac{4\pi(d-2)H_0 V}{\kappa_d}, \]
where $S_H$ is the Hubble entropy [48]. The Cardy formula puts a bound $S_H < S_B$, where
\[ S_B = \frac{2\pi r}{d-1} \tilde{E} \]
is the Bakenstein entropy. The “holographic bound” proposed by ’t Hooft and Susskind reads $S \leq S_{BH}$, where
\[ S_{BH} = \frac{4\pi (d-2)}{\kappa_d r} V. \]
This mimics that entropy is smaller than Bakenstein-Hawking entropy of the largest black hole that fits in the given volume. Another entropy bound is the “Hubble bound” proposed in [45], which reads $S \leq S_H$. This heralds that entropy is bounded by total entropy of Hubble size black hole.

For our purpose, we find entropy formula defined in (94) interesting. Though initially derived for two-dimensional CFT, this formula may be applied to $CFT_4/AdS_5$ [48], one regards $S_H$ then as the holographic entropy. One may worry about when applying this formula to non-CFTs, like a theory with (Riemann)$^2$ term. We therefore follow here an heuristic approach to estimate a difference between $S_H$ and $S_{BH}$. Note that, for $\gamma \neq 0$, a non-conformal behavior in entropy is seen also from the scaling relation between $\kappa_4$ and $\kappa_5$, the former one involves a non-trivial $\gamma$ via (87). Using the value $H_0 = 1/L$ and Eq. (87), the Hubble entropy $S_H$ in (94) takes the form, for $d = 4$,
\[ S_H = \frac{4\pi V_3 r_H^3}{\kappa_5} \left( 1 + \frac{4\gamma \kappa_5}{L^2} \right) \equiv \frac{N^2 V}{\pi L^3} \left( 1 + \frac{1}{16N} \right) \propto N^2VT^3. \]
Of course, in the large $N$ the correction term is negligible. We may calculate the difference in Bakenstein-Hawking entropy $S_{BH}$ (Eq. (62)) and Hubble entropy (Eq. (97)), which reads

$$ S_H - S_{BH} = \frac{4 \pi V_3 r_H^3}{\kappa_5} \left[ -\frac{4 \kappa_5 \gamma}{L^2} \left( 1 + \frac{5}{2} \frac{kL^2}{r_H^2} - \frac{3}{2} \frac{k^2L^4}{2r_H^2} \right) \left( 1 - \frac{kL^2}{2r_H^2} \right)^{-1} \right] $$

$$ = -48 \pi V_3 \gamma r_H \left( k + \frac{r_H^2}{3L^2} \right) \equiv -\frac{3N V_3 r_H}{4 \pi L} \left( k + \frac{r_H^2}{3L^2} \right). \quad (98) $$

Note that the difference in entropies can arise only in the order of $N$, though each entropy is proportional to $N^2$. Moreover, a negative sign in the difference implies that $S_H \leq S_{BH}$, since $\gamma > 0$. An obvious fact that $S_H$ does not coincide with $S_{BH}$ is not surprising with $\gamma \neq 0$, because an AdS dual theory is not conformal if $\gamma \neq 0$, but entropy formulas for the boundary field theory and the FRW equations can be expected to coincide only for a radiation dominated (brane) universe [48]. In particular, for $\gamma = 0$, one has $S_H = S_{BH}$. For the $k = -1$ case (which correspond to $H_0 r > 1$ case, i.e. $r_H/L > 1$ at $r = r_H$), one finds $S_H > S_{BH}$ if $r_H^2 > 3L^2$, but $S_H < S_{BH}$ if $L^2 < r_H^2 < 3L^2$. For the $k = 0$ and $k = 1$ cases (which correspond to the $H_0 r \leq 1$ cases, i.e. $r_H/L \leq 1$ at $r = r_H$), one has $S_H \leq S_{BH}$. Thus for a flat and spherical AdS black holes, we propose a new entropy bound $S_H \leq S_{BH}$. These results might be crucial to understand possible entropy bounds with a non-trivial (Riemann)$^2$ interaction.

### 7 Conclusions

A possibility is that holography beyond the AdS/CFT persists in true quantum gravity and unified field theory, which then requires inclusion of the higher order curvature derivative terms. With this motivation, we studied in details the thermodynamic properties of anti-de Sitter black holes for the Einstein-Gauss-Bonnet theory, and the Einstein term in the action corrected by general $R^2$ terms, and discussed their thermodynamic behavior, including the thermal phase transitions in AdS space. We obtained in useful form the expression of free energy for the Gauss-Bonnet black hole, and calculated entropy and energy. In Einstein theory, only the $k = 1$ case is plausible to explain the Hawking-Page phase transition. This is suggestive, because the AdS solution with $k = 1$ can be embedded in ten-dimensional IIB supergravity such that the supergravity background is of the form $AdS_5 \times S^5$, for which the boundaries of the bulk manifold has the same intrinsic geometry as the background of the dual theory at finite temperature. We noted that unlike to Einstein gravity, in EGB theory there may occur a Hawking-Page phase transition even when the event horizon is a negative ($k = -1$) curvature hypersurface. The free energy of such topological Gauss-Bonnet black hole starts from negative value, reaches a positive maximum at some $r = r_+$, and then again go to negative infinity as $r_+ \to \infty$. Thus the hyperbolic ($k = -1$) AdS black holes, though can have a thermal anti-de Sitter phase, globally would prefer a stable black hole phase for a large $r_H$. 

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We then investigated that contribution from the squared of Ricci scalar and Ricci tensor can be absorbed into free energy, entropy, and energy via a redefinition of the five dimensional gravitational constant and the radius of curvature of AdS space. By introducing the Gibbon-Hawking surface term and a boundary action corresponding to the vacuum energy on the brane, we recovered the RS type fine tunings as natural consequences of the variational principle. In (Riemann)$^2$-gravity, the Bakenstein-Hawking entropy agrees with the value obtained using similar formula when the black hole horizon $r_H$ is much larger than the AdS length scale $L$. The Bakenstein-Hawking entropy in the Gauss-Bonnet gravity coincides with the value directly evaluated using Wald’s formula. These are interesting and quite pleasing results. We also obtained formulas for free energy, entropy and energy of the AdS black holes with $R^2$ terms. Another interesting observation is that the thermodynamic properties of $k = -1$ AdS black holes, for $d + 1 = 5$, under a critical value of $\hat{\gamma}/L^2$ in (Riemann)$^2$ gravity are qualitatively similar to those of Gauss Bonnet black hole with a spherical event horizon ($k = 1$).

We established certain relations between the boundary field theory parameters defined on the brane and the bulk parameters associated with the Schwarzschild AdS black hole in five dimensions. Specifically, we found the FRW type equations on the brane, and exhibited that matter on the AdS boundary (brane) is not conformal if $\gamma$ is non-zero. Using a heuristic but viable approach, we calculated a difference between the holographic entropy and the Bakenstein-Hawking entropy, which do not coincide with $\gamma \neq 0$. The essential new ingredient in our analysis is the role of the higher curvature terms in explaining the essential features of the black hole thermodynamics, including thermal phase transitions, and holography.

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References


Figure 1: Einstein gravity ($\hat{\alpha} = 0$): The inverse temperature ($\beta_0$) versus the horizon radius ($r_H$). The three curves above from up to down correspond respectively to the cases $k = -1$, $k = 0$ and $k = +1$.

Figure 2: The Gauss-Bonnet black hole: The inverse temperature ($\beta_0$) versus black hole radius ($r_H$) for the case $k = 1$ (from up to down: $d + 1 = 4$, 5, 6, and 10. We have fixed $\hat{\alpha}/\ell^2 = (d-2)(d-3)\alpha\kappa_{d+1}/\ell^2$ at $\alpha\kappa_{d+1}/\ell^2 = (0.2)/81$. With $d + 1 = 5$, a new phase of locally stable small black hole is seen, and for $d \neq 4$, the thermodynamic behavior is qualitatively similar to that of $\hat{\alpha} = 0$ case.
Figure 3: The Gauss-Bonnet black hole: inverse temperature vs horizon radius ($r_H$) for the case $k = 0$ in $d = 4$, and $\hat{\alpha}/\ell^2 = (0.7)/0.81$. Only for $\hat{\alpha} = \ell^2$, $T_H = T_c$, and hence $F = 0$ (global AdS vacuum), otherwise free energy is always negative, since $\hat{\alpha} < \ell^2$ should hold for $\Lambda < 0$.

Figure 4: Gauss-Bonnet black hole: free energy ($F$) vs horizon radius $r_H$ for the case $k = 0$ with $d = 4$. The black holes evolve from global AdS phase ($F = 0$), to thermodynamically stable AdS black holes phase ($F < 0$). The three curves from left to right correspond respectively to the cases $\hat{\alpha}/\ell^2 = (0.3)$, $(0.3)/4$ and $\hat{\alpha}/\ell^2 = (0.3)/9$. 

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Figure 5: The Gauss-Bonnet black hole: inverse temperature ($T^{-1}$) vs horizon radius ($r_H$) for the case $k = 1$ in $d = 4$ and $\hat{\alpha}/\ell^2 = (0.7)/(0.9)^2$. The upper curve corresponds to $T_c^{-1}$ and the lower one to $T_H^{-1}$. The region where $T_H$ exceeds $T_c$ is shown.

Figure 6: The Gauss-Bonnet black hole: free energy ($F$) vs horizon radius ($r_H$) for the case $k = 1$ in $d = 4$. The three curves from up to down correspond respectively to the cases $\hat{\alpha}/\ell^2 = (0.3)/100$, $(0.3)/64$ and $\hat{\alpha}/\ell^2 = (0.3)/36$. The $k = 1$ is the most plausible situation for a Hawking-Page phase transition.
Figure 7: The Gauss-Bonnet black hole: inverse temperature \( (T^{-1}) \) vs horizon radius \( (r_H) \) for the case \( k = -1 \) in \( d = 4 \) and \( \hat{\alpha}/\ell^2 = (0.1)/(0.9)^2 \). The upper curve corresponds to \( T_c^{-1} \) and the lower one to \( T_H^{-1} \) (in the range of \( 0 \) to \(-1.5\)). For a small \( r \), one has \( F > 0 \) before hitting a singularity at \( r = r_s \) (see Fig. (8) below), other than at \( r = 0 \), and \( r_s \) is shielded by \( r_H \).

Figure 8: The Gauss-Bonnet black hole: free energy \( (F) \) vs horizon radius \( (r_H) \) for the case \( k = -1 \) in \( d = 4 \). The three curves from up to down correspond respectively to the values \( \hat{\alpha}/\ell^2 = (0.3)/(0.8)^2 \), \( (0.3)/(0.65)^2 \), \( (0.3)/(0.6)^2 \) and \( 0.3/4 \). For \( \hat{\alpha}/\ell^2 > (0.3)/(0.6)^2 \) and \( \hat{\alpha}/\ell^2 < (0.3)/(1.5)^2 \), the free energy is always negative.
Figure 9: The Gauss-Bonnet black hole: free energy \( F \) vs horizon radius \( r_H \) for the case \( k = -1 \) in \( d = 4 \). All the curves in between \( (0.3)/(0.7595)^2 \leq \hat{\alpha}/\ell^2 \leq (0.3)/(0.7905)^2 \) coincide each other. This is the region where a small topological black hole would prefer thermal AdS phase, \( F > 0 \).

Figure 10: The black hole in \( (\text{Riemann})^2 \) gravity: Hawking temperature vs horizon radius \( r_H \) for \( d+ = 5 \). We have actually plotted the graph between \( \pi T_H \) versus \( r_H \). The three curves from up to down correspond respectively to the cases \( k = +1, 0 \) and \( -1 \) at \( \hat{\gamma}/L^2 = (0.1)/25 \).
Figure 11: The black hole in (Riemann$^2$) gravity: inverse temperature vs horizon radius ($r_H$) in $d = 4$ for the case $k = -1$ at $\hat{\gamma}/L^2 = (0.1)/(0.325)^2$ (which corresponds to the value $\hat{\gamma}/\ell^2 = (0.1)/(0.390)^2$, so that $\hat{\gamma}/3\ell^2 < 1$ and hence $\Lambda < 0$. We have actually plotted the graph between $\pi \times \beta_0$ and $r_H$. The thermodynamic properties of such topological ($k = -1$) black hole under a critical value of $\hat{\gamma}/L^2$ are qualitatively similar to those of Gauss Bonnet black hole with a spherical event horizon ($k = 1$) in $d + 1 = 5$ (Fig. (2)).

Figure 12: The black hole in (Riemann$^2$) gravity: free energy ($F$) vs horizon radius ($r_H$) in $d = 4$. The three curves from up to down correspond respectively to the cases $k = +1, 0$ and $-1$ at $\hat{\gamma}/L^2 = (0.15)/25$. 