Nambu Quantum Mechanics: A Nonlinear Generalization of Geometric Quantum Mechanics

Djordje Minic\textsuperscript{1} and Chia-Hsiung Tze\textsuperscript{2}

\textit{Institute for Particle Physics and Astrophysics}
\textit{Department of Physics}
\textit{Virginia Tech}
\textit{Blacksburg, VA 24061}

\textbf{ABSTRACT}

We propose a generalization of the standard geometric formulation of quantum mechanics, based on the classical Nambu dynamics of free Euler tops. This extended quantum mechanics has in lieu of the standard exponential time evolution, a nonlinear temporal evolution given by Jacobi elliptic functions. In the limit where latter’s moduli parameters are set to zero, the usual geometric formulation of quantum mechanics, based on the Kahler structure of a complex projective Hilbert space, is recovered. We point out various novel features of this extended quantum mechanics, including its geometric aspects. Our approach sheds a new light on the problem of quantization of Nambu dynamics. Finally, we argue that the structure of this nonlinear quantum mechanics is natural from the point of view of string theory.

\textsuperscript{1}e-mail: dminic@vt.edu
\textsuperscript{2}e-mail: kahong@vt.edu
Generalizations of quantum mechanics are difficult. One idea which has often been exploited is to make the Schrödinger equation non-linear (see for example [1]). Another approach is to extend the quantum phase space from the usual complex projective space to an arbitrary Kahler manifold. Yet another avenue is to enlarge complex quantum mechanics by changing the coefficients on the Hilbert space to quaternions [2] or octonions [3, 4]. In this article a more radical approach to this question is investigated. What is proposed is a modification of the kinematics and dynamics, the very symplectic and Riemannian structure of geometric complex quantum mechanics.

Any search for generalizations of quantum mechanics has to have a well defined motivation. One possible general starting point is provided by the observation that the evolution of fundamental physical theories, characterized by appearance of new dimensionful parameters (new constants of nature), can be mathematically understood from the point of view of deformation theory [5]. In particular, relativity theory, quantum mechanics and quantum field theory can be understood mathematically as deformations of unstable structures [6]. An example of an unstable algebraic structure is non-relativistic classical mechanics. By deforming an unstable structure, such as classical non-relativistic mechanics, via dimensionful deformation parameters, the speed of light $c$ and the Planck constant $\hbar$, one obtains new stable structures - special relativity and quantum mechanics. Likewise, the relativistic quantum mechanics (quantum field theory) can be obtained through a double ($c$ and $\hbar$) deformation. It is natural to expect that there is a further deformation via one more dimensionful constant, the Planck length $l_P$. The resulting structure could be expected to form a stable structural basis for a quantum theory of gravity.

A closely related idea has appeared in open string field theory, as originally formulated by Witten [7]. There, the deformation parameters are $\alpha'$ and $\hbar$. The classical open string field theory lagrangian is based on the use of the string field (which involves an expansion to all orders in $\alpha'$) and a star product which is defined in terms of the world-sheet path integral, also involving $\alpha'$. The full quantum string field theory is thus, in principle, an example of a one-parameter ($\alpha'$) deformation of quantum mechanics.

In this letter we lay the basis for a generalized quantum mechanics based on the classical Nambu dynamics [9] of Euler’s asymmetric top. This Nambu quantum mechanics, naturally possesses besides Planck constant, new deformation parameters. One of its defining experimental signatures is a nonlinear time evolution generated by Jacobian elliptic functions, as compared to the standard exponential time evolution of standard quantum mechanics. The new deformation parameters are given by the moduli of the elliptic functions. In the limit

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3 An algebraic structure is termed stable (or rigid) for a class of deformations if any deformation in this class leads to an equivalent (isomorphic) structure.

4 Similarly, one can also intuit that string theory calls for a generalization of quantum mechanics from the existence of the minimal length uncertainty relations in the framework of perturbative string theory [8].
when these are set to zero, the usual geometric formulation of quantum mechanics, based
on the Kahler structure of the space of rays in a complex Hilbert space, is recovered. We
point out various features of and issues raised by this extended quantum mechanics, in-
cluding its nonstandard geometric aspects. Our approach sheds a new light on the problem of
quantization of Nambu dynamics and is natural from the point of view of string theory.

We begin with a brief recapitulation of the geometric formulation of quantum mechanics
as originally formulated by Kibble [10] (for reviews of this approach consult [11, 12, 13, 14]).
This geometric setting will be a springboard for our attempt to go beyond standard quantum
mechanics. The basic observation is that pure states of a quantum mechanical system
 correspond to rays in a complex linear Hilbert space $H$. The latter can also be seen as a
real vector space with a complex structure $J$. So the Hermitian inner product of two states
$<\psi|$ and $|\phi>$ in $H$ can be split into its real and imaginary parts:

$$<\psi|\phi> = g(\psi,\phi) + i\omega(\psi,\phi) = \delta_{ij}\psi_i\phi_j + i\epsilon_{ij}\psi_i\phi_j.$$  

(1)
with $i,j=1,2$, labelling the real and imaginary components of $\psi$ and $\phi$. The metric $g$ is the
scalar product and the antisymmetric $\omega$ is a symplectic 2-form, they are related through
$J$ as $g(\psi,\phi) = \omega(\psi, J\phi)$. The triple $(g, \omega, J)$ makes $H$ a Kahler space. Thus the curved
space of rays of $H$, called the projective Hilbert space $\mathcal{P}$, is the quantum phase space and
has the complex geometry of a Kahler manifold. Pure states of the quantum system are
represented as points of the manifold $\mathcal{P}$, which is endowed with natural symplectic and
Riemannian structures. This symplectic structure encodes the symplectic structure that
survives in the classical limit. One notes that the Riemannian structure, with which the
complex Kahler structure is said to be compatible, is absent in the classical phase space and
is in fact a key ingredient of geometric quantum theory as it encodes the information about
pure quantum mechanical properties, such as the measurement process and Heisenberg’s
uncertainty relations. Up to numerical factors, Planck constant is given by the inverse of
the constant holomorphic sectional curvature of $\mathcal{P}$. Observables $A = <\hat{A}>$, defined as the
expectation value of a hermitian linear operator $\hat{A}$, correspond to real valued differentiable
functions on $\mathcal{P}$. The derivative of such a Kahlerian function $A$ vanishes at an “eigenstate”
with the value of $A$ at such a point being the corresponding “eigenvalue”. The evolution of
states (the Schrödinger equation) represents a symplectic flow generated by a Hamiltonian.

More explicitly, let us consider a pure state $\psi = \sum_a e_a \psi_a$, where the $\psi_a$ are the generalized
Fourier components of $\psi$ in an orthonormal eigenbasis $\{e_a\}$ of the Hamiltonian $H$ of a given
system. Also, setting for convenience Planck constant equal to 1, we let $q^a = \sqrt{2} \text{Re}\psi_a$ and
$p_a = \sqrt{2} \text{Im}\psi_a$ with the $(q^a + ip_a)$ providing the homogeneous coordinates for $\mathcal{P}$. The natural
symplectic structure on $\mathcal{P}$ is then given by the closed, nondegenerate 2-form $\omega^{(2)} = dp^a \wedge dq^a$, $d\omega^{(2)} = 0$. The Poisson bracket is defined as usual: $\{f, g\} = \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p^a} - \frac{\partial f}{\partial p^a} \frac{\partial g}{\partial q^a} \equiv \omega^{AB} \frac{\partial f}{\partial x^A} \frac{\partial g}{\partial x^B}$. 

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where $\omega^{AB}$ is the inverse of the symplectic 2-form and the $x^A = (p_a, q^a)$ form a set of canonical coordinates. As to the Schrödinger equation, with $h = \langle \hat{H} \rangle$, it now takes the form of Hamilton’s equations:

$$\frac{dp_a}{dt} = \{h, p_a\}, \quad \frac{dq^a}{dt} = \{h, q^a\}. \number{2}$$

Here $h = \frac{1}{2} \sum_a [(p^a)^2 + (q_a)^2] \omega_i$, and $\omega_i$ denote eigenvalues of the given Hamiltonian $H$. Thus for a general quantum system in the specific basis $\{e_a\}$, the Hamiltonian $h$ describes an infinite set of abstract free harmonic oscillators! Thus we have the alternative view of quantum mechanics as a rather familiar classical Hamiltonian system, albeit one on a (generally infinite-dimensional) nonlinear, projective Hilbert space [15]. The Schrödinger equation follows from the variation of the action $S = \int (p_a dq^a - h dt)$. The time evolution is, of course, exponential: $(q_a + ip_a)(t) = \exp(-i\omega_a t)(q_a + ip_a)(0)$. In general, an arbitrary observable $O$ will evolve according to

$$\frac{dO}{dt} = \{h, O\}. \number{3}$$

It also turns out that the expectation values of commutators of operators acting on the Hilbert space $\mathcal{H}$ are the Poisson brackets of the corresponding Kahlerian functions in the geometric formulation!

The normalization of the wave function $\psi^* \psi = 1$, known as the Born rule (imposed by introducing a Lagrange multiplier into the above action) becomes, $\frac{1}{2} \sum_a [(p^a)^2 + (q_a)^2] = 1$. Moreover the points $\psi$ and $e^{i\alpha} \psi$ are to be identified. Thus, if we take our Hilbert space to be one of finite (e.g. for a spin system) complex dimension $n + 1$, namely $\mathcal{H} = \mathbb{C}^{n+1}$, the above kinematic constraint says that the quantum phase space of rays in $\mathcal{H}$ is the complex projective $\mathbb{C}P(n)$. The latter is thus the base space of the complex Hopf bundle $S^{2n+1}$ over $\mathbb{C}P(n)$ (which can be realized as the coset space of $U(n+1)/U(1) \times U(n)$) with a $U(1)$ fiber, the group of complex phases in quantum mechanics. The Riemannian metric on $\mathcal{P}$ is given by the natural Kahler metric. For $\mathcal{P} = \mathbb{C}P(n)$, it is the well-known Fubini-Study metric, $ds_{12}^2 = \left(1 - |\langle \psi_1 | \psi_2 \rangle|^2 \right)$. For example, the Heisenberg uncertainty relations arise from such a metric of the state manifold whose local properties also lead to a generalized energy-time uncertainty relation [13].

Finally, given a curve $\Gamma$ in the projective Hilbert space $\mathcal{P}$, the geometric (Berry) phase [16] is given by [13]

$$\int_{\Sigma} dp_a \wedge dq^a, \number{4}$$

where $\Sigma$ has as its boundary $\Gamma$. Its expression as a symplectic area enclosed by $\Sigma$ shows that it results solely from the geometry of the Hermitian inner product and is independent of the Hamiltonian and the equation of motion provided that the latter is first order in time.
In the foregoing it is clear from the perspective of geometric quantum mechanics that the process of quantization is superfluous. One takes quantum theory as given. Quantum theory is in fact mathematically equivalent to a special type of classical Hamiltonian phase-space dynamics, albeit with a key difference: namely the underlying phase-space is not the finite dimensional symplectic phase space of classical mechanics, but rather the (infinite dimensional) quantum mechanical Kahler state space itself, naturally endowed with a compatible Riemannian metric. This formulation points to specific ways of extending quantum mechanics. This is achieved by tempering with either the kinematics, or the dynamics and quantum phase space of standard geometric quantum mechanics or indeed with all of these elements simultaneously. Next we choose to substitute the dynamics of the above systems of non-interacting harmonic oscillators by a system of free Nambu tops and explore the ensuing implications.

We begin then by summing up the corresponding geometrical characteristics of the classical Nambu mechanics [9]. In this letter we make use specifically of a Nambu system of order 3, namely a Hamiltonian system defined with respect to a ternary bracket. Thus the fundamental analog of the symplectic 2-form of usual Hamiltonian dynamics is a closed non-degenerate 3-form [17]

$$\omega^{(3)} = dM_1 \wedge dM_2 \wedge dM_3,$$

and the action given as an integral of the corresponding Poincare-Cartan 2-form

$$S = \int M_1 dM_2 \wedge dM_3 - H_1 dH_2 \wedge dt.$$  

This form of the action shows that initial and final states in this type of Nambu dynamics are described by loops rather than points, because the integrand of the action is a two form, rather a one form, as in the usual Hamiltonian dynamics. The fact that loops appear in the phase space of triple Nambu mechanics thus points out to a possible application to string theory.

The equations of motion - Nambu equations - follow from $\delta S = 0$:

$$\frac{dF}{dt} = \{H_1, H_2, F\}$$

where $F$ is a function of $M_1, M_2, M_3$ and the Nambu-Poisson bracket $\{F, G, H\}$ is defined as

$$\{F, G, H\} = \epsilon^{ijk} \partial_{M_i} F \partial_{M_j} H \partial_{M_k} G.$$  

The Nambu bracket generates volume preserving diffeomorphisms on the phase space of the Nambu dynamics and the Liouville theorem is obeyed. The Poisson bracket is evidently a natural “contraction” of the Nambu bracket. The latter satisfies the following three key conditions: skew-symmetry, the Leibniz rule and the Fundamental Identity [17, 18], the counterpart of the Jacobi identity obeyed by Poisson brackets:
1. Skew-symmetry
\[ \{A_1, A_2, A_3\} = (-1)^{\epsilon(p)} \{A_{p(1)}, A_{p(2)}, A_{p(3)}\}, \]  
where \( p(i) \) is the permutation of indices and \( \epsilon(p) \) is the parity of the permutation,

2. Derivation (the Leibniz rule)
\[ \{A_1 A_2, A_3, A_4\} = A_1 \{A_2, A_3, A_4\} + \{A_1, A_3, A_4\} A_2, \]

3. Fundamental Identity
\[ \{\{A_1, A_2, A_3\}, A_4, A_5\} + \{A_3, A_4, A_1, A_2, A_5\} = \{A_1, A_2, A_3, A_4, A_5\}. \]

We should mention that an algebraic n-ary generalization of the Nambu bracket associated with a Nambu-Poisson manifold [17]. However in such a scheme, the associated fundamental identity is too restrictive in that the Nambu dynamics on a n-dimensional manifold is then determined by \((n-1)\) conserved Hamiltonian functions. This too large a number of integrals of motion is clearly most unsuitable for our specific generalization of geometric quantum dynamics where \( n \) is finite only for, say, spin systems but is generically infinite.

Now, we observe that, just as the simple harmonic oscillator is the prototype classical and quantum system of the standard Hamiltonian mechanics, the Euler asymmetric top is the prototypical representative of Nambu’s ternary mechanics. The time evolution of the Euler top in Nambu mechanics is described by two Hamiltonian functions given by the total energy and a Casimir invariant, the square of the angular momentum. These two conserved quantities are: \( H_1 = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{I_i} (M_i)^2 \) and \( H_2 = \frac{1}{2} \sum_{i=1}^{3} (M_i)^2 \), where \( I_i \) denote the principal moments of inertia. It is easy to see that the Nambu equations of motion give the equations written in \( \mathbb{R}^3 \) for the asymmetric Euler top, a rigid body fixed in the center of mass:
\[ \frac{dM_1}{dt} = \left( \frac{1}{I_3} - \frac{1}{I_2} \right) M_2 M_3, \quad \frac{dM_2}{dt} = \left( \frac{1}{I_1} - \frac{1}{I_3} \right) M_3 M_1, \quad \frac{dM_3}{dt} = \left( \frac{1}{I_2} - \frac{1}{I_1} \right) M_1 M_2. \]

where we take \( I_3 > I_2 > I_1 \). These equations have been reincarnated during recent decades in the celebrated Nahm equations [19] for the \( SU(2) \) self-dual Yang-Mills of relevance to theories of extended objects such as monopoles and membranes. Specifically the Euler equations for the asymmetric top naturally describe geodesic flows on a triaxial ellipsoid and can be solved in terms of Jacobi elliptic function [20]. For later comparison, we elaborate briefly on this last point. The equations for the asymmetric top are closely linked to the \( SO(3) \) algebra, the group \( SO(3) \) being that of proper orthogonal transformations in \( \mathbb{R}^3 \). \( \mathbb{R}^3 \) can then be identified with the \( SO(3) \) Lie algebra since the latter is isomorphic to that of vectors in \( \mathbb{R}^3 \). It was shown [21] that the left invariant metric tensor on \( SO(3) \), compatible
with the top motion is given by \( g(X, Y) = -\frac{1}{2}K(IX, Y) \) where \( K \) is the appropriate Killing form, \( I \) is the moment of inertia operator and \( X \) and \( Y \) are tangent vectors to \( \text{SO}(3) \). Then, in an \( \text{SO}(3) \) eigenbasis of \( I \) the above top equations are the geodesics of the given left-invariant Riemannian metric of the group \( \text{SO}(3) \), on a reduced phase space to be identified as a 2-sphere. Thus in terms of the components of angular velocity \( \Omega \), Eq(12) also reads

\[
\frac{d\Omega^i}{dt} + \Gamma_{jk}^i \Omega^j \Omega^k = 0, \tag{13}
\]

where the Christoffel symbols are \( \Gamma_{jk}^i = \frac{1}{2} \epsilon_{jki} (1 - \frac{I_j}{I_i} - \frac{I_k}{I_i}) \). Indeed by being so embedded in \( \text{SO}(3) \) the Euler top equations of motion become a Hamiltonian flow on the (co-)adjoint orbits of the action of \( \text{SO}(3) \) which are 2-spheres, \( S^2 \). As a Hamiltonian system on \( \mathbb{R}^3 \), the time evolution of any observable \( G \) is given through a Lie-Poisson bracket \( \{G, H_1\} \):

\[
\frac{dG}{dt} = \{G, H_1\} = \sum \epsilon_{ijk} M_i \frac{\partial H_1}{\partial M_j} \frac{\partial G}{\partial M_k}. \tag{14}
\]

In fact its very form readily motivates the definition of the triple Nambu bracket if one has another conserved quantity quadratic in the \( M_i \)'s as is the case of \( H_2 \) for the top. However, due to the existence of another integral of motion, \( H_2 \), which is an invariant level surface, a 2-sphere, this bracket is degenerate. This 2-sphere is then the reduced phase space on which the Lie-Poisson bracket is restricted and then nondegenerate. The top equations can be written in the canonical Hamiltonian form, using Darboux theorem [22] by introducing two canonical variables \( \alpha \) and \( \beta \) such that

\[
M_1 = \sqrt{H_2 - \beta^2 \sin \alpha}, \quad M_2 = \sqrt{H_2 - \beta^2 \cos \alpha}, \quad M_3 = \beta. \tag{15}
\]

The top equations have then the canonical form

\[
\frac{d\alpha}{dt} = \frac{\partial H}{\partial \beta}, \quad \frac{d\beta}{dt} = -\frac{\partial H}{\partial \alpha}, \tag{16}
\]

where the Hamiltonian \( H \) reads as follows

\[
H = \frac{1}{2} \left( \frac{\sin \alpha^2}{I_1} + \frac{\cos \alpha^2}{I_2} \right) (H_2 - \beta^2) + \frac{\beta^2}{I_3}. \tag{17}
\]

The symplectic structure is given by \( d\alpha \wedge d\beta \).

Returning to the dynamics of the Euler top, we underscore the case of the symmetric (or Lagrange) top \( (I_1 = I_2) \) for which \( M_3 = \text{const} \). In that case one obtains effectively a simple harmonic oscillator with the characteristic frequency \( \left( \frac{1}{I_1} - \frac{1}{I_3} \right) M_3 \). Also in this limit the Jacobi elliptic functions become ordinary trigonometric functions. We mention in passing that there exists an obvious generalization of this system in the case of an \( n \)-ary Nambu bracket \( (n > 3) \), generated by \( (n - 1) \) quadratic Nambu Hamiltonians. The
resulting generalization of the Euler top can be solved in terms of automorphic functions on a Riemann surface of genus \((n - 1)\), given the results of [23].

Now we are set to state our proposed generalization of the geometric formulation of quantum mechanics. We call it for short - Nambu quantum mechanics.

The key to our scheme is the embedding of the standard quantum mechanics represented as a Hamiltonian system describing a collection of \((N+1)\) abstract harmonic oscillators into a Nambu Hamiltonian system describing an abstract collection of \((N+1)\) Euler tops. The exponential evolution from standard quantum mechanics is then immediately generalized into a time evolution described by the Jacobi elliptic functions and the geometric formulation of quantum mechanics is naturally generalized into a larger geometric structure, possessing extra deformation parameters, one per top used - the modulus of the Jacobi elliptic functions which is the measure of the asymmetry of a top.

We consider a phase space described by real functions of \((3N + 3)\) variables or \((N+1)\) Nambu triplets, \(m_1^a, m_2^a, m_3^a\) (\(a = 1,2,..N+1\)). The fundamental equations of motion for a general observable \(O(m_1^a, m_2^a, m_3^a)\) are assumed to be of the Nambu type

\[
\frac{dO}{dt} = \{h_1, h_2, O\},
\]

where the Nambu bracket \(\{F, G, H\}\) is defined as above \(\{F, G, H\} = \epsilon^{ijk} \partial_{m_i^a} F \partial_{m_j^a} G \partial_{m_k^a} H\). The summation over \(a\) is understood. The generalized Schrödinger equation (the Nambu-Schrödinger equation) is given by the Nambu-Hamilton equation

\[
\frac{dm_1^a}{dt} = (\alpha_3^a - \alpha_2^a) m_2^a m_3^a, \quad \frac{dm_2^a}{dt} = (\alpha_1^a - \alpha_3^a) m_3^a m_1^a, \quad \frac{dm_3^a}{dt} = (\alpha_2^a - \alpha_1^a) m_1^a m_2^a.
\]

As is well known, these equations can be integrated in closed form in terms of Jacobi elliptic functions \(sn, cn, dn\):

\[
m_1^a(t) = K_1^a \text{sn}(c(t - t_0)), \quad m_2^a(t) = K_2^a \text{cn}(c(t - t_0)), \quad m_3^a(t) = K_3^a \text{dn}(c(t - t_0)),
\]

where the modulus \(k\) of the Jacobi elliptic functions is given by

\[
k^2 = \frac{\alpha_1 - \alpha_2}{\alpha_2 - \alpha_3} \frac{2h_1 - \alpha_3 h_2}{h_2 \alpha_1 - 2h_1}.
\]
be more precise we display the simple asymptotics of the Jacobi functions [24]:

\[
\begin{align*}
\sin u & = \sin u - \frac{1}{4}k^2(u - \sin u \cos u) \cos u + O(k^4) \\
c\cos u & = \cos u + \frac{1}{4}k^2(u - \sin u \cos u) \sin u + O(k^4) \\
d\cos u & = 1 - \frac{1}{4}k^2 \sin^2 u + O(k^4),
\end{align*}
\]

(22)

where in our case \( u = c(t - t_0) \). In the \( \alpha_1^a = \alpha_2^a \) limit the Euler top equations describe a (generally infinite) collection of harmonic oscillators and the time evolution is exponential. Also \( m_3^a = const \) and \( m_1^a \) and \( m_2^a \) can be identified with the real and imaginary parts of the usual wave function, as described above (\( m_1^a = q^a, m_2^a = p^a \)). This motivates the general expression for what we call the Nambu wave function

\[
\Psi^a = \sum_i m_i^a e_i
\]

(25)

where \( e_i \) are the usual quaternions imaginary units such that \( e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \). The quaternion conjugate Nambu wave function is \( \bar{\Psi}^a = \sum_i m_i^a \bar{e}_i = -\bar{\Psi}^a \). The inner product reads

\[
\bar{\Psi} \Phi = \delta_{ij} \Psi_i \Phi_j - \epsilon_{ijk} \epsilon_k \bar{\Psi}_i \Phi_j = \bar{\Psi} \cdot \bar{\Phi} - \bar{\epsilon} \cdot (\bar{\Psi} \times \bar{\Phi})
\]

(26)

where in complete correspondence with Eq.(1), the quaternionic real part is the scalar vector product and the imaginary part is the antisymmetric vector product. The second term in the above equation is the quaternionic counterpart of the symplectic 2-form. It is at the basis of the 3-form, characteristic of Nambu’s original mechanics [9]. Due to the non-linear nature of the Nambu-Schrödinger equation the superposition principle apparently no longer holds for \( \Psi \) (See however ref.[25]).

It is also useful to introduce the following two \( 3 \times 3 \) matrices \( L_{rs}^a = -L_{sr}^a, (L_{rr}^a = 0, L_{12}^a = m_3^a, L_{13}^a = -m_2^a, L_{23}^a = m_1^a) \), and \( J_{rs}^a \) \( (J_{rr}^a = \alpha_r^a) \). Then the Nambu-Schrödinger equation can be written in the following Lax form

\[
\frac{dL}{dt} = [L, (JL + LJ)].
\]

(27)

These equations are integrable (in complete analogy with the Euler top equations). From the corresponding Lax equations, one can deduce an infinite number of conservation laws [26].

Notice that the analogs of the commutators of the Nambu quantum mechanics are precisely given in terms of the classical Nambu bracket! Thus this formulation circumvents some of the well known problems encountered in the quantization of the Nambu dynamics [27].
Given this abstract structure of the Nambu-Schrödinger quantum mechanics, it is natural to write down an appropriate operator version. The operator form of the Nambu-Schrödinger equations we suggest reads as follows

\[
\frac{dm^a_1}{dt} = \mathcal{H}m^a_2m^a_3 - \mathcal{H}m^a_3m^a_2, \tag{28}
\]
\[
\frac{dm^a_2}{dt} = \mathcal{H}m^a_3m^a_1 - \mathcal{H}m^a_1m^a_3, \tag{29}
\]
\[
\frac{dm^a_3}{dt} = \mathcal{H}m^a_1m^a_2 - \mathcal{H}m^a_2m^a_1, \tag{30}
\]

where the Hamiltonian is denoted by \(\mathcal{H}\). Similarly, for an operator \(W\) the eigenvalue problem can defined by the equations

\[
Wm^a_2m^a_3 - Wm^a_3m^a_2 = wm^a_1, \tag{31}
\]
\[
Wm^a_3m^a_1 - Wm^a_1m^a_3 = wm^a_2, \tag{32}
\]
\[
Wm^a_1m^a_2 - Wm^a_2m^a_1 = wm^a_3. \tag{33}
\]

Equations of this type appear in the theory of multidimensional determinants [28].

It is interesting to note that the Nambu-Heisenberg commutator

\[ [P, Q, R,] \equiv PQR - PRQ + QRP - QPR + RPQ - RQP = i\hbar \] (34)

can have both finite and infinite dimension Hilbert space realizations [9, 17, 29]. This Nambu-Heisenberg relation suggests the following cubic form of the Nambu-Heisenberg uncertainty principle \(\Delta P \Delta Q \Delta R \sim \hbar_N\), which is similar to the suggested generalization of the space-time uncertainty relation in M-theory [30]. Notice though, that it is not clear that a Hilbert space formulation of the Nambu quantum mechanics is physically appropriate, mainly because of the lack of the superposition principle. Perhaps one should expect an appearance of the 2-Hilbert space structure [31].

Now we discuss the basics of the geometry of the Nambu quantum mechanics. While Nambu mechanics can be locally embedded as a constrained system in the canonical Hamiltonian phase space framework, we shall take it as a stand alone new mechanics. Here tailored to our very purpose is the formulation of [33] for a system made up with \(n\) Nambu triplets. It closely parallels the characteristics of Hamiltonian systems though there are some key differences. Referring to [33] for the details, it is worth gathering the key features of that

\footnote{It has been argued in [31] as well in [32] that background independent formulation of quantum gravity might call for an algebraic structure larger than the category of Hilbert spaces, in which Hilbert spaces of different dimensionality naturally appear[32].}
formulation. The counterpart of standard symplectic phase space is the 3n dimensional non
symplectic Nambu manifold $M^{3n}$ where a nondegenerate closed 3-form takes the place of
the symplectic 2-form. A $C^\infty$ mapping $F : M^{3n} \to N^{3n}$ is a canonical transformation if it
leaves the 3-form invariant. In particular, the Nambu equations of motion realize a phase
flow through two Hamiltonian functions $h_1$ and $h_2$ with the time evolution of the system be-
ing a unfolding of successive 3-form preserving canonical transformations. In such a Nambu
system, a Poisson bracket of 2-forms closing on an SO(3) algebra is the counterpart of the
Poisson bracket of 1-forms or vector fields of usual Hamilton dynamics. On the other hand
the triple brackets for functions $F, G, H$ has a non-associative structure as observed by [9]
and [4]. There is a Nambu-Darboux theorem whereby at every point $P$ in $M^{3n}$ there is a co-
ordinate chart $(U, \phi)$ in which $\omega^3$ has the Darboux form $\omega^{3} = \sum_{i=0} d\alpha_{3i+1} \wedge d\alpha_{3i+2} \wedge d\alpha_{3i+3}$
i(0,1,..n-1).

It is specially noteworthy that the normalization of the Nambu wave function or Born
rule $\sum_a \bar{\Psi}^a \Psi^a = \text{const}$ does not have to be imposed. It is in fact automatically conserved and
given by the value of $h_2$! Thus the usual normalization condition for wave functions follows
from the dynamical set-up and is strongly indicative (after appropriate normalization) of
a probabilistic or stochastic interpretation for the generalized Nambu quantum mechanics.
On the other hand, the fact that the superposition principle may fail suggests that the
concept of measurement and the standard statistical interpretation has to be reevaluated.
The condition

$$C = \sum_a \left[ (m^a_1)^2 + (m^a_2)^2 + (m^a_3)^2 \right],$$

(35)

where $C$ is a constant, defines the space of states of the Nambu quantum mechanics together
with the requirement that the points in this space of states are identified under the action
of $SO(3)$. In other words, $\Psi$ and $U\Psi U^\dagger$, where $U$ is an $SO(3)$ matrix, are identified as
physical states. Thus there is a non-abelian $SO(3)$ phase, which generalizes the usual $U(1)$
phase of quantum mechanics. (It is a fascinating possibility that the usual $U(1)$ phase could
emerge dynamically from the non-abelian $SO(3)$ phase.) More precisely, the normalization
condition as in the U(1) phase case of quantum mechanics, gives rise to the structure of a
$SO(3)$ principal fibre bundle of spheres over spheres, namely $S^{3n+2} \to S^{3n-1}$. We call these
Nambu bundles and readily check that these bundles sit between the complex Hopf bundles
$S^{2n+1} \to CP^n$ of quantum mechanics and the quaternionic Hopf bundles $S^{4n+3} \to HP^n$
of quaternionic quantum mechanics, $HP^n$ being the n-dimensional (4n real) quaternionic
projective space. Namely that, for a given n, the complex line bundles are embedded as
they should within the Nambu bundles, which are themselves embedded in turn in the
quaternionic line bundles. In this sense Nambu quantum mechanics as formulated here is
intermediate between complex and quaternionic quantum mechanics. The three quantum
phase spaces are nested within one another as $CP(n) \subset S^{3n-1} \subset HP(n)$. The Nambu
 manifold, a \((3n-1)\)-sphere is generally neither symplectic nor complex. It is endowed with an invariant closed nondegenerate 3-form which descends from a closed 4-form, characteristic of \(HP^n\) and giving the latter a quaternionic Kahler structure. The metric of the "round" sphere \(S^{3n-1}\) is the extension of the Fubini-Study metric of the complex quantum mechanics.

\[
ds^2_{12} = (1 - |\langle \Psi_1 | \Psi_2 \rangle|^2).
\]  

(36)

Finally, given the basic equations of the Nambu quantum mechanics, we see that there exists a geometric phase generalizing the usual Berry phase. The generalized geometric phase, which should be independent of the equation of motion of first order in time, is given in terms of the Nambu wave function,

\[
\int \bar{\Psi} d\Psi.
\]  

(37)

Clearly, many fundamental questions have yet to be answered about the above Nambu quantum mechanics, the foremost one being its physical interpretation. Thus the bundle structure endowed with a closed nondegenerate 3-form suggests the formulation of a) some generalized Poisson structure and b) hypercomplex structure for the Nambu manifold, something intermediate between complex and quaternionic structure. The existence of some kind of hypercomplex structure may yet provide a nonlinear superposition principle, if any exist. This hope is not out of reach as the classical Euler top is completely integrable with a very apparent Lax structure. The fact that a generalized Born rule is built in through the second Hamiltonian supports our view that geometrizing Nambu mechanics is a natural step since it is already quantized, that we are in the presence of a stochastic dynamical system where the phase space \(S^{3n-1}\) is also a probability space. A n-sphere probability space was considered in [34] in a spinorial generalization on quantum mechanics. Moreover, being the prototype Blaschke manifold [35], the round hyperspheres \(S^n\) share with the projective (Blaschke) spaces \((RP^n, CP^n, HP^n, CaP^2 [3])\) the defining property that their geodesics exhibit the most regular behavior. It is known [13, 25] that this regular geodesic structure is at the basis of the generalized energy-time uncertainty relation.

One of the fundamental questions to be answered concerns the nature and properties of the “observables” of Nambu quantum mechanics. We know that they must be a rather restricted class of functions over the Nambu manifold since in the symmetric top limit where standard quantum mechanics is recovered, they must coincide with the Kahler functions. The latter are in one to one correspondence with observable hermitian operators. Obviously the technology of the operatorial formalism for our scheme has to be developed further and applied to simple physical systems [29]. Similarly, the physical meaning of the generalized uncertainty relations should be understood. Another question is whether there exists a Weyl-Wigner-Moyal-like deformation quantization [36] formulation of geometric Nambu quantum mechanics? We hope to address some of these issues in future work.
Finally, we believe that the correct arena for the application of our formalism is to be found in string theory. In particular, in references [30] and [37] it has been argued that the structure of the Nambu bracket appears quite naturally in the problem of a covariant formulation of Matrix theory [38] based on the analogy with the eleven-dimensional membrane [39]. One of the obstacles in this approach was rooted in the quantization problem of the Nambu bracket. (Similarly the quantization of the topological open membrane is closely related to the problem of the quantization of the triple Nambu bracket [40].) As pointed out in this article, our geometric formulation of the Nambu bracket allows for an identification of the classical Nambu bracket with the quantum Nambu bracket provided one works on the configuration space of the Nambu quantum mechanics. Thus, we believe a stage is set for an application of the Nambu bracket to Matrix theory.

Acknowledgments: We thank V. Balasubramanian, L. N. Chang, M. di Ventri, M. Gunaydin, T. Hubsch, F. Larsen, T. Mizutani, N. Okamura, B. Schmittmann, A. Sen, J. Slawny, U. Tauber, T. Takeuchi, and Royce Zia for useful discussions. D.M. thanks to H. Awata, M. Li and T. Yoneya as well as B. Pioline for many discussions on the Nambu dynamics.

References


