Comment on “Time-like flows of energy-momentum and particle trajectories for the Klein–Gordon equation”

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Abstract

Horton, Dewdney, and Nesteruk [1] have proposed Bohm-type particle trajectories accompanying a Klein–Gordon wave function $\psi$ on Minkowski space. From two vector fields on space-time, $W^+$ and $W^-$, defined in terms of $\psi$, they intend to construct a timelike vector field $W$, the integral curves of which are the possible trajectories, by the following rule: at every space-time point, take either $W = W^+$ or $W = W^-$ depending on which is timelike.

This procedure, however, is ill-defined as soon as both are timelike, or both spacelike. Indeed, they cannot both be timelike, but they can well both be spacelike, contrary to the central claim of [1]. We point out the gap in their proof, provide a counterexample, and argue that, even for a rather arbitrary wave function, the points where both $W^+$ and $W^-$ are spacelike can form a set of positive measure.

Let $\psi = e^{P+iS}$ (where $P$ and $S$ are real) solve the Klein-Gordon equation, $-\Box \psi = m^2 \psi$. Set $P_\mu = \partial_\mu P$, $S_\mu = \partial_\mu S$, and

$$\theta = \sinh^{-1} \frac{P_\mu P_\mu - S_\mu S_\mu}{2 P_\mu S_\mu}.$$ 

That $P_\mu$ and $S_\mu$ are orthogonal is an exceptional case that we neglect, like the authors of [1]. For $W_\mu$ one is supposed to take either $W^+_\mu = e^{\theta} P_\mu + S_\mu$.

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or \( W^- = -e^{-\theta}P_\mu + S_\mu \), depending on which is timelike; they cannot both be timelike since they are orthogonal. The question is, could they both be spacelike?

The authors of \cite{1} declare that \( W^+ \) and \( W^- \) cannot both be spacelike and argue like this: otherwise there exists a Lorentz frame such that \( W^+_0 = 0 \) and \( W^-_0 = 0 \), thus \( e^\theta P_0 = -e^{-\theta}P_0 \), from which they conclude \( e^\theta = -e^{-\theta} \), which is impossible.

There are two mistakes in this argument. On the one hand, that two vectors are both spacelike does not mean they are both contained in a spacelike hyperplane (corresponding to \( x^0 = 0 \) in the appropriate Lorentz frame): think of \((1, 2, 0, 0)\) and \((0, 2, 0, 0)\), two spacelike vectors whose difference is timelike, so they cannot lie in a spacelike hyperplane. On the other hand, even if \( W^+ \) and \( W^- \) do lie in a spacelike hyperplane, no contradiction would arise since in this case \( P_0 \) would be 0 (in the frame described above).

\( W^+ \) and \( W^- \) lying in the same spacelike hyperplane amounts to \( P_\mu \) and \( S_\mu \) lying in that hyperplane. Can this case occur? Clearly: since the Klein–Gordon equation is of second order, one may choose \( \psi \) and \( \partial_0 \psi \) ad libitum on the \( x^0 = 0 \) hyperplane. Can it also occur for the first-order Klein–Gordon equation \( -i\partial_0 \psi = \sqrt{m^2 - \Delta} \psi \), or, equivalently, for functions from the positive-energy subspace? Here is an example: let \( \psi \) be a superposition of three\(^1\) plane waves

\[
\psi(x) = \sum_{i=1}^{3} c_i e^{ik_{\mu}^{(i)} x^\mu}
\]

with wave vectors \( k_{\mu}^{(1)} = (m, 0, 0, 0) \), \( k_{\mu}^{(2)} = (\sqrt{27m}, \sqrt{26m}, 0, 0) \), \( k_{\mu}^{(3)} = (\sqrt{27m}, 0, \sqrt{26m}, 0) \), and \( c_1 = 3, c_2 = -1/\sqrt{3} - i, c_3 = i \). Then, at the coordinate origin, we find \( P_\mu = (0, \alpha, -\alpha, 0) \) and \( S_\mu = (0, -\beta, 0, 0) \) with \( \alpha = \sqrt{26m}/\gamma, \beta = \alpha/\sqrt{3} \) and \( \gamma = 3 - 1/\sqrt{3} \). This example could also be made square-integrable by replacing the plane waves \( \exp(ik_{\mu}^{(i)} x^\mu) \) by positive-energy \( L^2 \) Klein–Gordon functions \( \varphi^{(i)}(x) \) with the properties \( \varphi^{(i)}(0) = 1 \) and \( \partial_\mu \varphi^{(i)}(0) = ik_{\mu}^{(i)} \).

One may suspect, however, that perhaps this particular wave function \( \psi \) is very exceptional, and perhaps even that for this special wave function the coordinate origin is a rather atypical point, so that the sort of situation

\(^1\)Two will not suffice for an example since \( P_\mu \) and \( S_\mu \) are linear combinations of the \( k_\mu \) vectors.
just described can be ignored. After all, we would be willing to ignore the case where $P_\mu$ and $S_\mu$ are orthogonal because in the 8-dimensional space of all possible pairs of vectors $P_\mu, S_\mu$ it corresponds to a subset of dimension 7, and therefore one would expect that the space-time points where this happens form a set of measure zero.

But since $W_\mu^+ W^{\mu+}$ and $W_\mu^+ W^{\mu-}$ are continuous functions of $P_\mu$ and $S_\mu$, the set of pairs $P_\mu, S_\mu$ where both $W^+$ and $W^-$ are spacelike is open and thus has positive measure in 8 dimensions. (The same is true of the set of pairs $P_\mu, S_\mu$ such that the 2-plane they span contains solely spacelike vectors.)

I know of nothing precluding any pairs $P_\mu, S_\mu$ from arising from a Klein–Gordon wave function, so it seems reasonable to expect that the space-time points with spacelike $W^+$ and $W^-$ form a set of positive measure for many wave functions, perhaps for most.

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References