Hard Thermal Effects in Noncommutative $U(N)$ Yang-Mills Theory

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We study the behaviour of the two- and three-point thermal Green functions, to one loop order in noncommutative $U(N)$ Yang-Mills theory, at temperatures $T$ much higher than the external momenta $p$. We evaluate the amplitudes for small and large values of the variable $\theta p T$ ($\theta$ is the noncommutative parameter) and exactly compute the static gluon self-energy for all values of $\theta p T$. We show that these gluon functions, which have a leading $T^2$ behaviour, are gauge independent and obey simple Ward identities. We argue that these properties, together with the results for the lowest order amplitudes, may be sufficient to fix uniquely the hard thermal loop effective action of the noncommutative theory.

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I. INTRODUCTION

Noncommutative manifolds have been used in physics for quite some time [1] and were more recently applied in the context of string theories [2]. In certain circumstances, the low-energy behaviour of these theories may be described in terms of gauge fields defined on a space-time where the coordinates do not commute, so that

$$[x_\mu, x_\nu] = i \theta_{\mu\nu}. \quad (1)$$

The antisymmetric tensor $\theta_{\mu\nu}$, which has the canonical dimension of inverse mass squared, is assumed to be independent of the space-time coordinates. Such gauge field theories are non-local and exhibit many intriguing properties, which have been much studied in recent years (for reviews and a complete list of references see, for example, [3, 4]). In particular, several thermal effects in noncommutative theories have been already examined in a series of interesting papers [5, 6, 7, 8].

In gauge theories at finite temperature, a consistent perturbative expansion requires the resummation of a set of diagrams called hard thermal loops [9]. These arise from one-loop diagrams in the region where the internal momenta are of order of the temperature $T$, which is large compared with all the external momenta. The purpose of this work is to study in the non-commutative $U(N)$ Yang-Mills theory, the high temperature behaviour of the two- and three-point gluon functions. Our method of calculation employs an analytic continuation of the imaginary-time formalism [10]. Using this approach, we relate the Green functions to forward scattering amplitudes of on-shell thermal particles, a technique that has been previously applied in the SU($N$) gauge theory as well as in gravity [11, 12, 13]. In contrast to the situation in the commutative theory, where the hard thermal loop contributions are completely determined in the region where $T \gg p$, one has to consider in the noncommutative case also the value of the independent parameter $\theta p T$. For arbitrary values of this parameter, the explicit calculation of these non-local amplitudes is, in general, very difficult. Only in certain limiting cases, such as $\theta p T \ll 1$ or $\theta p T \gg 1$, is that their evaluation becomes more transparent. An exception occurs in the static limit, where one can evaluate, as shown in section II, the gluon self-energy $\Pi_{\mu\nu}^{AB}$ in a closed form for all values of $\theta p T$. The corresponding result shows a significant difference from the behaviour in the commutative case, where only the $\Pi_{00}^{AB}$ component survives in the static limit. On the other hand, in the noncommutative theory, we find that also the $\Pi_{0i}^{AB}$ components receives leading $T^2$ contributions in the static case. This behaviour reflects the effect of extra magnetic fields which are induced by the noncommutative character of the theory. Furthermore, we show in the section III that the gluon self-energy is generally transverse with respect to the external momenta, and that all the $\theta$-dependence resides in the U(1) subgroup of U($N$).

The three-point gluon function is discussed in section IV, where we also provide its full $\theta$-dependence which arises in both the U(1) and SU($N$) sectors. Although the inherent angular integrations are extremely involved and cannot be performed in closed form, we demonstrate that, in general, the leading $T^2$ contributions to the three-point amplitude are related to those of the two-point function in a manner dictated by the Ward identity.

Our main motivation for this investigation is that, under certain conditions, these gauge invariant amplitudes, together with the Ward identity, may be sufficient to determine the effective action which sums up the effects of all hard thermal loops. We discuss this issue in section V and present more details on our calculations in the appendices.
II. THE TWO-POINT FUNCTION

In accordance with the approach initiated in [14] and extended to the Yang-Mills theory in [11] we can compute the Feynman graphs of Fig. (1) by considering the on-shell forward scattering amplitudes $A_{AB}^{\mu\nu}$ of Fig. (2). These amplitudes are related to the corresponding two-point function by the equation

$$\Pi_{\mu\nu}^{AB} = -\frac{1}{(2\pi)^3} \int \frac{d^3k}{2|k|} N(|k|) A_{\mu\nu}^{AB} \big|_{k_0=|k|}: \quad A, B = 0, 1, 2, \cdots, N^2-1,$$

where

$$N(|k|) = \frac{1}{e^{|k|/T} - 1}$$

is the Bose-Einstein distribution function.

![Diagram](image1.png)

FIG. 1: One-loop diagrams which contribute to the self-energy in the noncommutative U($N$) theory. Wavy and dashed lines denote respectively gauge particles and ghosts. The external momenta are inward.

Using the Feynman rules shown in the appendix A, the contribution of the graph of Fig. (2) a) to the forward scattering amplitude is given by

$$B_{AB(a)}^{\mu\nu} = -3g^2\eta^{\mu\nu} \left[ \text{Tr} D_A D_B (1 - c_p) - \text{Tr} F_A F_B (1 + c_p) \right].$$

We are employing the definitions

$$(F_A)_{BC} \equiv f_{BAC}, \quad (D_A)_{BC} \equiv d_{BAC}$$

as well as the abbreviations

$$c_p \equiv \cos(p \times k), \quad c_{pq} \equiv \cos(p \times q), \quad c_{\frac{1}{2} p} \equiv \cos\left(\frac{1}{2} p \times k\right), \quad s_{\frac{1}{2} p} \equiv \sin\left(\frac{1}{2} p \times k\right), \quad c_{\frac{1}{2} p \times q} \equiv \cos\left(\frac{1}{2} p \times q\right); \quad p \times q \equiv p_{\mu} \theta^{\mu\nu} q_{\nu}$$
Using the results (see \[15\] for similar formulas in the context of one-loop renormalization of noncommutative theories)

\[
\begin{align*}
\text{Tr } F_A F_B &= -N(1 - \delta_{A,0}) \delta_{AB} \\
\text{Tr } D_A D_B &= N(1 + \delta_{A,0}) \delta_{AB} \\
\text{Tr } D_A F_B &= 0,
\end{align*}
\]

Eq. (4) reduces to

\[
B_{AB}^{\mu\nu} = -6g^2 N \delta_{AB} \eta^{\mu\nu} (1 - \delta_{A,0} c_p).
\]

From the graph of Fig. (2 bi) we receive a total contribution

\[
B_{AB}^{\mu\nu} = \frac{g^2}{4} N \delta_{AB} \left[5p^2 \eta^{\mu\nu} - 2p^\mu p^\nu + 5(k^\mu p^\nu + k^\nu p^\mu) + 10k^\mu k^\nu + 2k \cdot p \eta^{\mu\nu}\right] \frac{1}{p^2 + 2k \cdot p}
\times [D_A D_B(1 - c_p) - F_A F_B(1 + c_p)] (k \rightarrow -k).
\]

The contribution of Fig. (2 bii) is identical to that of Fig. (2 bi) with the momentum reversed. Consequently, totalling Figs. (2 bi) and (2 bii) results in

\[
B_{AB}^{\mu\nu} = g^2 N \delta_{AB} (1 - \delta_{A,0} c_p) \left[5p^2 \eta^{\mu\nu} - 2p^\mu p^\nu + 5(k^\mu p^\nu + k^\nu p^\mu) + 10k^\mu k^\nu + 2k \cdot p \eta^{\mu\nu}\right] \frac{1}{p^2 + 2k \cdot p} (k \rightarrow -k).
\]

In a completely analogous fashion, we find that the amplitude receives the following contributions from Figs. (2 ci) and (2 cii),

\[
B_{AB}^{\mu\nu} = -g^2 N \delta_{AB} (1 - \delta_{A,0} c_p) \frac{k^\mu (k + p)^\nu}{p^2 + 2k \cdot p} (k \rightarrow -k).
\]

and

\[
B_{AB}^{\mu\nu} = -g^2 N \delta_{AB} (1 - \delta_{A,0} c_p) \frac{k^\mu (k - p)^\nu}{p^2 + 2k \cdot p} (k \rightarrow -k).
\]

In computing (8), (10), (11) and (12) the gauge parameter \(\xi\) in Eq. (A1) has been taken to be arbitrary, but it cancels completely in the final result for the \(T^2\) terms. In the regime in which \(p \ll k \sim T\), we make the expansion

\[
\frac{1}{p^2 + 2k \cdot p} = \frac{1}{2k \cdot p} - \frac{p^2}{(2k \cdot p)^2} + \cdots,
\]

so that at the leading order in \((2k \cdot p)^{-1}\), the total contribution to \(A_{\mu\nu}^{AB}\) coming from (8), (10), (11) and (12) is

\[
A_{\mu\nu}^{AB} = -4g^2 N \delta^{AB} \left[1 - \delta^{A,0} \cos(\bar{p} \cdot k)\right] G_{\mu\nu}; \quad \bar{p}_\mu \equiv p^\rho \theta_{\rho\mu}
\]

where

\[
G_{\mu\nu} = \frac{p^2 k_\mu k_\nu}{(k \cdot p)^2} - \frac{p_\mu k_\nu + p_\nu k_\mu}{k \cdot p} + \eta_{\mu\nu}.
\]

One can easily verify that the transversality property

\[
p^\mu \Pi_{\mu\nu}^{AB} = 0
\]

is satisfied. Indeed, this is a direct consequence of \(p^\nu G_{\mu\nu} = 0\). Therefore, we can express the self-energy in terms of the following decomposition

\[
\Pi_{\mu\nu}^{AB} = \Pi_1^{AB} \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \Pi_2^{AB} \left(p_\mu - \frac{p^2}{p \cdot u} u_\mu\right) \left(p_\nu - \frac{p^2}{p \cdot u} u_\nu\right) \frac{1}{p^2}
\]

\[
+ \Pi_3^{AB} p^2 \bar{p}_\mu \bar{p}_\nu + \Pi_4^{AB} \left[\left(p_\mu - \frac{p^2}{p \cdot u} u_\mu\right) \bar{p}_\nu + \left(p_\nu - \frac{p^2}{p \cdot u} u_\nu\right) \bar{p}_\mu\right],
\]
where \( u_\mu \) represents the heat bath four velocity \([u = (1, 0, 0, 0)]\). A straightforward calculation gives

\[
\Pi_{1}^{AB} = \Pi^\mu_{\mu}^{AB} + (\sigma^2 - 1)\Pi_{00}^{AB} - \frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB} ; \quad \sigma^2 \equiv \frac{p^2_0}{p^2}
\] (18)

\[
\Pi_{2}^{AB} = \sigma^2 \Pi^\mu_{\mu}^{AB} + 2\sigma^2(\sigma^2 - 1)\Pi_{00}^{AB} - \sigma^2 \frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB}
\] (19)

\[
\Pi_{3}^{AB} = -\frac{1}{p^2} \Pi^\mu_{\mu}^{AB} - \frac{\sigma^2 - 1}{p^2} \Pi_{00}^{AB} + 2\frac{\bar{p}^\mu \bar{p}^\nu}{p^4} \Pi_{\mu\nu}^{AB}
\] (20)

\[
\Pi_{4}^{AB} = (\sigma^2 - 1)\frac{p_0 \bar{p}_0}{p^2} \Pi_{\mu0}^{AB} .
\] (21)

In order to ensure unitarity [3, 4], we have taken

\[
\theta_{0\mu} = 0.
\] (22)

The computation of the integrals appearing in Eqs. (18) to (21) is carried out in the appendix B. We find

\[
\bar{p}_\mu \Pi_{\mu0}^{AB} = 0,
\] (23)

\[
\Pi^\mu_{\mu}^{AB} = N \delta^{AB} \frac{g^2 T^2}{3} \left[ 1 - \frac{6}{\pi^2} \delta^{A,0} \sum_{n=1}^{\infty} \frac{1}{n^2 + \tau^2} \right],
\] (24)

\[
\Pi_{00}^{AB} = \frac{2g^2N \delta^{AB} T^2}{(2\pi)^2} \int_{-1}^{1} d\zeta \left[ 1 - \frac{2\sigma}{\sigma - \zeta} + \frac{\sigma^2 - 1}{(\sigma - \zeta)^2} \right] \times \left[ \frac{\pi^2}{6} - \delta^{A,0} \sum_{n=1}^{\infty} \frac{n}{n^2 + \tau^2(1 - \zeta^2)^2} \right],
\] (25)

and

\[
\frac{\bar{p}_\mu \bar{p}_\nu}{p^2} \Pi_{\mu\nu}^{AB} = \frac{-2g^2N \delta^{AB} T^2}{(2\pi)^2} \int_{-1}^{1} d\zeta \left\{ \left[ \frac{\sigma^2 - 1}{(\sigma - \zeta)^2} \right] - \frac{2\sigma}{\sigma - \zeta} + \frac{\sigma^2 - 1}{(\sigma - \zeta)^2} \right\} \times \left[ \frac{\pi^2}{6} - \delta^{A,0} \sum_{n=1}^{\infty} \frac{n}{n^2 + \tau^2(1 - \zeta^2)^2} \right] + \delta^{A,0} \frac{\sigma^2 - 1}{(\sigma - \zeta)^2} \frac{1}{2\tau^2} \sum_{n=1}^{\infty} \left( \frac{n - \sqrt{n^2 + \tau^2(1 - \zeta^2)^2}}{n^2 + \tau^2(1 - \zeta^2)^2} \right)^2 \left( n + 2\sqrt{n^2 + \tau^2(1 - \zeta^2)^2} \right)
\] (26)

We have defined \( \tau^2 \equiv \bar{p}_\mu \bar{p}_\nu T^2 = (\theta_{\mu\nu} p^\nu)^2 T^2 \). However, in view of condition (22), this parameter is actually independent of the energy \( p_0 \).

As \( G_{\mu\nu} \) in Eq. (13) is homogeneous of degree zero in \( k \), \( \Pi_{\mu\nu}^{AB} \) acquires to leading order an overall factor of \( T^2 \).

We now consider the various limits referred to in the introduction. With the limit \( \theta p T \gg 1 \), we see that all terms proportional to \( \delta^{A,0} \) vanish, so that non-planar graphs no longer contribute. This leaves us with some straightforward integration that leads to

\[
\Pi^\mu_{\mu}^{AB} = N \delta^{A,B} \frac{g^2 T^2}{3},
\] (27)

\[
\Pi_{00}^{AB} = N \delta^{A,B} \frac{g^2 T^2}{3} \left( 1 - \frac{\sigma}{2} \log \frac{\sigma + 1}{\sigma - 1} \right)
\] (28)
and
\[
\frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB} = -N\delta^{AB} \frac{g^2 T^2}{3} \left[ \frac{1}{2} \left( \sigma^2 - 1 \right) \log \frac{\sigma + 1}{\sigma - 1} - \sigma \right]
\]
(29)

Furthermore, in the limit \(\theta p T \ll 1\), it is possible to extract the contribution to the sums in Eqs. (24) to (28) that are of leading order in \(\tau^2\). Using the standard results for the Riemann zeta function, \(\zeta(2) = \pi^2/6\) and \(\zeta(4) = \pi^4/90\), we find that
\[
\Pi_{\mu\nu}^{AB} \approx N\delta^{AB} \frac{g^2 T^2}{3} \left[ 1 - \delta^{A,0} + \frac{(\pi \tau)^2}{15} \delta^{A,0} \right],
\]
(30)
\[
\Pi_{00}^{AB} \approx N\delta^{AB} \frac{g^2 T^2}{3} \left[ \left( 1 - \frac{\sigma}{2} \log \frac{\sigma + 1}{\sigma - 1} \right) (1 - \delta^{A,0}) \right]
+ \frac{(\pi \tau)^2}{15} \left( 2 - 3\sigma^2 + \frac{3}{2}\sigma(\sigma^2 - 1) \log \frac{\sigma + 1}{\sigma - 1} \right) \delta^{A,0}
\]
(31)
and
\[
\frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB} = -N\delta^{AB} \frac{g^2 T^2}{3} \left\{ \sigma \left[ \frac{1}{2} \left( \sigma^2 - 1 \right) \log \frac{\sigma + 1}{\sigma - 1} - \sigma \right] \left( 1 - \delta^{A,0} \right) \right.
+ \frac{(\pi \tau)^2}{15} \left[ 1 - \frac{15}{4}\sigma^2 + \frac{9}{4}\sigma^4 - \frac{9}{8}\sigma(\sigma^2 - 1) \log \frac{\sigma + 1}{\sigma - 1} \right] \delta^{A,0} \right\}.
\]
(32)

Terms of order \(\tau^4\) and beyond can be similarly computed. We note that all terms involving \(\theta\) in Eqs. (23) to (28) are proportional to \(\delta^{A,0}\).

The integration over \(\zeta\) in Eqs. (23) and (26) can be performed and in the static limit when \(\sigma, p_0 \to 0\). We find in this case that
\[
\lim_{\sigma \to 0} \Pi_{00}^{AB} = N\delta^{AB} \frac{g^2 T^2}{3} \left[ 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \tau^2)^2} \delta^{A,0} \right],
\]
(33)
\[
\lim_{\sigma \to 0} \frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB} = -2N\delta^{AB} \frac{g^2 T^2}{\pi^2} \delta^{A,0} \sum_{n=1}^{\infty} \frac{\tau^2}{(n^2 + \tau^2)^2}.
\]
(34)

(Of course, Eq. (24) remains unchanged in the static limit.) The sums appearing in Eqs. (24), (33) and (34) are standard [13]:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2},
\]
(35)
\[
\sum_{n=1}^{\infty} \frac{1}{(n^2 + a^2)^2} = \frac{\pi}{4a^3} \coth(\pi a) + \frac{\pi^2}{4a^2} \csch^2(\pi a) - \frac{1}{2a^2}.
\]
(36)

We consequently are led to
\[
\Pi_{\mu\nu}^{AB} = N\delta^{AB} \frac{g^2 T^2}{3} \left[ 1 - 3 \left( \frac{1}{\pi \tau} \coth(\pi \tau) - \frac{1}{(\pi \tau)^2} \right) \delta^{A,0} \right],
\]
(37)
\[
\lim_{\sigma \to 0} \Pi_{00}^{AB} = N\delta^{AB} \frac{g^2 T^2}{3} \left\{ 1 - \frac{3}{2} \left[ \frac{1}{\pi \tau} \coth(\pi \tau) - \csch^2(\pi \tau) \right] \delta^{A,0} \right\}
\]
(38)
and
\[
\lim_{\sigma \to 0} \frac{\bar{p}^\mu \bar{p}^\nu}{p^2} \Pi_{\mu\nu}^{AB} = -N\delta^{AB} \frac{g^2 T^2 \csch^2(\pi \tau)}{2(\pi \tau)^2} \left[ 1 + (\pi \tau)^2 - \cosh(2\pi \tau) + \frac{\pi \tau}{2} \sinh(2\pi \tau) \right] \delta^{A,0}.
\]
(39)
From the previous results we can now obtain the static limit of Eqs. (18), (19) and (20) (we show in the appendix that $\Pi^{AB}_{i0} = 0$). From Eq. (18), we have

$$\lim_{\sigma \to 0} \Pi^{AB}_{1} = \lim_{\sigma \to 0} \Pi^{\mu AB}_{\mu} - \lim_{\sigma \to 0} \Pi^{AB}_{00} - \lim_{\sigma \to 0} \frac{\tilde{p}^{\mu} \tilde{p}^{\nu}}{\tilde{p}^2} \Pi^{AB}_{\mu\nu}. \tag{40}$$

Using Eqs. (24), (33) and (34), we obtain

$$\lim_{\sigma \to 0} \Pi^{AB}_{1} = 0. \tag{41}$$

The static limit of $\Pi_2$ in Eq. (19) behaves as $\sigma^2$. Therefore, $\Pi_2$ will contribute only to $\Pi^{AB}_{i0}$ in Eq. (17) but not to either $\Pi^{AB}_{ij}$ or $\Pi^{AB}_{ii}$ ($i, j = 1, 2, 3$) [of course, we do not need $\Pi_2$ in order to obtain $\Pi^{AB}_{00}$ which has already been obtained explicitly in Eq. (38)]. Using Eq. (20), we obtain for the static limit of $\Pi_3$

$$\lim_{\sigma \to 0} \Pi^{AB}_{3} = -\frac{1}{|\tilde{p}|^2 |\tilde{p}|^2} \lim_{\sigma \to 0} \left[ \Pi^{\mu AB}_{\mu} - \Pi^{AB}_{00} - 2 \frac{\tilde{p}^{\mu} \tilde{p}^{\nu}}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} \right]$$

$$= -3 \lim_{\sigma \to 0} \frac{\tilde{p}^{\mu} \tilde{p}^{\nu}}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} \tag{42}$$

Finally, using Eq. (13) in Eq. (42) as well as simple functional relations involving the hyperbolic functions, we obtain

$$\lim_{\sigma \to 0} \Pi^{AB}_{3} = N \delta^{A,B} \delta^{A,0} \frac{3 g^2 T^2}{2} \left[ \frac{1}{\sinh^2(\pi \tau)} - \frac{2}{(\pi \tau)^2} + \frac{\cosh(\pi \tau)}{\pi \tau \sin(\pi \tau)} \right]. \tag{43}$$

Inserting Eq. (13) into Eq. (17) one can easily obtain $\Pi^{AB}_{ij}$. It is interesting to note that in commutative Yang-Mills theory $\Pi^{AB}_{00}$ is non-zero but $\Pi^{AB}_{ij}$ vanishes, while in the noncommutative case, we see from Eq. (43) that $\Pi^{AB}_{ij} \neq 0$ when then colour $A$ is in the U(1) sector ($A = 0$). This is consistent with the additional magnetic interactions appearing in the initial Lagrangian. However, since $\Pi^{AB}_{ii}(p_0 = 0, \tilde{p} \to 0) = 0$, the magnetic mass vanishes also in the noncommutative theory.

It also proves possible to examine $\Pi^{AB}_{\mu\nu}$ in the long wave length limit $\sigma \to \infty$ and $\tau \to 0$. In this case, we see from Eqs. (24), (25) and (26) that

$$\Pi^{AB}_{00} \to 0,$$

$$\Pi^{AB}_{\mu} \to N \delta^{A,B} \frac{3 g^2 T^2}{9} (1 - \delta^{A,0}),$$

$$\frac{\tilde{p}^{\mu} \tilde{p}^{\nu}}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} \to N \delta^{A,B} \frac{3 g^2 T^2}{9} (1 - \delta^{A,0}). \tag{44}$$

III. THE THREE-POINT FUNCTION

The Feynman graphs we consider are those of Fig. 3. These are associated with the amplitudes of Fig. 4. The calculation of the diagrams in figure 4 is straightforward but very tedious. After some algebra, the amplitude in figure 4 (a) can be written as (we have employed the Maple version of the symbolic computer package HIP)

$$A^{gl}_{\mu\nu\lambda} = \left[ \sin \left( \frac{p_1 \times p_2}{2} \right) C^{gl}_{\sin} ABC + \cos \left( \frac{p_1 \times p_2}{2} \right) C^{gl}_{\cos} ABC \right] L^{gl}_{\mu\nu\lambda}. \tag{45}$$

Similarly the sum of the diagrams in figures 4 (b) and 4 (c) give

$$A^{gh}_{\mu\nu\lambda} = \left[ \sin \left( \frac{p_1 \times p_2}{2} \right) C^{gh}_{\sin} ABC + \cos \left( \frac{p_1 \times p_2}{2} \right) C^{gh}_{\cos} ABC \right] L^{gh}_{\mu\nu\lambda}. \tag{46}$$

The factors $C^{gl}_{\sin}, C^{gl}_{\cos}, C^{gh}_{\sin}, C^{gh}_{\cos}$ are trigonometric functions of the internal momentum $k$ and involve the colour factors. At any specific order in the hard thermal loop expansion, the Lorentz factors $L^{gl}_{\mu\nu\lambda}$ and
$L^{gh}_{\mu\nu\lambda}$ will be odd or even in $k$. Terms which are odd in $k$ will be multiplied by the following antisymmetric factors

$$A_{gl}^{ABC} = \frac{1}{2} \left( C_{gl}^{ABC}(k) - C_{gl}^{ABC}(-k) \right)$$

$$A_{gh}^{ABC} = \frac{1}{2} \left( C_{gh}^{ABC}(k) - C_{gh}^{ABC}(-k) \right)$$

$$A_{gs}^{ABC} = \frac{1}{2} \left( C_{gs}^{ABC}(k) - C_{gs}^{ABC}(-k) \right)$$

Terms which are even in $k$ will be multiplied by the following symmetric factors

$$S_{gl}^{ABC} = \frac{1}{2} \left( C_{gl}^{ABC}(k) + C_{gl}^{ABC}(-k) \right)$$

$$S_{gh}^{ABC} = \frac{1}{2} \left( C_{gh}^{ABC}(k) + C_{gh}^{ABC}(-k) \right)$$

$$S_{gs}^{ABC} = \frac{1}{2} \left( C_{gs}^{ABC}(k) + C_{gs}^{ABC}(-k) \right)$$

A straightforward calculation gives

$$A_{gs}^{ABC} = -\text{Tr} \left[ F_A^{A} D^{B} D^{C} c_{\frac{1}{2}p_1} c_{\frac{1}{2}p_2} s_{\frac{1}{2}p_3} - F^{A} F^{B} F^{C} c_{\frac{1}{2}p_1} s_{\frac{1}{2}p_2} c_{\frac{1}{2}p_3} ight. \right.$$  

$$+ F^{C} D^{A} D^{B} s_{\frac{1}{2}p_1} c_{\frac{1}{2}p_2} c_{\frac{1}{2}p_3} - F^{B} D^{C} D^{A} s_{\frac{1}{2}p_1} s_{\frac{1}{2}p_2} s_{\frac{1}{2}p_3} \right].$$
In contrast to the antisymmetric coefficient of \( \sin[(p_1 \times p_2)/2] \), using the relations \( \frac{1}{2} \) and \( \frac{1}{2} \), we can write
\[
\text{Tr} F^A F^B F^C = -\frac{N}{2} f^{ABC} = -\text{Tr} F^A D^B D^C,
\]
as well as the cyclic property of the trace, we can write
\[
A_s^{\text{gl} ABC} = -\frac{N}{2} f^{ABC} \left[ c_{p_1}^p c_{p_2}^p s_{p_3}^p + c_{p_2}^p s_{p_1}^p c_{p_3}^p + s_{p_1}^p s_{p_2}^p c_{p_3}^p \right] = -\frac{N}{2} f^{ABC} \sin \left( \frac{p_1 + p_2 + p_3}{2} \times k \right) = 0,
\]
where we have used the momentum conservation \( p_1 + p_2 + p_3 = 0 \). A similar calculation for the ghost part also gives
\[
A_s^{\text{gh} ABC} = 0.
\]
The Eqs. \( \frac{1}{2} \) and \( \frac{1}{2} \) imply that there is no contribution which is odd in the temperature \( T \) and proportional to \( \sin[(p_1 \times p_2)/2] \).

Let us now consider the coefficients of \( \cos[(p_1 \times p_2)/2] \), which are antisymmetric functions of \( k \), namely \( A_s^{\text{gl} ABC} \) and \( A_s^{\text{gh} ABC} \). The colour traces which are involved now are \( \frac{1}{2} \). We have
\[
\text{Tr} D^A D^B D^C = \frac{N}{2} \eta_{ABC} d^{ABC}; \quad \eta_{ABC} = d_{AB} d_{C} - 4 \delta_{A+B+C,0}
\]
\[
\text{Tr} F^A F^B F^C = -\frac{N}{2} c_{ABC} d^{ABC}; \quad c_A = 1 - \delta_{A,0}; \quad d_A = 1 + \delta_{A,0}
\]
A straightforward calculation gives
\[
A_s^{\text{gl} ABC} = -\text{Tr} \left[ F^A F^B D^C c_{p_1}^p c_{p_2}^p s_{p_3}^p + F^A F_B F^C s_{p_1}^p c_{p_2}^p c_{p_3}^p + F^C F_B F^D s_{p_1}^p s_{p_2}^p s_{p_3}^p \right] \quad \text{(54)}
\]
In contrast to the antisymmetric coefficient of \( \sin[(p_1 \times p_2)/2] \), \( A_s^{\text{gl} ABC} \) does not vanish by itself. Using trace cyclicity, one can write
\[
A_s^{\text{gh} ABC} = \text{Tr} \left[ -F^A F^B D^C c_{p_1}^p c_{p_2}^p s_{p_3}^p + \frac{1}{3} D^A D^B D^C s_{p_1}^p s_{p_2}^p s_{p_3}^p \right] + \text{cyclic permut.}
\]

FIG. 4: Forward scattering amplitudes corresponding to the diagrams in Figure 3. The direction of the ghost momentum is the same as the corresponding internal gluon line. Permutations of the vertices are understood.
Let us consider the specific case of the superleading contribution, which would be proportional to \(T^3\). The Lorentz factor for this piece is the same for the gluon and for the ghost diagrams, being proportional to \([\text{see the diagrams in figure 1(a), (b) and (c)}]\):

\[
\frac{k_\mu k_\nu k_\lambda}{k \cdot p_1 k \cdot p_2 k \cdot p_3}
\]

Including the colour and the \(\cos\left(\frac{p_1 \times p_2}{2}\right)\) factors, this leads to a contribution

\[
\cos\left(\frac{p_1 \times p_2}{2}\right) A_{\cos}^{gl ABC}(p_1, p_2, p_3) k_\mu k_\nu k_\lambda \frac{1}{k \cdot p_1 k \cdot p_2 k \cdot p_3}
\]

The only part of Eq. (57) which will change when we add together all cyclic permutations is the denominator. Using momentum conservation, we have

\[
\frac{1}{k \cdot p_1 k \cdot p_3} + \frac{1}{k \cdot p_1 k \cdot p_2} + \frac{1}{k \cdot p_2 k \cdot p_3} = 0,
\]

and thus we can easily see that the superleading contribution (viz., those that are proportional to \(T^3\)) will vanish. The same property is also true for the ghost diagram. This cancellation of the superleading contribution is similar to what happens in QCD [1]. However, in the case of noncommutative theories one has to employ the cyclicity property of the colour/trigonometric factor, rather than simply relying in the cancellation which occurs when we add the contributions with \(k \rightarrow -k\), as is the case in QCD. (In QCD, there is no \(k\)-dependent trigonometric factor which is odd in \(k\).)

In a similar way, it is straightforward to show that

\[
S_{\cos}^{gl ABC} = \text{Tr} \left[ F^B F^B F^C c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} c_{\frac{1}{2} p_3} + F^A D^B D^C c_{\frac{1}{2} p_1} s_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} 
+ F^B D^C D^A c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} + F^C D^B D^A s_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right]
= -\frac{N}{2} f^{ABC} \cos \left[ \frac{1}{2} (p_1 + p_2 + p_3) \times k \right] = -\frac{N}{2} f^{ABC},
\]

\[
S_{\sin}^{gl} = \text{Tr} \left[ F^A F^B F^C \left( c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} c_{\frac{1}{2} p_3} - c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} - c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} + c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right) - F^A D^B D^C \left( c_{\frac{1}{2} p_1} s_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} + c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right) 
- F^B D^C D^A \left( s_{\frac{1}{2} p_1} s_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} - c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right) - F^C D^B D^A \left( s_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} + c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right) \right]
= -\frac{N}{2} f^{ABC},
\]

\[
S_{\sin}^{gl ABC} = \text{Tr} \left[ -F^A F^B D^C c_{\frac{1}{2} p_1} s_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} - F^B F^C D^A c_{\frac{1}{2} p_1} s_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} 
+ F^C F^A D^B c_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} c_{\frac{1}{2} p_3} + F^A D^B D^C s_{\frac{1}{2} p_1} c_{\frac{1}{2} p_2} s_{\frac{1}{2} p_3} \right]
= -\frac{N}{2} d^{ABC} \left[ 1 + c_{\frac{1}{2} p_1} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta B, 0 \delta C, 0 - \delta A, 0) 
- c_{\frac{1}{2} p_2} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta A, 0 \delta B, 0 - \delta B, 0) 
- c_{\frac{1}{2} p_3} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta A, 0 \delta B, 0 - \delta C, 0) \right],
\]

and

\[
S_{\sin}^{gh ABC} = \frac{1}{4} \text{Tr} \left[ F^A F^B D^C + F^A D^B F^C + D^A F^B F^C - D^A D^B D^C 
+ c_{\frac{1}{2} p_1} (F^A F^B D^C + F^A D^B F^C - D^A F^B F^C + D^A D^B D^C) 
+ c_{\frac{1}{2} p_2} (F^A F^B D^C + F^A D^B F^C - D^A F^B F^C - D^A D^B D^C) 
+ c_{\frac{1}{2} p_3} (F^A F^B D^C + F^A D^B F^C + D^A F^B F^C + D^A D^B D^C) \right]
= -\frac{N}{2} d^{ABC} \left[ 1 + c_{\frac{1}{2} p_1} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta B, 0 \delta C, 0 - \delta A, 0) 
- c_{\frac{1}{2} p_2} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta A, 0 \delta B, 0 - \delta B, 0) 
+ c_{\frac{1}{2} p_3} (\delta A, 0, \delta B, 0 \delta C, 0 - \delta A, 0 \delta B, 0 - \delta C, 0) \right].
\]
By Eqs. (69) to (72) it is evident that the full amplitude associated with Figs. (a), (b) and (c) is

\[
A_{\mu \nu \lambda}^{ABC} \mid_{(\alpha),(\beta),(\gamma)} = -\frac{N}{2} \left[ f^{ABC} \cos \left( \frac{p_1 \times p_2}{2} \right) + d^{ABC} \left( 1 + O^{ABC} \right) \sin \left( \frac{p_1 \times p_2}{2} \right) \right] \\
\times \left[ L_{\mu \nu \lambda}^{gl} + L_{\mu \nu \lambda}^{gh} \right],
\]

where

\[
O^{ABC} = \frac{1}{\theta p T} \left[ \delta^{A,0} \delta^{B,0} \delta^{C,0} - \delta^{B,0} \delta^{C,0} - \delta^{A,0} \right] \\
- \cos(p_2 \times k) \left( \delta^{A,0} \delta^{B,0} \delta^{C,0} - \delta^{A,0} \right) + \cos(p_3 \times k) \left( \delta^{A,0} \delta^{B,0} \delta^{C,0} - \delta^{A,0} \delta^{B,0} \right) - \cos(p_1 \times k) \left( \delta^{A,0} \delta^{B,0} \delta^{C,0} - \delta^{A,0} \delta^{B,0} \right)
\]

is an oscillatory part which gives a subleading contribution for \( \theta p T \gg 1 \). Furthermore, explicit computation gives

\[
L_{\mu \nu \lambda}^{gl} + L_{\mu \nu \lambda}^{gh} = \frac{i}{k \cdot p_3} \left[ \frac{4 p_1^2 k_{\mu} k_{\nu} k_{\lambda}}{(k \cdot p_1)^2} + \frac{4 k_{\mu} k_{\nu} p_{3\lambda}}{k \cdot p_1} + \frac{4 p_{3\nu} k_{\lambda}}{k \cdot p_1} - k_{\mu} \eta_{\lambda,\nu} - k_{\nu} \eta_{\lambda,\mu} \right] \\
- \left( \{[\mu, p_1], (\lambda, p_3)\} \leftrightarrow \{[(\lambda, p_3), (\mu, p_1)]\} \right).
\]

(As in the case of the two-point function, in the leading thermal contributions, all dependence of the three-point function on the gauge parameter \( \xi \) cancels completely.) The full contribution to the three-point function is obtained adding to Eq. (63) two cyclic permutations of \((\mu, p_1), (\nu, p_2) (\lambda, p_3)\). In the limit \( \theta p T \gg 1 \), \( O^{ABC} \) can be neglected, the colour/trigonometric factor in Eq. (63) does not change under cyclic permutations. Therefore, we can write

\[
\lim_{(\theta p T) \to \infty} A_{\mu \nu \lambda}^{ABC} \mid_{(\alpha),(\beta),(\gamma)} = \frac{N}{2} C^{ABC}(p_1, p_2) \left[ L_{\mu \nu \lambda}^{gl} + L_{\mu \nu \lambda}^{gh} \right] + \text{cyclic perm. of } (\mu, p_1), (\nu, p_2) (\lambda, p_3),
\]

where \( C^{ABC}(p_1, p_2) \) is given by Eq. (A3). Since the factor \( C^{ABC}(p_1, p_2) \) does not change under cyclic permutations, the Lorentz factor \( L_{\mu \nu \lambda}^{gl} + L_{\mu \nu \lambda}^{gh} \), plus its cyclic permutations simplifies to an expression without terms involving the metric tensor \( \eta \), so that the full expression from the diagrams in Figs. (a), (b) and (c) can be written as

\[
\lim_{(\theta p T) \to \infty} A_{\mu \nu \lambda}^{ABC} \mid_{(\alpha),(\beta),(\gamma)} = -\frac{i N}{k \cdot p_3} \left[ \frac{2 k_{\mu} k_{\nu} k_{\lambda}}{(k \cdot p_1)^2} + \frac{2 k_{\mu} p_{3\nu}}{k \cdot p_1} + (\mu \leftrightarrow \nu) \right] \\
+ \left( \{[\mu, p_1], (\lambda, p_3)\} \leftrightarrow \{[(\lambda, p_3), (\mu, p_1)]\} \right).
\]

The remaining contribution, associated with the amplitudes in Figs. (d), (e) and (f), is purely oscillatory and hence will not contribute when \( \theta p T \gg 1 \). Including the two identical contributions which arise as a result of reversing the momentum flow of \( k \) in Fig. (d), (e), and (f) and also by interchanging the two vertices appearing there, we obtain

\[
A_{\mu \nu \lambda}^{ABC} \mid_{(d),(e),(f)} = -\frac{2i N}{k} \frac{d^{ABC}}{2} \sin \left( \frac{p_1 \times p_2}{2} \right) \cos(p_1 \times k) \left( \delta^{A,0} \delta^{B,0} \delta^{C,0} - \delta^{B,0} \delta^{C,0} - \delta^{A,0} \right) \\
k_{\mu} \eta_{\nu,\lambda} + k_{\nu} \eta_{\mu,\lambda} + 4 k_\lambda \eta_{\mu,\nu} \\
+ \left( \{[\mu, p_1], (\nu, p_2)\} \leftrightarrow \{[(\nu, p_2), (\mu, p_1)]\} \right) \\
+ \text{cyclic perm. of } (\mu, p_1), (\nu, p_2) (\lambda, p_3).
\]

After using (A3) and (A4) in conjunction with

\[
\Gamma^{ABC}_{\mu \nu \lambda}(p_1, p_2, p_3) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2 |k|} \; N(|\vec{k}|) \; A_{\mu \nu \lambda}^{ABC} \mid_{k_0 = |\vec{k}|}; \quad A, B, C = 0, 1, 2, \ldots, N^2 - 1
\]

to compute the three-point function in the hard thermal limit, one is confronted with very complicated angular integrals. However, it is apparent that because \( A_{\mu \nu \lambda}^{ABC}(k) \) in Eq. (63) is homogeneous of zero degree in \( k \), the three-point function is quadratic in \( T \), as in commutative Yang-Mills theory. Furthermore, the simple Ward identity

\[
p^3_3 \Gamma^{ABC}_{\mu \nu \lambda}(p_1, p_2, p_3) = i C^{ABC}(p_1, p_2) \left( \Pi^{BC}_{\mu \lambda}(p_1) - \Pi^{BC}_{\nu \lambda}(p_2) \right)
\]

(70)
can be seen to be satisfied in the hard thermal limit when \( \theta p T \gg 1 \), without having to perform the integration over \( \vec{k} \). This can be verified directly, at the integrand level, using the explicit forms of the two- and three-point amplitudes given by Eqs. (14) (with \( \theta p T \gg 1 \)) and (69). Actually, this Ward identity should be satisfied for all values of \( \theta p T \). This is because in the hard thermal limit amplitudes with external ghost lines do not have a \( T^2 \) behaviour and hence BRS identities reduce to simple Ward identities such as those in (70). The above Ward identity, together with the results given in Eqs. (68) and (69), implies that the leading \( T^2 \) contributions to the static three point amplitude are non-vanishing. This behaviour contrasts with the one in the commutative theory, where the gluon self-energy is the only static amplitude with a hard thermal loop.

IV. DISCUSSION

An essential ingredient of the resumation program is the computation of the effective action for the hard thermal loops. In this work, we have addressed the problem of obtaining the two- and three-point functions in noncommutative U(\( N \)) Yang-Mills theory at high temperature. These calculation are much more difficult than the corresponding ones in commutative SU(\( N \)) theory, due to the presence of the tensor \( \theta_{\mu\nu} \), which appears in the interaction vertices. It is interesting to remark at this point that, as \( \theta_{\mu\nu} \rightarrow 0 \), the U(1) sector decouples, so that the usual results of the SU(\( N \)) gauge theory are recovered. This can be understood by noting that the temperature does provide a natural ultraviolet cut-off for the thermal part of the amplitudes (in contrast, such limit is singular at \( T = 0 \), due to the phenomenon of the UV/IR mixing). This fact enables one to take, for the leading thermal contributions in the noncommutative theory, the limit \( \theta_{\mu\nu} \rightarrow 0 \) in a smooth way.

The approach which relates the hard thermal loops to the angular integrals of forward scattering amplitudes of on-shell thermal particles, allows one to infer much useful information about their high-temperature behaviour. From an examination of these amplitudes, where the leading terms are all of order \( T^2 \), one learns the following properties of the angular integrands:

(a) The non-localities involve, in configuration space, products of \( (k \cdot \partial)^{-1} \).

(b) Apart from the trigonometric factors involving the noncommutative parameter, the integrands are homogeneous functions of \( k \) of zero degree and Lorentz covariant.

(c) They are gauge invariant and satisfy simple Ward identities analogous to those of the tree amplitudes.

Using similar arguments to the ones employed in reference [17], we expect that these properties, together with the results for the lowest order amplitudes, may be sufficient to determine the effective action for hard thermal loops. This issue of the noncommutative Yang-Mills theory is currently being considered.

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APPENDIX A: FEYNMAN RULES

The propagators of the gauge and ghost particles are respectively given by

\[
\begin{align*}
\mathbf{A} & \quad \mathbf{P} \quad \mathbf{V} \quad \mathbf{B} : \quad -i \, \frac{\delta^{A,B}}{(p^2 + i\epsilon)} \left( \eta_{\mu\nu} - (1 - \xi) \frac{p_{\mu} p_{\nu}}{p^2} \right) \\
\mathbf{A} & \quad \mathbf{P} \quad \mathbf{B} : \quad i \, \frac{\delta^{A,B}}{p^2 + i\epsilon}
\end{align*}
\]

(A1)
The vertices are

\[
\begin{align*}
\text{A} & \quad -g C^{ABC}(p_1, p_2) \left[ (p_1 - p_2)^\mu \eta^{\nu\sigma} + (p_2 - p_3)^\mu \eta^{\rho\sigma} + (p_3 - p_1)^\sigma \eta^{\lambda\mu} \right] \\
\text{B} & \quad g C^{ABC}(p_2, p_3) p_2^\mu \\
\text{C} & \quad -i g^2 \left[ C^{ABX}(p_1, p_2) C^{XCD}(p_3, p_4)(\eta^{\mu\lambda} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\lambda}) \\
& \quad + C^{ACX}(p_1, p_3) C^{XBD}(p_4, p_2)(\eta^{\mu\rho} \eta^{\lambda\nu} - \eta^{\mu\nu} \eta^{\lambda\rho}) \\
& \quad + C^{ADX}(p_1, p_4) C^{XBC}(p_2, p_3)(\eta^{\mu\nu} \eta^{\lambda\rho} - \eta^{\mu\lambda} \eta^{\nu\rho}) \right],
\end{align*}
\]

where

\[
C^{ABC}(p_1, p_2) = f^{ABC} \cos(\frac{p_1 \times p_2}{2}) + d^{ABC} \sin(\frac{p_1 \times p_2}{2}); \quad p_1 \times p_2 \equiv p_1^\mu \theta_{\mu\nu} p_2^\nu
\]

and all momenta are inward. Dirac delta functions for the conservation of momenta are understood.

**APPENDIX B: INTEGRALS**

In this appendix, the integrals appearing in Eqs. (18) to (21) are evaluated, leaving us in general with answers in terms of infinite sums.

First of all, from Eqs. (2), (14) and (15) we obtain

\[
\begin{align*}
p_0 \tilde{p}^\mu \Pi^{AB \mu} = 2g^2 N \delta^{AB} \int \frac{d^3 \vec{k}}{|\vec{k}|} N(|\vec{k}|) \left[ 1 - \delta^{A.0} \cos(K \cdot \tilde{p}) \right] K \cdot \tilde{p} \left[ \frac{p^2}{(K \cdot p)^2} - \frac{p_0}{K \cdot p} \right]; \quad K = \left( 1, \frac{\vec{k}}{|\vec{k}|} \right)
\end{align*}
\]

One can perform the previous integral using a coordinate system such that two of the three orthogonal directions are \( \frac{\vec{p}}{|\vec{p}|} \) and \( \frac{\vec{k}}{|\vec{k}|} \). Since the integrand is an odd function of \( K \cdot \tilde{p} \), the integral will vanish. Therefore, we conclude that

\[
\Pi^{A.4}_4 = 0.
\]

Let us now consider the quantity \( \Pi^{A..B}_\mu \). Using Eqs. (2) and (14) we obtain, since \( G^\mu_\mu = 2 \),

\[
\Pi^{A..B}_\mu = 4g^2 N \delta^{A..B} \int \frac{d^3 \vec{k}}{|\vec{k}|} N(|\vec{k}|) \left[ 1 - \delta^{A.0} \cos(K \cdot \tilde{p}) \right].
\]
Using spherical coordinates as shown in the figure, so that

$$k \cdot \hat{p} = -|\hat{p}| \theta |\vec{k}| \cos(\psi),$$

we can write (with $u \equiv |\vec{k}|$)

$$\Pi^\mu_{AB} = \frac{4g^2N \delta^{AB}}{(2\pi)^3} \int_0^\infty \frac{udu}{e^{u} - 1} \int_0^{2\pi} d\phi \int_{-1}^1 d\zeta \left[ 1 - \delta^{A,0} \cos(\tau u \zeta) \right]$$

$$= \frac{8g^2N \delta^{AB} T^2}{(2\pi)^2} \int_0^\infty \frac{udu}{e^{u} - 1} \left[ 1 - \delta^{A,0} \frac{\sin(\tau u)}{\tau u} \right].$$

Expressing the Bose distribution in terms of the geometrical series,

$$\frac{1}{e^u - 1} = \sum_{n=1}^\infty e^{-nu} u \frac{d}{du} e^{-nu},$$

$$\int_0^\infty \frac{udu}{e^u - 1} = -\sum_{n=1}^\infty \frac{d}{dn} \int_0^\infty du e^{-nu} = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

and similarly, expanding

$$\int_0^\infty \frac{\sin(bu)du}{e^u - 1} = \sum_{n=1}^\infty \int_0^\infty \sin(bu) e^{-nu} du$$

$$= \sum_{n=1}^\infty \int_0^\infty \frac{1}{2i} \left( e^{(ib-n)u} - e^{(-ib-n)u} \right) du = b \sum_{n=1}^\infty \frac{1}{b^2 + n^2},$$

we obtain from (B3)

$$\Pi^\mu_{AB} = \frac{8g^2N \delta^{AB} T^2}{(2\pi)^2} \left[ \frac{\pi^2}{6} - \delta^{A,0} \sum_{n=1}^\infty \frac{1}{n^2 + \tau^2} \right].$$
Let us now consider the component $\Pi_{00}^{AB}$. Using Eqs. (2), (14) and (15) and the spherical coordinates indicated in figure 6, we obtain

$$
\Pi_{00}^{AB} = \frac{2g^2N \delta_{A,B} T^2}{(2\pi)^3} \int_{-1}^{1} d(cos(\psi)) \int_{0}^{\infty} \frac{du}{e^u - 1} \int_{0}^{2\pi} d\phi \left[ 1 - \frac{2\sigma}{\sigma - \cos(\psi)} + \frac{\sigma^2 - 1}{(\sigma - \cos(\psi))^2} \right] 
\times [1 - \delta^{A,0}(\tau u \sin(\psi) \cos(\phi))].
$$

(B10)

Using the standard result [18],

$$
\int_{0}^{2\pi} d\phi \cos(\alpha \cos(\phi)) = 2\pi J_0(\alpha),
$$

we obtain

$$
\Pi_{00}^{AB} = \frac{2g^2N \delta_{A,B} T^2}{(2\pi)^2} \int_{-1}^{1} d(cos(\psi)) \int_{0}^{\infty} \frac{du}{e^u - 1} \left[ 1 - \delta^{A,0}(\tau u \sin(\psi)) \right] 
\times \left[ 1 - \frac{2\sigma}{\sigma - \cos(\psi)} + \frac{\sigma^2 - 1}{(\sigma - \cos(\psi))^2} \right].
$$

(B12)

But now we take

$$
\int_{0}^{\infty} \frac{du}{e^u - 1} J_0(au) = -\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \approx \frac{\pi^2}{6} \left( 1 - \frac{\pi^2 a^2}{10} \right).
$$

(B13)

which can be written as [18]

$$
= -\sum_{n=1}^{\infty} \frac{d}{dn} \frac{1}{\sqrt{n^2 + a^2}} = \sum_{n=1}^{\infty} \frac{n}{(n^2 + a^2)^{3/2}} \approx \frac{\pi^2}{6} \left( 1 - \frac{\pi^2 a^2}{10} \right).
$$

(B14)

By Eqs. (B7) and (B14), (B12) becomes

$$
\Pi_{00}^{AB} = \frac{2g^2N \delta_{A,B} T^2}{(2\pi)^2} \int_{-1}^{1} d\zeta \left[ 1 - \frac{2\sigma}{\sigma - \zeta} + \frac{\sigma^2 - 1}{(\sigma - \zeta)^2} \right] 
\times \left[ \frac{\pi^2}{6} - \delta^{A,0} \sum_{n=1}^{\infty} \frac{n}{n^2 + \tau^2(1 - \zeta^2)^{3/2}} \right].
$$

(B15)
Finally, let us compute \( \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} \). Using Eq. (18) and the spherical coordinates indicated in figure 2, we obtain

\[
\frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} = -\frac{2g^2N\delta^{AB}T^2}{(2\pi)^3} \int_{-1}^{1} d(\cos(\psi)) \int_0^\infty \frac{du}{e^u - 1} \int_0^{2\pi} d\phi \left( \begin{array}{c} \sigma^2 - 1 \\
(\sigma - \cos(\psi))^2 \end{array} \right) \sin(\psi) \cos(\phi) \left( \begin{array}{c} 0 \\
-1 \end{array} \right)
\]

Performing the integration over \( \phi \) \( \text{(18)} \)

\[
\int_0^{2\pi} d\phi \cos^2(\phi) \cos(a \cos(\phi)) = \frac{1}{2} \int_0^{2\pi} d\phi (1 + \cos(2\phi)) \cos(a \cos(\phi)) = \pi [J_0(a) - J_2(a)], \quad (B17)
\]

we obtain

\[
\frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} = -\frac{2g^2N\delta^{AB}T^2}{(2\pi)^3} \int_{-1}^{1} d(\cos(\psi)) \left( \begin{array}{c} \sigma^2 - 1 \\
(\sigma - \cos(\psi))^2 \end{array} \right) \sin(\psi) \cos(\phi) \left( \begin{array}{c} 0 \\
-1 \end{array} \right)
\]

The integral over \( u \) in Eq. (B18) is first expanded. Similarly to the Eqs. (B7) and (B14), we now have

\[
\int_0^\infty \frac{du}{e^u - 1} J_2(au) = -\sum_{n=1}^\infty \frac{d}{dn} \int_0^\infty e^{-nu} J_2(au) \quad (B19)
\]

which by standard formula in \( \text{(18)} \) becomes

\[
-\sum_{n=1}^\infty \frac{d}{dn} \int_0^\infty e^{-nu} J_2(au) = -\sum_{n=1}^\infty \frac{d}{dn} \frac{a^{-2} \sqrt{n^2 + a^2} - n^2}{\sqrt{n^2 + a^2}}
\]

\[
= \frac{1}{a^2} \sum_{n=1}^\infty \frac{(n - \sqrt{n^2 + a^2})^2 (2\sqrt{n^2 + a^2} + n)}{(n^2 + a^2)^{3/2}}
\]

\[
\approx \frac{3}{4a^2} \sum_{n=1}^\infty \frac{1}{n^3} = \frac{\pi^4 a^2}{120}. \quad (B20)
\]

One can now reduce \( \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2} \Pi^{AB}_{\mu\nu} \) to the expression involving a sum and an integral over \( \zeta = \cos(\psi) \) given in Eq. (20).

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