Abstract

In heterotic string theory compactified to four dimensions with N=2 supersymmetry, string-loop corrections to the low-energy effective action and to solutions of the equations of motion are studied. Within the framework of N=2 supersymmetric formulation of the theory, the effective action is obtained in the first order in string coupling constant. Starting from a tree-level dyonic black hole solution, in the first order in string coupling constant, we solve the system of Killing spinor and Maxwell equations and obtain solutions for the string-loop-corrected moduli and metric of the dyonic black hole.

PACS: 04.70.Dy,04.50.+h,11.25.Db,11.25.Mj
Keywords: string theory, black holes, N=2 supergravity
1 Introduction

In perturbative approach to string theory the amplitudes are calculated as a sum of contributions from the string world sheets of different genera. The low-energy effective action of light modes also receives contributions from higher genera. If gravity is considered within the framework of superstring theory, we face with a problem of stringy corrections to the standard Einstein gravity. In this paper, we discuss the string-loop-corrected effective action and, starting from a tree-level dyonic black-hole, obtain the string-loop-corrected dyonic solution.

To be concrete, we consider 6D effective theory obtained by compactification of the heterotic string theory on the manifold $K3$ or on the suitable orbifold yielding $N = 1$ supersymmetry in 6D. Dyonic black hole is a particular solution of the tree-level equations of motion of 6D theory. On the other hand, it can be interpreted as a solution to the equations of motion of 4D theory obtained by compactification of the 6D theory on the torus $T^2$. The resulting 4D theory has $N = 2$ local supersymmetry.

In Sect.2 we review the structure of the bosonic part of the universal sector of the effective theory obtained by compactification of 6D heterotic string theory on the torus $T^2$ with the boundary conditions on the torus $T^2$ untwisted. Coordinates $X^n$ for the untwisted directions have non-zero classical parts producing dependence on the backgrounds of the torus $T^2$.

We briefly discuss calculation of the loop corrections from the world sheets of torus topology to different terms of bosonic part of the effective action keeping only the terms up to the second order in derivatives. The Einstein term receives no correction, the gauge couplings and kinetic terms of the internal metric are modified.

Instead of direct path-integral calculation of loop corrections, in Sect.3 we consider formulation of the theory based on $N = 2$ local supersymmetry. In this approach, dynamics of the theory is defined in terms of the prepotential of the theory. The prepotential of $N = 2$ locally supersymmetric theory receives string-loop correction only from the world sheets of torus topology. Using the loop-corrected prepotential, we calculate the gauge couplings in the first order in string coupling. To make contact with the solution of equations of motion derived from the string effective action, we discuss the form of the loop-corrected symplectic transformation connecting the holomorphic section with the prepotential and that associated with the heterotic string compactification. We review the derivation of the Killing spinor equations which are conditions for the supersymmetry variations of the gravitino and gaugino to vanish and present them in different forms.

In Sects.4 and 5 we solve the combined system of Killing spinor equations. To solve the loop-corrected equations, we specify a particular tree-level dyonic black-hole solution with the constant metric of the two-torus $T^2$ and vanishing antisymmetric tensor. With this choice, the tree-level moduli $T$ and $U$ are independent of coordinates, but dilaton and space-time metric are coordinates-dependent. The loop correction to the prepotential depends only on the moduli $T$ and $U$. In perturbative approach, in the first order in string coupling constant, the loop corrections to the gauge couplings and to the Kähler potential are calculated by substituting the tree-level moduli and thus the functions of the $T$ and $U$ are independent of coordinates also. This technical simplification makes possible to avoid too cumbersome expressions for solutions to the system of Maxwell and Killing spinor equations and to obtain explicit expressions for
the loop corrections to the metric and moduli.

In Sect.6 we discuss purely electric and magnetic limits of dyonic black hole.
In Sect.7 we consider the loop-corrected expression for the BPS mass of the black hole.
In Sect.8 the equations for the axions are presented.

2 String-loop corrected $N = 2$ supersymmetric effective action

The bosonic part of the universal sector of the 6$D$ effective action of the heterotic string theory compactified to six dimensions with $N = 1$ supersymmetry on the manifold $K3$ or on a suitable orbifold is

$$I_6 = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-G^{(6)}_e^{-\Phi}} \left[ R^{(6)} + (\partial\Phi)^2 - \frac{H^2}{12} \right] + \ldots. \quad (1)$$

Further compactification on a two-torus yields the 4$D$ theory with $N = 2$ supersymmetry. The standard decomposition of the 6$D$ metric is

$$G^{(6)} = \begin{pmatrix} G_{\mu\nu} + A^m_{\mu} A^n_{\nu} G_{mn} & A^m_{\mu} G_{mn} \\ A^n_{\nu} G_{mn} & G_{mn} \end{pmatrix}. \quad (2)$$

Dimensional reduction of the action (1) yields [1]

$$I_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-G^{(4)}_e^{-\phi}} \left[ R(G') + (\partial\phi)^2 - \frac{(H')^2}{12} + \frac{1}{4} F(LML)F + \frac{1}{8} Tr(\partial ML\partial ML) \right], \quad (3)$$

where $\mu, \nu = 0, \ldots, 3$ and $m, n = 1, 2$. The second pair of the vector fields are the components $B_{mn}$ of the antisymmetric field $B^{(6)}$.

Here

$$M = \begin{pmatrix} G^{-1} & G^{-1} B \\ -BG^{-1} & G \end{pmatrix}, \quad L = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad (4)$$

and

$$G' \equiv G'_{\mu\nu} = G_{\mu\nu} + A^m_{\mu} A^n_{\nu} G_{mn}, \quad G \equiv G_{mn},$$

$$H' \equiv H'_{\mu\nu\lambda} = H_{\mu\nu\lambda} - (A^{(1)}_m H_{m\nu\lambda} - A^{(1)}_m A^{(1)}_n H_{mn\lambda} + \text{cycl. perms.}).$$

A direct way to calculate corrections to the string-tree-level effective action from integration over the string world sheets of torus topology is to perform the path integrals for correlators yielding the relevant structures in the effective action. For heterotic string theory, The part of the world-sheet action of the heterotic string theory which depends on background fields from the universal sector is

$$I_{1,0} = \int d^2z d\theta (G_{MN} + B_{MN})(\hat{X})D\hat{X}^M \partial \hat{X}^N. \quad (5)$$
Here $I_{1,0}$ is the action with the $(1,0)$ supersymmetry in the left supersymmetric sector, $\hat{X}^M = X^M + \vartheta \psi^M$, $M = 0, ..., 9$, $D = \partial_\vartheta + \vartheta \partial_z$.

Performing integration over $\vartheta$ one arrives at the action

$$I_{1,0} = \int d^2 z (G_{\mu\nu} + B_{\mu\nu}) (\partial X^\mu \bar{\partial} X^\nu - \psi^\mu \bar{\partial} \psi^\nu) + (G_{\mu n} + B_{\mu n}) (\partial X^\mu \bar{\partial} X^n - \psi^\mu \bar{\partial} \psi^n) + (G_{mn} + B_{mn}) (\partial X^m \bar{\partial} X^n - \psi^m \bar{\partial} \psi^n) + (G_{\mu \rho} + B_{\mu \rho}) \psi^\rho \psi^\mu \bar{\partial} X^\nu + G_{mn} \psi^\rho \psi^m \bar{\partial} X^n.$$

To be concrete, we have in view heterotic string theory compactified to four dimensions on the orbifold $T^4/Z_2 \times T^2$. The partition function has the form of the sum of terms, where each term is the product of contributions from integration over bosonic and fermionic variables.

Correlator of free bosons on the world sheet of torus topology with the Teichmüller parameter $\tau$ is given by [2]

$$<X(z, \bar{z})X(0)> = - \log |\vartheta_1(z)|^2 - \frac{\pi}{2\tau_2} (\text{Im} z)^2$$

(7)

Correlator of fermions with even spin structures $a, b \neq 1, 1$ is

$$S[a \ b](z) = <\psi(z)\psi(0)> = \frac{\vartheta[a \ b](z)\vartheta'[0]}{\vartheta[a \ b](0)\vartheta_1(0)},$$

(8)

where $\vartheta[1] = \vartheta_1$.

Contribution from the integration over two left-moving fermions carrying the 4D space-time indices (in the light-cone gauge) yields the blocks of the form (see, for example, [3])

$$\sum_{a,b=0}^1 (-)^{a+b+ab} \psi^2[a \ b](0) \vartheta[a+h \ b+g](0) \vartheta[a-h \ b-g](0) \eta^4$$

(9)

In untwisted case, eight left-moving fermions have the same spin structures, and the resulting 4D theory has $N = 4$ supersymmetry. Shiftings $h, g$ taking values $0, 1$ appear from the orbifold construction and reduce supersymmetry to $N = 2$.

Performing in (9) summation over spin structures and using the Jacobi identities, one has

$$\frac{1}{2} \sum_{a,b=0}^1 (-)^{a+b+ab} \psi^2[a \ b](0) \vartheta[a+h \ b+g](0) \vartheta[a-h \ b-g](0) \eta^4 = -\vartheta^2[1 \ 1] \vartheta[1-h \ 1-g] \vartheta[1+h \ 1+g](0) = 0,$$

(10)

because $\vartheta_1(0) = 0$.

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2We consider only fermions with even spin structures which yield non-vanishing contributions to the correlators.
Let us show that the Einstein term receives no corrections from the world sheets of torus topology. The graviton vertex function is \[ V_G = \int d^2 z \, G_{\mu\nu}(X)(\partial X^\mu + i(p\psi)\psi^\mu)\bar{\partial}X^\nu e^{ipX}. \] (11)

Let us calculate the two-point correlator of the vertex functions (11) and take the \( O(p^2) \) piece. Due to (10), purely bosonic part of the correlator \( \langle V_G V_G \rangle \) vanishes. The four-fermion part is proportional to the expression

\[
\frac{1}{2} \sum_{a,b \neq 1,1} (-)^{a+b+ab} \eta \frac{\partial^2[a]}{\partial [a]}(0)\frac{\partial [a+h]}{\partial [g]}(0)\frac{\partial [a-h]}{\partial [b-g]}(0)\, S^2(a\, b)(z) = 4\pi^2 \eta^2 \partial_1 [1 - h] \partial_1 [1 + g](0),
\]

Expression (12) is independent of \( z \). Thus, the \( O(p^2) \) piece of the correlator \( \langle V_G V_G \rangle \) vanishes, and there is no correction to the Einstein term. The same is true for the vertex \( V_B \) with \( G_{\mu\nu} \) substituted by \( B_{\mu\nu} \).

Let us consider the correlators of the gauge vertices \[ V_A = \int d^2 z \, G_{pq} \left( A^p_\mu(X) \partial X^\mu + \frac{1}{2} F^p_{\mu\nu} \psi^\mu \bar{\psi}^\nu \right) \bar{\partial}X^q e^{ipX} \] (the same with \( A^p_\mu \) substituted by \( B_{pq} \)), where \( p,q = 1,2 \) label the directions of the untwisted torus \( T^2 \). Non-vanishing contribution of the left-moving fields is produced by the fermionic terms in the vertices \( V_A \). In contrast to the graviton vertex, in the present case, the right-moving bosons \( X^p \) carry not \( 4D \), but an internal \( T^2 \) index. The fields \( X^p \) can be split into the classical

\[
\tilde{X}^p(m, n) = \pi R \left[ (m^p - n^p\tau)\frac{z}{i\tau_2} - (m^p - n^p\tau)\frac{\bar{z}}{i\tau_2} \right]
\]

and a quantum parts \( Y^p \): \( X^p = \tilde{X}^p + Y^p \). Here it is assumed that the classical fields \( \tilde{X}^p \) take values on the circles of the radius \( R \): \( \tilde{X}^p \sim \tilde{X}^p + 2\pi Rk \). Contribution of the right-moving bosons is

\[
\sum_{(m,n)} e^{-S(m,n)-I(Y)} \left[ \bar{\partial}X^p \bar{\partial}X^q + \langle \bar{\partial}Y^p \bar{\partial}Y^q \rangle \right],
\]

where the classical action is

\[
S(m, n) = \frac{\pi R^2 G_{pq}}{\tau_2} (m^p - n^p\tau)(m^q - n^q\tau) - I(Y) = \int d^2 z \, G_{pq} \partial Y^p \partial Y^q .
\]

As in the case of the graviton vertex, the integral over bosonic correlator \( \langle \bar{\partial}Y^p \bar{\partial}Y^q \rangle \) vanishes. The classical part is non-zero and produces a one-loop correction to the tree-level term \( G_{pq} F_{\mu\nu}^p F^{q\mu\nu} \).

In the following, instead of performing path-integral calculations of the string-loop corrections to various terms of the action, we shall make use of the \( N = 2 \) formulation of the theory, in which different loop corrections are expressed via the loop-corrected prepotential.  

\textsuperscript{3}It is understood that the wave function factors making the amplitude vanishing on-shell are removed.
3 \ N=2 formulation of the effective field theory

Written in the Einstein frame, where \( g_{\mu \nu} = e^{-\phi} G'_{\mu \nu} \), the action (3) is \[1\]

\[
I_4 = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{e^{-\phi}}{4} \mathcal{F}(LML) \mathcal{F} + \frac{a_1}{4 \sqrt{-g}} \mathcal{F} L^* \mathcal{F} + \frac{1}{8} Tr(\partial ML \partial ML) \right].
\] (17)

The 4D dilaton \( \phi \) and axion \( a_1 \) are defined as

\[
\phi = \Phi - \frac{1}{2} \ln \det(G_{mn}), \quad \partial_{\rho} a_1 = -H^\mu_{\nu\lambda} e^{-2\phi} \sqrt{-g} e_{\mu\nu\lambda\rho}.
\] (18)

The bosonic part of the 4D action written in the standard form of \( N = 2 \) supersymmetric theory is (for example, \[6, 7, 8, 9\] and references therein)

\[
I_4 = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + (\bar{N}_{IJ} \mathcal{F}^{-I} \mathcal{F}^{-J} - N_{IJ} \mathcal{F}^{+I} \mathcal{F}^{+J}) + k_{ij} \partial_{\mu} z^i \partial^{\mu} \bar{z}^j + \ldots \right].
\] (19)

The gauge coupling constants, \( N_{IJ} \), are defined below,

\[
\mathcal{F}^{\pm}_{\mu \nu} = \frac{1}{2} (\mathcal{F}_{\mu \nu} \pm i \sqrt{-g}^{*} \mathcal{F}_{\mu \nu}) = \frac{1}{2} (\mathcal{F}_{\mu \nu} \pm i \varepsilon_{\mu \nu \rho \lambda} \mathcal{F}^{\rho \lambda})
\]

Here \( \mathcal{F}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \lambda} \mathcal{F}^{\rho \lambda} \), where \( \varepsilon_{\mu \nu \rho \lambda} \) is the flat antisymmetric tensor, \( \varepsilon_{0123} = -1 \), \( z^i \) are the moduli identified below, \( k_{ij} \) is the Kaehler metric

\[
k_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j},
\]

where \( K \) is the Kähler potential calculated below.

In the case of the \( N = 2 \) supersymmetric compactification of superstring theory, the prepotential of the resulting STU model receives only one-string-loop correction and is of the form \[10, 11\]

\[
F = -\frac{X^1 X^2 X^3}{X^0} - i X^0 h(-i \frac{X^2}{X^0}, -i \frac{X^3}{X^0}) + \ldots,
\] (20)

where

\[
\frac{X^1}{X^0} = z^1 = iy_1 = i \left( e^{-\phi} + ia_1 \right),
\]

\[
\frac{X^2}{X^0} = z^2 = iy_2 = i \left( e^{\gamma + \sigma} + ia_2 \right),
\]

\[
\frac{X^3}{X^0} = z^3 = iy_3 = i \left( e^{\gamma - \sigma} + ia_3 \right)
\] (21)

and dots stand for contributions from other moduli. Here and below \( I, J = 0, \ldots, 3 \) and \( i, j = 1, 2, 3 \). The one-loop part of the prepotential is independent of the modulus \( X^1 \). In the following, we do not use the explicit form of the loop correction calculated in \[11, 14, 15\].
The moduli $z^i$ and the vector fields are identified by comparing definitions (21) with parametrization of the metric components of two-torus
\begin{equation}
G_{mn} = e^{2\sigma} \begin{pmatrix} e^{2\gamma-2\alpha} + a_3^2 & a_3 \\ -a_3 & 1 \end{pmatrix}
\end{equation}
and the antisymmetric tensor $B_{12} = a_2$.

The moduli $y_i$ are equal to conventional moduli $S, T, U$:
\begin{equation}
(y_1, y_2, y_3) = (S = e^{-\phi} + ia_1, T = \sqrt{G} + iB_{12}, U = (\sqrt{G} + iG_{12})/G_{22}).
\end{equation}

The dilaton $\phi$ can be split into the sum of the constant part $\phi_0$ and a term vanishing at spatial infinity $\phi = \phi_0 + \phi_1$. In string perturbation theory, higher order contributions enter with the factor $e^{\epsilon\chi\phi}$, where $\chi$ is the Euler characteristic of the string world sheet [12]. The exponent $e^{\epsilon\phi_0} \equiv \epsilon$ can be considered as a string-loop expansion parameter. In the following, we include the factor $\epsilon$ in string-loop corrections, and use the notation $\phi$ for the non-constant part of the dilaton.

The gauge part of the action (3) with $B_{12} = 0$ and diagonal metric $G_{mn}$ is
\begin{equation}
-\frac{1}{4} G_{11} (\mathcal{F}^{(1)})^2 - \frac{1}{4} G_{22} (\mathcal{F}^{(1)})^2 - \frac{1}{4} G_{11} (\mathcal{F}^{(2)})^2 - \frac{1}{4} G_{22} (\mathcal{F}^{(2)})^2,
\end{equation}
where
\begin{equation}
\mathcal{F}_{\mu\nu}^{(1)} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu, \quad \mathcal{F}_{\mu\nu}^{(2)} = \partial_\mu B_{m\nu} - \partial_\nu B_{m\mu}.
\end{equation}

It is convenient to relabel the vector fields in correspondence with the moduli with which they form the superfields
\begin{equation}
A_\mu^1 = \sqrt{8} A^0_\mu, \quad B_1^\mu = \sqrt{8} A^1_\mu, \quad A^2_\mu = \sqrt{8} A^2_\mu, \quad B_2^\mu = \sqrt{8} A^3_\mu.
\end{equation}
The factor $\sqrt{8}$ appears because of different normalizations of the gauge field in the actions (17) and (19).

Let us turn to the $N = 2$ supersymmetric action (19). In sections which admit the pre-potential, the gauge coupling constants in the action (19) are calculated using the formula [13]
\begin{equation}
N_{IJ} = F_{IJ} + 2i \frac{(Im F_{IK} X^K)(Im F_{JL} X^L)}{(X^I Im F_{IJ} X^J)},
\end{equation}
where $F_I = \partial^I X^J F, F_{IJ} = \partial^2_{X^I X^J} F$, etc. In the first order in string coupling constant, we obtain the gauge couplings $N_{IJ}$ as
\begin{align}
N_{00} &= iy_3 \left(-1 + \frac{n}{4y_3}\right), \quad N_{01} = -\frac{n + 2v}{4y_1} - ia_1 \frac{y_2 y_3}{y_1}, \\
N_{02} &= -\frac{n + 2v - 2y_2 h_1 y_3 + 4y_2 h_2}{4y_2} - ia_2 \frac{y_1 y_3}{y_2}, \quad N_{03} = -\frac{n + 2v - 2y_3 h_1 y_2 + 4y_3 h_3}{4y_3} - ia_3 \frac{y_1 y_2}{y_3},
\end{align}
Here we kept notations $y_i$ for the real parts of the moduli and introduced $y^3 = y_1 y_2 y_3$, $hy = h_a y_a = h_2 y_2 + h_3 y_3$, $h_a = \partial_{y_a} h$, $h_{ab} = \partial_{y_a} \partial_{y_b} h$ and

$$v = h - y_a h_a, \quad n = h - h_a y_a + y_a h_{ab} y_b, \quad y_2 h y = y_2 h_2 a y_b. \quad (27)$$

The terms of the first order in string coupling $\epsilon$ are linear functions of the loop correction to the prepotential $h$ and are calculated by substituting the tree-level moduli. Tree-level moduli are obtained from the input tree-level solution to the equations of motion which is dyonic black hole solution [22, 23] and also [8, 9, 20, 27] and refs. therein. The imaginary parts $a_i$ of the moduli $y_i$ which are absent in the classical dyonic solution can appear in loop-corrected solutions of the field equations the first order in the string coupling constant.

The field equations and the Bianchi identities for the gauge field strengths are

$$\partial_{\mu} \left( \sqrt{-g} \text{Im} G^{-\mu \nu} \right) = 0$$
$$\partial_{\mu} \left( \sqrt{-g} \text{Im} F^{-\mu \nu} \right) = 0 \quad (28)$$

where $G^{-\mu \nu} = N_{IJ} F^{-J \mu \nu}$. Eqs.(28) and are invariant under the symplectic $Sp(8, \mathbb{Z})$ transformations

$$O = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (29)$$

where

$$A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = 1. \quad (30)$$

Under symplectic transformations the couplings $N_{IJ}$ are transformed as

$$\hat{N} = (C + DN)(A + BN)^{-1}. \quad (31)$$

In sections which do not admit a prepotential (including that which naturally appears in 4D compactification of the heterotic string), the gauge couplings are calculated by making a symplectic transformation of the couplings (26) calculated in the section with the prepotential. Specifically, at the tree level, the section connected with compactification of the heterotic string from 6D to 4D is obtained from that with the prepotential (20) by symplectic transformation [10]

$$A = \text{diag}(1, 0, 1, 1), \quad B = \text{diag}(0, 1, 0, 0), \quad C = \text{diag}(0, -1, 0, 0), \quad D = \text{diag}(1, 0, 1, 1). \quad (32)$$
At the one-loop level, we look for a symplectic transformation connecting heterotic section with that with the prepotential in the form

\[ A = \text{diag}(1, 0, 1, 1) + (a_{ij}), \quad B = \text{diag}(0, 1, 0, 0) + (b_{ij}), \]
\[ C = \text{diag}(0, -1, 0, 0) + (c_{ij}), \quad D = \text{diag}(1, 0, 1, 1) + (d_{ij}), \]

where all the entries \((a_{ij}), (b_{ij}), (c_{ij})\) and \((d_{ij})\) are of the first order in the string coupling \(\epsilon\).

The matrices \(a, b, c\) and \(d\) are constrained by relations (30).

In the case of tree-level solutions with vanishing axionic parts, the tree-level gauge couplings \(N_{00} = -iy_1y_3, \quad N_{22} = -\frac{\phi'}{y_2}, \quad N_{33} = -\frac{\phi'}{y_3}\) are proportional to \(e^{-\phi}\), whereas the expression \(N_{11} = -\frac{\phi''}{y_1}\) contains the factor \(e^{\phi}\). In the heterotic section, the tree-level gauge couplings are proportional to \(y_1 = e^{-\phi}\). Loop corrections to the tree-level couplings as well as non-diagonal couplings absent at the tree level appear with the extra factor \(\epsilon e^{\phi}\), i.e. are \(O(\epsilon^0)\).

We require that the loop-corrected symplectic transformations (33) produce the same structure of loop corrections to the gauge couplings in the heterotic section as that which appear in path-integral calculation of loop corrections, i.e. loop corrections have the extra factor \(\epsilon e^{\phi}\) as compared to the tree-level expressions. Because we perform calculations with accuracy up to the terms of the first order in string coupling, corrections to the gauge couplings due to the one-loop term in the prepotential and to the loop corrections to the tree-level symplectic transformations can be treated independently.

From the relations (30) it follows that admissible non-zero entries are \(c_{ij}\) with \(c_{11} = c_{11} = 0\) and \(d_{11}\).

Calculating the matrix of the loop-corrected gauge couplings in the heterotic section and requiring that it is of the form discussed above, we find

\[
\hat{N}_{IJ} = \begin{pmatrix}
N_{00} + c_{00} - \frac{N_{01}^2}{N_{11}} & N_{01} - \frac{N_{01}N_{12}}{N_{11}} & N_{02} - \frac{N_{01}N_{12}}{N_{11}} & N_{03} - \frac{N_{01}N_{13}}{N_{11}} \\
\frac{N_{01}}{N_{11}} & \frac{1}{N_{11}} + d_{11} & \frac{N_{12}}{N_{11}} & \frac{N_{13}}{N_{11}} \\
N_{20} + c_{20} - \frac{N_{01}N_{10}}{N_{11}} & \frac{N_{21}}{N_{11}} & N_{22} - \frac{N_{20}^2}{N_{11}} & N_{23} - \frac{N_{21}N_{13}}{N_{11}} \\
N_{30} + c_{30} - \frac{N_{01}N_{10}}{N_{11}} & \frac{N_{31}}{N_{11}} & N_{32} + c_{32} - \frac{N_{31}N_{12}}{N_{11}} & N_{33} + c_{33} - \frac{N_{32}N_{12}}{N_{11}}
\end{pmatrix}
\]

(34)

where \(c_{ij}\) and \(d_{11}\) are real constants.

Since at the tree level there are only diagonal couplings, the non-diagonal terms are of next order in \(\epsilon\). The terms of the form \(\frac{N_{11}N_{1J}}{N_{11}}\) are also of the next order in the string coupling.

From the symplectic transformation of the field strengths

\[
\begin{pmatrix}
\hat{F}^- \\
\hat{G}^-
\end{pmatrix} = O \begin{pmatrix}
F^- \\
G^-
\end{pmatrix}
\]

(35)

we obtain the relation between the field strengths in the section with the prepotential and those which appear in the heterotic string effective action in the form

\[
\hat{F}^- = F^-, \quad \hat{F}^2 = F^2, \quad \hat{F}^- = F^- \]

9
where the Green-Schwarz function \( V \). Here \( \omega \) depends on the moduli \( y_i \) is given by

\[
K = -\ln[(y_1 + \bar{y}_1 + V)(y_2 + \bar{y}_2)(y_3 + \bar{y}_3)],
\]

where the Green-Schwarz function \( V \) [10, 11, 14, 16]

\[
V(y_2, \bar{y}_2, y_3, \bar{y}_3) = \frac{Re h - Re y_2 Re \partial_y h - Re y_3 Re \partial_y h}{Re y_2 Re y_3}
\]

is of the first order in string coupling.

To write the supersymmetry transformations, one introduces symplectic invariants [7, 17, 18]

\[
S_{\mu\nu} = X I m N_{IJ} F_{\mu\nu}^{-J},
\]

\[
T_{\mu\nu} = 2i e^{K/2} S_{\mu\nu} = 2i e^{K/2} X I m N_{IJ} F_{\mu\nu}^{-J}
\]

and

\[
G_{\mu\nu}^{-i} = -k^{ij} \bar{f}_j I m N_{IJ} F_{\mu\nu}^{-J}.
\]

Here \( k^{ij} \) is the inverse Kähler metric, and \( f_i^I = (\partial_{z_i} + \frac{1}{2} \partial_{\bar{z}_i} K) e^{K/2} X^I \). The supersymmetry transformations of the chiral gravitino \( \psi_{\alpha\mu} \) and gaugini \( \lambda^i_\alpha \) are

\[
\delta \psi_{\alpha\mu} = D_\mu \epsilon_\alpha - T_{\mu\nu}^{-\gamma} \epsilon_{\alpha\beta} \epsilon^\beta,
\]

\[
\delta \lambda^i_\alpha = i \gamma^\mu \partial_{\mu} \epsilon^\alpha + G_{\mu
u}^{-i} \gamma^\nu \epsilon^{\alpha\beta} \epsilon_\beta,
\]

where

\[
D_\mu \epsilon_\alpha = (\partial_{\mu} - \frac{1}{4} w^{\hat{a} \hat{b}} \gamma_{\hat{a} \hat{b}} + \frac{i}{2} Q_\mu) \epsilon_\alpha.
\]

Here \( w^{\hat{a} \hat{b}} \) and \( Q_\mu \) are the spin and Kähler connections; \( \hat{a}, \hat{b}, \ldots \) are the tangent space indices, \( a, b, \ldots \) are the space-time indices.

The metric of a stationary spherically-symmetric configuration is

\[
ds^2 = -e^{2U} (dt + w_i dx^i)^2 + e^{-2U} dx^i dx_i.
\]

The only non-vanishing components of the spin connection \( w_0^{\hat{a}} \) are \( w_0^{\hat{a} \hat{b}} = \frac{1}{2} \partial_{\hat{b}} e^{2U} \) and \( w_0^{\hat{a}} = \frac{1}{2} (\partial_{\hat{a}} w_0 - \partial_0 w_0) \).
Requiring the supersymmetry variations of gravitino and gaugini to vanish, we obtain a system of equations. We look for a solution of this system with the supersymmetry parameter satisfying the relation $\epsilon^\alpha = \gamma_0^\alpha \epsilon^\beta \epsilon_\beta$ \footnote{More exactly, following [21], one must extract from the supersymmetry parameter the coordinate-dependent factor.}. The $\mu = 0$ component of Eq.(42) takes the form

$$\frac{1}{4} w_0^{-\dot{a}\dot{b}} \gamma_\dot{a} \gamma_\dot{b} \epsilon^\alpha + T_{0n}^i e^{bn} \gamma_\dot{b} \epsilon_{\alpha \beta} \epsilon^\beta = 0. \tag{44}$$

Here, using antisymmetry of $w_0^{-\dot{a}\dot{b}}$ in upper indices and chirality of the spinor $\epsilon^\alpha$, we transformed $w_0^{-\dot{a}\dot{b}}$ into $w_{\dot{a} \dot{b}}^0$. Using the relations

$$G_{\dot{m}\dot{n}} = i \epsilon_{\dot{m}\dot{n}\dot{b}0} G^{-\dot{b}0}, \quad G_{\mu\nu} \gamma^\mu \gamma^\nu \epsilon^\alpha = 4 G_{0n}^{-\dot{b}0} \gamma^\dot{b} \epsilon^\alpha \tag{45}$$

valid for any self-dual tensor and chiral spinor, Eq.(44) can be rewritten as

$$\left( w_{\dot{0}0}^{-\dot{a}0} - T_{0n}^i e^{bn} \right) \gamma^\dot{b} \epsilon_{\alpha \beta} \epsilon^\beta = 0. \tag{46}$$

To have a nontrivial solution for the supersymmetry parameter, we require that $w_{\dot{0}0}^{-\dot{a}0} - T_{0n}^i e^{bn} = 0$. Below we consider static configurations (see Sects.5 and 8). Sufficient condition to have a static metric is $ImT_{0n}^- = 0$ in which case

$$\frac{1}{2} w_{\dot{0}0}^0 - e^U T_{0n}^- = 0. \tag{47}$$

Using the relations (45), conditions of the gaugini supersymmetry transformation to vanish are written as

$$(i \gamma^n \partial_n z^i + 4 G_{0n}^{-\dot{b}0} \gamma^\dot{b} \epsilon^\alpha \epsilon^\beta = 0 \tag{48}$$

Transforming the index $n$ in $\gamma^n$ to the tangent space index and the indices in $G_{0n}^{-\dot{a}0}$ to the world ones, we find that there is a nontrivial solution provided

$$i \partial_n z^i + 4 e^{-U} G_{0n}^{-\dot{a}0} = 0. \tag{49}$$

Convoluting the equation (49) with the function $f_I^I$ and using the relation of special $N = 2$ geometry

$$k^{ij} f_i^I f_j^J = -\frac{1}{2} (ImN)^{IJ} - e^K \bar{X}^I X^J,$$

it is obtained in the form (cf.[19, 20])

$$i f_I^I \partial_n z^i + 4 e^{-U} \left( \frac{1}{2} \mathcal{F}_{0n}^- + e^K \bar{X}^I S_{0n}^I \right) = 0. \tag{50}$$

Eq.(50) can be recast to a form which contains the fields $G_{I0n}$ and $\bar{F}_I$. Convoluting Eq.(50) with $\bar{F}_I$ and using the identities

$$\mathcal{F}^{-I} \bar{F}_I = G_I^- \bar{X}^I, \quad G_I^I f_I^I = \bar{X}^I g_{II},$$
where \( g_{Ii} = (\partial_i + \frac{1}{2} \partial_i K) e^{K/2} F_I \), which follow from definitions of \( f^I_i \) and \( G_{I\mu
u}^- \), we have

\[
i\bar{X}^i g_{Ii} z^i + 4 e^{-U} \left( \frac{1}{2} G_{I0n}^- \bar{X}^i + e^K F_{I} \bar{X}^i S_{0n} \right) = 0.
\]

Removing the functions \( \bar{X}^i \), we obtain the symmetric equation

\[
i g_{Ii} z^i + 4 e^{-U} \left( \frac{1}{2} G_{I0n}^- + e^K F_{I} S_{0n} \right) = 0. \tag{51}
\]

Using the gravitino equation, the Eqs.(50) and (51) are presented as (cf. [8])

\[
-2\mathcal{F}_{0n}^I = i \left[ e^U \partial_n (e^{K/2} X^I) - (e^{K/2} \bar{X}^I) \partial_n e^U \right] - i M (\partial_i K \partial_n y_i) e^{K/2+U} X^I, \tag{52}
\]

\[
-2 G_{I0n}^- = i \left[ e^U \partial_n (e^{K/2} F_I) - (e^{K/2} \bar{F}_I) \partial_n e^U \right] - i M (\partial_i K \partial_n y_i) e^{K/2+U} F_I. \tag{53}
\]

Here we used the equality \( \partial_i K \partial_n y_i = \frac{1}{2} \partial_n K + i M (\partial_i K \partial_n y_i) \). Eqs.(52) and (53) are not independent, but one set can be obtained from the other. One can also take some equations from the first set, and the remaining equations from the second. In this paper we shall use the first set of Eqs.(50) or (52). Another useful choice for practical calculations is to take for \( I = 0, 1 \) the equations from the first set (52) and for \( I = 3, 4 \) from the second (53).

## 4 Solution of the tree-level equations

In this section, to fix notations for the following, we solve the combined tree-level system of the Maxwell equations and Killing spinor equations for the moduli (cf. [19, 21, 27]). We look for a solution in the holomorphic section associated with the compactified heterotic string with the metric in the form (43), two magnetic fields \( \mathcal{F}^0_{\mu
u} \) and \( \mathcal{F}^1_{\mu
u} \) and two electric fields \( \mathcal{F}^2_{\mu
u} \) and \( \mathcal{F}^3_{\mu
u} \). We consider the case of purely real moduli \( y_i \) (21), i.e. \( a_i = 0 \). The basis of the heterotic holomorphic section is expressed via the moduli in the section with the prepotential as [10]

\[
(\bar{X}^I, \bar{F}_I) = (1, y_{2y_3}, iy_2, iy_3; -iy_1 y_2 y_3, -iy_1, y_1 y_3, y_1 y_2). \tag{54}
\]

Solving the Maxwell equations and the Bianchi identities we obtain the tree-level magnetic

\[
\mathcal{F}_{0n}^{-0} = i \frac{\hat{P}_0}{2 \sqrt{-g^r}} \frac{x^n}{r^n} = i \frac{e^{2U} \hat{P}_0}{2} \frac{x^n}{r^3}, \quad \mathcal{F}_{0n}^{-1} = i \frac{\hat{P}_1}{2 \sqrt{-g^r}} \frac{x^n}{r^n} = i \frac{e^{2U} \hat{P}_1}{2} \frac{x^n}{r^3} \tag{55}
\]

and electric

\[
\mathcal{F}_{0n}^{-2} = -i \frac{\hat{Q}_2}{2 \sqrt{-g} \text{Im} \bar{N}_{22}} \frac{x^n}{r}, \quad \mathcal{F}_{0n}^{-3} = -i \frac{\hat{Q}_3}{\sqrt{-g} \text{Im} \bar{N}_{33}} \frac{x^n}{r} \tag{56}
\]

field strengths, where \( \sqrt{-g} = r^2 e^{-2U} \) and the gauge couplings are

\[
\hat{N}_{00} = -iy_1 y_2 y_3, \quad \hat{N}_{11} = -iy_1 y_3, \quad \hat{N}_{22} = -iy_1 y_3, \quad \hat{N}_{33} = -iy_1 y_2, \quad \hat{N}_{01} = -iy_1 y_2 y_3.
\]
In the case of real moduli $y_i$, comparing with the action (1), we have

$$Im(\hat{N}_0, \hat{N}_1, \hat{N}_2, \hat{N}_3) = -e^{-\phi}(G_{11}, G_{12}, G_{22}, G_{22}).$$

The charges $\sqrt{8}\hat{p}_I$ and $\sqrt{8}\hat{q}_I$ are constrained to lie on an even self-dual lattice [1]. The tree-level Kähler potential is

$$K = -\ln 8y_1y_2y_3. \quad (57)$$

For the symplectic invariant combination $S_{on}$ we obtain

$$S_{on} = (Im\hat{N}_0 \mathcal{F}_{0n}^{-0} + y_2y_3 Im\hat{N}_1 \mathcal{F}_{0n}^{-1} + iy_2 Im\hat{N}_2 \mathcal{F}_{0n}^{-2} + iy_3 Im\hat{N}_3 \mathcal{F}_{0n}^{-3})$$

$$= -i\frac{y_1y_2y_3}{2} \left( \hat{p}_0^0 + \frac{\hat{p}_1^1}{y_2y_3} + \frac{\hat{q}_2^2}{y_1y_3} + \frac{\hat{q}_3^3}{y_1y_3} \right) e^{2U} \frac{x^n}{r^3}. \quad (58)$$

Gravitini Eq.(47) takes the form

$$\frac{1}{4} \partial_\alpha e^{2U} - \left( \frac{y_1y_2y_3}{8} \right)^{1/2} e^{3U} \left( \hat{p}_0^0 + \frac{\hat{p}_1^1}{y_2y_3} + \frac{\hat{q}_2^2}{y_1y_3} + \frac{\hat{q}_3^3}{y_1y_3} \right) \frac{x^n}{r^3} = 0. \quad (59)$$

The tree-level gaugini equations (50) written in the section associated with the prepotential are

$$I = 0 : \frac{ie^{K/2}}{2} \partial_\alpha y_1y_2y_3 - 4e^{-U} \left( \frac{1}{2} \mathcal{F}_{0n}^{-0} + e^K S_{on} \right) = 0$$

$$I = 1 : y_1e^{K/2} \partial_\alpha y_1y_2y_3 + 4e^{-U} \left( \frac{\mathcal{F}_{0n}^{-1}}{2N_{11}} - iy_1e^K S_{on} \right) = 0$$

$$I = 2 : y_2e^{K/2} \partial_\alpha y_1y_2y_3 + 4e^{-U} \left( \frac{\mathcal{F}_{0n}^{-2}}{2N_{11}} - iy_2e^K S_{on} \right) = 0$$

$$I = 3 : y_3e^{K/2} \partial_\alpha y_1y_2y_3 + 4e^{-U} \left( \frac{\mathcal{F}_{0n}^{-3}}{2N_{11}} - iy_3e^K S_{on} \right) = 0. \quad (60)$$

The field strengths in the section with the prepotential are defined from those (55) and (56) by using (36) and (37).

In the following, we consider a particular extremal dyonic solution of the Eqs.(59) with arbitrary constant moduli $y_2$ and $y_3$ and with the charges subject to relations

$$\hat{p}_0^0 = \frac{\hat{p}_1^1}{y_2y_3}, \quad \hat{q}_2^2 = \hat{q}_3^3. \quad (61)$$

Four gaugini equations reduce to one equation. The system of the gravitini and gaugini equations is

$$\partial_\alpha e^{2U} - (8y_1y_2y_3)^{1/2} e^{3U} \left( \hat{p}_0^0 + \frac{\hat{q}_2^2}{y_1y_3} \right) \frac{x^n}{r^3} = 0,$$

$$\partial_\alpha y_1 - (8y_1y_2y_3)^{1/2} e^{2U} \left( \hat{p}_0^0 - \frac{\hat{q}_2^2}{y_1y_3} \right) \frac{x^n}{r^3} = 0. \quad (62)$$
Introducing the charges $P$ and $Q$ as

$$P = \sqrt{8y_2 y_3} \hat{P}^0, \quad Q = \hat{Q}_2 \sqrt{\frac{8y_2}{y_3}},$$ (63)

for the metric and dilaton we obtain

$$e^{-2\nu} = \frac{(P + r)(Q + r)}{r^2}, \quad y_1^{-1} = e^\phi = \frac{P + r}{Q + r} \equiv f_0$$ (64)

which is a particular case of a general dyonic BPS saturated solution [22, 23] The components of the metric $G_{\alpha\beta}$ are

$$G_{11} = y_2 y_3, \quad G_{22} = \frac{y_2}{y_3}.$$ (65)

The factor $\sqrt{8}$ appears because of different normalizations of the gauge terms in the actions (3) and (19).

## 5 Solution of the system of the loop-corrected equations

Our next aim is to solve the system of Maxwell and Killing spinor equations for the loop-corrected metric and moduli using (64) as the the tree-level solution. We look for a solution in the first order in the string coupling constant. The loop corrections to the gauge coupling constants are calculated with the tree-level moduli, and the terms which depend on the moduli $y_2$ and $y_3$ are independent of coordinates. Dependence on coordinates enters through the modulus $y_1 = f_0^{-1}$.

In the basis associated with the heterotic string compactification, the part of the matrix $\hat{N}_{\alpha\beta}$ which contains the imaginary parts of the moduli, $a_i$, is

$$\begin{pmatrix}
0 & a_1 & -ia_2 y_3^2/y_2^3 & -ia_3 y_3^2/y_2^3 \\
a_1 & 0 & ia_2 y_3^2/y_2^3 & ia_2 y_3^2/y_2^3 \\
-ia_2 y_3^2/y_2^3 & ia_3 y_3^2/y_2^3 & 0 & a_3 \\
-ia_3 y_3^2/y_2^3 & ia_2 y_3^2/y_2^3 & a_3 & 0
\end{pmatrix}.$$ (65)

Let us solve the Maxwell equations (in the heterotic holomorphic section) which can be rewritten in the form

$$\partial_\mu (\sqrt{-g} \text{Im} \hat{N}_{\alpha\beta} \hat{F}^\alpha_{\beta})^{\mu\nu} = 0.$$ (66)

With the accuracy of the terms of the first order in string coupling, Eqs.(66) written in spherical coordinates are

$$I = 0 : \partial_r [\sqrt{-g}(\text{Im} \hat{N}_{00} \hat{F}^0 + \text{Im} \hat{N}_{02} \hat{F}^2 + \text{Im} \hat{N}_{03} \hat{F}^3) + \text{Re} \hat{N}_{00} \hat{F}^0 + \text{Re} \hat{N}_{01} \hat{F}^1]^{0r} = 0$$ (67)

$$I = 1 : \partial_r [\sqrt{-g}(\text{Im} \hat{N}_{11} \hat{F}^1 + \text{Im} \hat{N}_{12} \hat{F}^2 + \text{Im} \hat{N}_{13} \hat{F}^3) + \text{Re} \hat{N}_{10} \hat{F}^0 + \text{Re} \hat{N}_{11} \hat{F}^1]^{1r} = 0$$ (68)

$$I = 2 : \partial_r [\sqrt{-g}(\text{Im} \hat{N}_{22} \hat{F}^2 + \text{Im} \hat{N}_{23} \hat{F}^3) + \text{Re} \hat{N}_{20} \hat{F}^0 + \text{Re} \hat{N}_{21} \hat{F}^1]^{2r} = 0$$ (69)

$$I = 3 : \partial_r [\sqrt{-g}(\text{Im} \hat{N}_{33} \hat{F}^3 + \text{Im} \hat{N}_{32} \hat{F}^2) + \text{Re} \hat{N}_{30} \hat{F}^0 + \text{Re} \hat{N}_{31} \hat{F}^1]^{3r} = 0$$ (70)
Only the diagonal gauge couplings $\hat{N}_{II}$ contain terms of zero order in string coupling. The field strengths $\hat{F}^{0,10r}$, absent at the tree level, are of the first order in the string coupling. Substituting the tree-level field strengths (55) and (56) in equations (67) and (68), we have

\[
\begin{align*}
\hat{F}^{00r} &= \frac{\hat{q}_0 - \text{Re}\hat{N}_{00} \hat{P}^0 - \text{Re}\hat{N}_{01} \hat{P}^1 - \hat{Q}_2 \text{Im}\hat{N}_{02} - \hat{Q}_3 \text{Im}\hat{N}_{03}}{\sqrt{-g} \text{Im}\hat{N}_{00}}, \\
\hat{F}^{10r} &= \frac{\hat{q}_1 - \text{Re}\hat{N}_{10} \hat{P}^0 - \text{Re}\hat{N}_{11} \hat{P}^1 - \hat{Q}_2 \text{Im}\hat{N}_{12} - \hat{Q}_3 \text{Im}\hat{N}_{13}}{\sqrt{-g} \text{Im}\hat{N}_{11}}, \\
\end{align*}
\]

where $\hat{q}_{0,1}$ are arbitrary constants which have a meaning of electric charges of the first order in string coupling. Since the numerators are of the first order in string coupling, denominators are taken in the leading order. Substituting explicit expressions for the couplings, we obtain

\[
\begin{align*}
\hat{F}^{00r} &= \frac{\hat{q}_0 - c_{00} \hat{P}^0 - a_1 \hat{P}^1 - a_2 \hat{Q}_a}{\sqrt{-g} \text{Im}\hat{N}_{00}}, \\
\hat{F}^{10r} &= \frac{\hat{q}_1 - d_{11} \hat{P}^1 - a_1 \hat{P}^0 + a_3 \hat{Q}_2/y_3^2 + a_2 \hat{Q}_3/y_3^2}{\sqrt{-g} \text{Im}\hat{N}_{11}}, \\
\end{align*}
\]

The Eqs.(69) and (70) yield

\[
\begin{align*}
\hat{F}^{20r} &= \frac{\hat{Q}_2 - \text{Re}\hat{N}_{20} \hat{P}^0 - \text{Re}\hat{N}_{21} \hat{P}^1 - \text{Im}\hat{N}_{22} \hat{Q}_3}{\sqrt{-g} \text{Im}\hat{N}_{22}}, \\
\hat{F}^{30r} &= \frac{\hat{Q}_3 - \text{Re}\hat{N}_{30} \hat{P}^0 - \text{Re}\hat{N}_{31} \hat{P}^1 - \text{Im}\hat{N}_{32} \hat{Q}_2}{\sqrt{-g} \text{Im}\hat{N}_{33}}, \\
\end{align*}
\]

At this point there are two options. Either we can allow for appearance of the fields absent at the tree level with the charges of the first order in string coupling, or we can require that the loop-corrected solution, as the tree level one, contains two electric and two magnetic fields. Since magnetic fields $^*\hat{F}^{2,3}$ with the charges $\hat{p}_{2,3}$ of the first order in string coupling enter the Maxwell equations multiplied by the non-diagonal gauge couplings $\hat{N}_{IJ}$ which are of the first order in string coupling and thus cannot be fixed with the accuracy of the first order in string coupling, we set the charges $\hat{p}_{2,3}$ equal to zero, but retain the electric charges $\hat{q}_0$ and $\hat{q}_1$.

In the section associated with the prepotential, the field strengths can be obtained either by direct solution of the system of Maxwell equations and Bianchi identities or by using relations (36) and (37).

In the basis with the prepotential the charges are

\[
(P^I, Q_I) = (\hat{P}^0, -\hat{q}_1 + d_{11} \hat{P}^1, 0, 0; -c_{00} \hat{P}^0 + \hat{q}_0, \hat{P}^1, \hat{Q}_2 - c_{20} \hat{P}^0, \hat{Q}_3 - c_{30} \hat{P}^0),
\]

and in the expressions for the electric field strengths $F^{10r}$ the constants $c_{0j}$ in the transformed charges cancel those in couplings.
Using the formula (37) for $F^{-1}$, we have

$$F^{-1}_{0i} = \frac{1}{2\sqrt{-g}ImN_{11}} \left[ P^{0}ReN_{10} - \hat{P}^{1} - iq_{1}ImN_{11} + \frac{ImN_{12}Q_{2}}{ImN_{22}} + \frac{ImN_{13}Q_{3}}{ImN_{33}} \right].$$  \hspace{1cm} (75)

Here we introduced $q_{1} \equiv -P^{1} = \hat{q}_{1} - d_{11}\hat{P}^{1}$ and $q_{0} \equiv Q_{0} = \hat{q}_{0} - c_{00}\hat{P}^{0}$. In the first order in the string coupling constant, using solutions for the field strengths (71) and (73), we calculate the expression $S_{0n}$ (40) as

$$S_{0n} = \left\{ \left[ P^{0}(ImN_{00} + y_{i}ReN_{i0} + c_{0a}ya) - \hat{P}^{1}y_{1} - (\hat{Q}_{2}y_{2} + \hat{Q}_{3}y_{3}) \right] - i[P^{0}(a_{1}y_{2}y_{3} + a_{2}y_{1}y_{3} + a_{3}y_{1}y_{2}) + a_{a}\hat{Q}_{a} + a_{1}\hat{P}^{1} + \hat{q}_{0} - c_{00}\hat{P}^{0} + (\hat{q}_{1} - d_{11}\hat{P}^{1})y_{2}y_{3}] \right\} \frac{i}{2}e^{2U}x^{n}r^{3}$$  \hspace{1cm} (76)

Only the the couplings $N_{00}$ and $N_{0i}$, $i = 1, 2, 3$ enter the expression (76). Substituting the loop-corrected gauge couplings (26), we obtain

$$ImN_{00} + y_{i}ReN_{i0} = -(y^{3} + 2v + h_{a}ya).$$  \hspace{1cm} (77)

All the terms containing second derivatives of the loop correction to the prepotential have canceled. The electric charges $\hat{q}_{0}$ and $\hat{q}_{1}$ enter only the equations for the imaginary parts of the moduli. Written in terms of the charges (74), the expression (76) is

$$S_{0n} = \left\{ \left[ -\hat{P}^{0}(y^{3} + 2v + h_{a}ya) - \hat{P}^{1}y_{1} - Q_{a}ya \right] - i[P^{0}(a_{1}y_{2}y_{3} + a_{2}y_{1}y_{3} + a_{3}y_{1}y_{2}) + a_{a}Q_{a} + a_{1}\hat{P}^{1} - q_{0} - q_{1}y_{2}y_{3}] \right\} \frac{i}{2}e^{2U}x^{n}r^{3}$$  \hspace{1cm} (78)

Because the Killing spinor equations (49) are linear in derivatives of the moduli $\partial z^{i}$, the equations for real parts of the moduli decouple from imaginary parts. To have a static metric, we must chose a solution for which $ImT^{0}_{0n} = 0$ (see. Sect.3), i.e. we take $S_{0n}$ purely imaginary.

Let us introduce notations for the loop-corrected metric and moduli. The functions $\phi, \gamma$ and $\sigma$ which enter the moduli (22) are split into the tree-level $\phi_{0}, \gamma_{0}$ and $\sigma_{0}$ and first-order parts in string coupling $\phi_{1}, \gamma_{1}$ and $\sigma_{1}$: $\phi = \phi_{0} + \phi_{1}$, etc. The function $U$ which enters the metric will be written as $2U_{0} + u_{1}$. The tree-level expressions for the functions $U_{0}$ and $\phi_{0}$ are given by (64), where $\gamma_{0}$ and $\sigma_{0}$ are arbitrary constants.

Using the Kähler potential (38), we calculate the combinations

$$B_{n}^{i} = f_{i}^{l}\partial_{n}z^{i}$$

which enter the gaugini Killing spinor equations (49). We have

$$B_{n}^{0} = \frac{1}{2}e^{K/2} \left( 1 - \frac{V}{2y_{1}} \right) \partial_{n}lny_{1}y_{2}y_{3}$$

$$B_{n}^{i} = iy_{i} \left( B_{n}^{0} + e^{K/2}\partial_{n}lny_{i} \right), \hspace{1cm} i = 1, 2, 3,$$  \hspace{1cm} (79)

where the Green-Schwarz function $V$ is

$$V = e^{-2\gamma}v.$$
The expressions (79) are calculated in the first order in the string coupling. In particular, all the factors multiplying the Green-Schwarz function $V$ which itself is of the first order in string coupling are taken at the zero order in the string coupling.

The charges $\hat{P}^{0,1}$ and $\hat{Q}_{2,3}$ are expressed via the charges $P$ and $Q$ as

$$
\hat{P}^{0} = \frac{Pe^{-\gamma_0}}{\sqrt{8}}, \quad \hat{P}^{1} = \frac{Pe^{\gamma_0}}{\sqrt{8}}, \quad \hat{Q}_{2} = \frac{Qe^{-\sigma_0}}{\sqrt{8}}, \quad \hat{Q}_{3} = \frac{Qe^{\sigma_0}}{\sqrt{8}}. \tag{80}
$$

Expanding the expression for the Kähler potential to the first order in string coupling, we obtain

$$
e^K = \frac{f_0 e^{-2\gamma_0}}{8} \left[ 1 + \left( \phi_1 - 2\gamma_1 - \frac{Vf_0}{2} \right) \right]. \tag{81}
$$

Using the expression for the Kähler potential (81) and expanding in the function $T_{0n}^2 = 2ie^K \frac{S_0}{2}$ which enters the gravitini equation (43) all the terms to the first order in string coupling, we have

$$
T_{0n}^2 = \frac{f_0^{-1/2}}{4} \left[ P \left( 1 - \frac{\phi_1}{2} + \left( \frac{3V}{4} + L \right) f_0 \right) + Qf_0 \left( 1 + \frac{\phi_1}{2} - \frac{Vf_0}{4} \right) \right] e^{2U/x^n} r^3. \tag{82}
$$

We used the relation $y_2 y_3 = e^{2\gamma}$ which follows from the definition (21) and introduced

$$
L = \frac{h_0 y_0 e^{-2\gamma_0}}{2}. \tag{83}
$$

Because the terms $V$ and $L$ are of the first order in string coupling, the factors $e^\phi$ multiplying these expressions can be substituted by their tree-level values $e^{\phi_0}$.

It is convenient to introduce the functions $q'$ and $l'$ as

$$
q' = \frac{f_0'}{f_0} = \frac{Q - P}{(r + P)(r + Q)}, \quad l' = 2U_0' = \frac{2PQ + Pr + Qr}{r(r + P)(r + Q)}. \tag{84}
$$

Now the gravitini equation (47) takes the form

$$
\frac{u'}{l'} - \frac{u}{2} + \frac{\phi_1}{2} \frac{P - Qf_0}{P + Qf_0} - \left( \frac{3V}{4} + L \right) f_0 + \frac{Qf_0}{P + Qf_0} (L + V) f_0 = 0. \tag{85}
$$

The leading-order terms have canceled due to the Eqs. (59).

Let us solve the gaugini equations. The equation (50) for $I = 0$ contains the combination

$$
\frac{1}{2} \mathcal{F}_{0n}^0 + e^K S_{0n} = \frac{e^{-\gamma_0}}{4\sqrt{8}} \left[ P \left( 1 + \gamma_1 - \left( \frac{V}{2} + L \right) f_0 \right) - Qf_0 (1 + \phi_1 - \gamma_1 - \frac{Vf_0}{2}) \right] i e^{2U/x^n} r^3. \tag{86}
$$

The one-loop-corrected expression for $B_{n}^0$ is

$$
B_{n}^0 = \frac{q' f_0^{1/2} e^{-\gamma_0}}{2\sqrt{8}} \left[ 1 + \frac{\phi_1'}{q'} - 2\gamma_1' + \frac{\phi_1 - 2\gamma_1}{2} - \frac{3Vf_0}{4} \right] \frac{x^n}{r}. \tag{87}
$$
Using Eqs.(86) and (87), we obtain the $I = 0$ equation in the form
\[
\frac{\phi'_1 - 2\gamma_1 q'}{q'} + \phi_1 P + \frac{Q f_0}{2} - \frac{u_1}{2} - 2\gamma_1 \frac{P}{P - Q f_0} + \left(-\frac{V}{4} + L\right) f_0 + \frac{Q f_0 L f_0}{P - Q f_0} = 0
\] (88)

To write the equation for $I = 1$, we must calculate the field strength $F^{-1}_{0n}$ (75). We have
\[
F^{-1}_{0n} = \frac{y_1 e^{-\gamma_0}}{2\sqrt{8}} \left[P \left(1 - 2\gamma_1 + \frac{V f_0}{2}\right) - Q f_0 \frac{V f_0}{2}\right] \frac{i e^{2U x^n r}}{r^3}.
\] (89)

The loop-corrected combination $B^1_n$ is
\[
B^1_n = -i \frac{q' f_0^{-1/2} e^{-\gamma_0}}{2\sqrt{8}} \left[1 + \frac{\phi_1' + 2\gamma_1'}{q'} - \frac{\phi_1 + 2\gamma_1}{2} - \frac{V f_0}{4} \right] x^n / r.
\] (90)

Using this expression and the field strength (89) we obtain the gaugini equation for $I = 1$
\[
\frac{\phi'_1 + 2\gamma_1 q'}{q'} + \phi_1 P + Q f_0 - \frac{u_1}{2} + 2\gamma_1 \frac{P}{P - Q f_0} + \left(-\frac{V}{4} + L\right) f_0 + \frac{Q f_0 L f_0}{P - Q f_0} = 0.
\] (91)

The Eqs.(88) and (91) split into the pair of equations
\[
\frac{\phi'_1}{q'} + \frac{\phi_1 P + Q f_0}{2} - \frac{u_1}{2} + \left(-\frac{V}{4} + L\right) f_0 + \frac{Q f_0 L f_0}{P - Q f_0} = 0
\] (92)

and
\[
\gamma'_1 + \gamma_1 \frac{P}{P - Q f_0} = 0.
\] (93)

Using the expressions (84) for $q'$ and (64) for $f_0$, we rewrite the Eq.(93) as
\[
\gamma'_1 - \gamma_1 \frac{P}{r(r + P)} = 0.
\] (94)

Substituting in the formula (73) for the field strength $F^{-2}_{0n}$ the explicit expressions for the loop-corrected couplings $\tilde{N}_{IJ}$, we obtain
\[
F^{-2}_{0n} = \frac{y_2 e^{-\gamma_0}}{2\sqrt{8}} f_0 \left[P \left(\frac{V}{2} + L_2\right) + Q \left(1 + \phi_1 - \gamma_1 + \sigma_1 - \frac{V f_0}{2}\right)\right] e^{2U x^n r^3}.
\] (95)

and similar expression for $F^{-3}_{0n}$ with the constant $L_3$ and the inverse sign of $\sigma_1$. Here
\[
L_a = h_a y_a e^{-2\gamma_0}.
\] (96)

Combining the Eqs.(50) with $I = 0$ and $I = 2$ (the same with $I = 3$), and using the expressions (79) for $B^I_n$, we have
\[
ie^{K/2} \frac{\partial S_{0n}}{y_2} + 4 e^{-U} \left(\frac{F^{-2}_{0n}}{2 y_2} - \frac{1}{2} F^{-0}_{0n} - 2 e^K S_{0n} \right) = 0.
\] (97)
Since the tree-level moduli $y_2$ and $y_3$ are constants, $\partial_n y_2$ and $\partial_n y_3$ are of the first order in string coupling. The combination \( \left( \frac{e^{-2\phi_0}}{2e^2} - \frac{1}{2} e^{-\phi_0} - 2e^K S_{0n} \right) \) is also of the first order in string coupling. Substituting in (97) the explicit expressions, we have

\[
\begin{align*}
\gamma_1' + \sigma_1' + \left( \frac{f_0}{r^2} \right)^{-1/2} e^{U_0} [P(L - L_2) f_0 - \gamma_1] - Q f_0 \sigma_1 &= 0, \\
\gamma_1' - \sigma_1' + \left( \frac{f_0}{r^2} \right)^{-1/2} e^{U_0} [P(L - L_3) f_0 - \gamma_1] + Q f_0 \sigma_1 &= 0.
\end{align*}
\] (98)

Combining the Eqs. (98) we obtain

\[
\begin{align*}
\gamma_1' - \gamma_1 \frac{P}{r(r + P)} &= 0, \\
\sigma_1' + \frac{f_0}{r^2} e^{U_0} \left[ \frac{1}{2} P(L_3 - L_2) - Q \sigma_1 \right] &= 0,
\end{align*}
\] (99)

where we used definitions (83) and (96) which yield

\[
2L - L_2 - L_3 = 0.
\] (100)

The first Eq. (100) is identical to (94). Thus, although the number of equations exceeds the number of variables, the system of equations is consistent. Substituting expressions for $f_0$ and $e^{U_0}$, we obtain the equation for $\sigma_1$ in the form

\[
\sigma_1' - \frac{1}{r(Q + r)} \left[ \frac{P}{2} (L_2 - L_3) - Q \sigma_1 \right] = 0.
\] (101)

Solving equations for the loop corrections to the moduli, we obtain

\[
\begin{align*}
\gamma_1 &= C_1 \left( \frac{r}{r + P} \right), \\
\sigma_1 &= C_2 \left( \frac{r}{r + P} \right) - \frac{(L_2 - L_3)}{2} \left( P \frac{P}{r + Q} \right),
\end{align*}
\] (102)

where $C_1$ and $C_2$ are arbitrary constants.

Let us solve the system of equations (85) and (92) for the loop corrections to the metric and dilaton. Using the relation

\[
l' = - \frac{P + Q f_0}{P - Q f_0} q',
\] (103)

we transform the system to the form

\[
\begin{align*}
\phi_1' - \frac{u_1}{2} q' - \frac{\phi_1}{2} l' + \left( L - \frac{V}{4} \right) f_0' + \frac{Q f_0}{P - Q f_0} f_0 = 0, \\
\phi_1' - \frac{u_1}{2} l' - \frac{\phi_1}{2} q' + \left( L + \frac{3V}{4} \right) f_0' + \frac{Q f_0}{P - Q f_0} \left( L + \frac{V}{2} \right) f_0 = 0.
\end{align*}
\] (104)

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Adding and subtracting the Eqs.(104), we rewrite the system as

\[ u' + \phi_1' - (u_1 + \phi_1) \frac{l' + q'}{2} + \left( 2L + \frac{V}{2} \right) \frac{Pf_0'}{P - Qf_0} = 0, \]
\[ u_1' - \phi_1' - (u_1 - \phi_1) \frac{l' - q'}{2} + Vf_0' \left( 1 + \frac{Qf_0}{2(P - Qf_0)} \right) = 0. \] (105)

Solving the Eqs.(105), we obtain

\[ u_1 + \phi_1 = \frac{r}{r + Q} \left[ c_1 - \left( 2L + \frac{V}{2} \right) \frac{P}{r} \right], \]
\[ u_1 - \phi_1 = \frac{r}{r + P} \left[ c_2 + \frac{V}{2Q} \left( \frac{(P - Q)^2}{r + Q} - \frac{P^2}{r} \right) \right]. \] (106)

Requiring that at large distances the loop-corrected metric and dilaton are asymptotic to the Lorentzian metric and constant dilaton equal to unity, we fix the constants \( c_1 \) and \( c_2 \): \( c_1 = c_2 = 0 \) and obtain

\[ u_1 = \frac{V}{4} \frac{r}{r + P} \left( \frac{(P - Q)^2}{Q(r + Q)} - \frac{P^2}{Qr} \right) - \frac{P}{r + Q} \left( L + \frac{V}{4} \right), \]
\[ \phi_1 = -\frac{V}{4} \frac{r}{r + P} \left( \frac{(P - Q)^2}{Q(r + Q)} - \frac{P^2}{Qr} \right) - \frac{P}{r + Q} \left( L + \frac{V}{4} \right). \] (107)

6 Purely electric and magnetic limits

Purely magnetic and electric solutions are obtained in the limits \( Q = 0 \) and \( P = 0 \), correspondingly.

In the limit \( Q \to 0 \) the combinations (106) become

\[ u_1 + \phi_1 = c_1 - \left( 2L + \frac{V}{2} \right) \frac{P}{r}, \]
\[ u_1 - \phi_1 = \frac{r}{r + P} \left[ c_2 - \frac{V}{2} \left( \frac{2P}{r} + \frac{P^2}{r^2} \right) \right]. \] (108)

Introducing \( f_0 = 1 + \frac{P}{r} \) and shifting the constants \( c_1 \) and \( c_2 \), we rewrite these relations as

\[ u_1 + \phi_1 = c_1 - \left( 2L + \frac{V}{2} \right) f_0 \]
\[ u_1 - \phi_1 = \frac{1}{f_0} \left( c_2 - \frac{Vf_0^2}{2} \right). \] (109)

which are the same as those obtained previously for magnetic black hole [24] up to the substitution \( L \to C = \frac{1}{2} \left[ (h_\alpha - c_{2\alpha})y_\alpha + \frac{C_{a\alpha}}{P_{a\alpha}} \right] e^{-2\gamma_0} \). The extra piece \( \frac{C_{a\alpha}}{P_{a\alpha}} e^{-2\gamma_0} \) appeared because when solving the loop-corrected system of Maxwell equations in the purely magnetic case, we
introduced arbitrary constants $C_{1,2}$ of the first order in the string coupling which have a meaning of electric charges. Solving the equations with non-zero electric charges and then letting the electric charge go to zero, we obtain the solution without the constants $C_a$.

Requiring that our solution becomes flat in the limit $r \to \infty$, we find that $c_1 = 2C + \frac{V}{2}$ and $c_2 = \frac{V}{2}$, and the loop corrections to the metric and dilaton are solution is

$$u_1 = -\left(\frac{V}{2} + C\right)\frac{P}{r} - \frac{V}{4} \frac{P}{r + P}, \quad \phi_1 = -\frac{C_P}{r} + \frac{V}{4} \frac{P}{r + P}.$$  \hspace{1cm} (110)

In the limit $P \to 0$ we obtain the purely electric black hole, and the loop corrections to the metric and dilaton are

$$u_1 = \frac{V}{4} \frac{Q}{r + Q}, \quad \phi_1 = -\frac{V}{4} \frac{Q}{r + Q}.$$  \hspace{1cm} (111)

Both electric and dyonic solutions have non-singular limit $r \to 0$. In particular, in dyonic case

$$u_1|_{r\to0} = -\frac{P}{Q} \left(L + \frac{V}{2}\right), \quad \phi_1|_{r\to0} = -\frac{P}{Q} L.$$  \hspace{1cm} (112)

Note that the limits $r \to 0$ and $Q \to 0$ do not commute.

7 BPS mass and asymptotics of the loop-corrected solution

BPS condition relating the mass of a solution with the central charge of the $N=2$ supersymmetry algebra should retain its form in perturbation theory provided the perturbation theory does not violate supersymmetry. The BPS spectrum of $N=2$ supersymmetric theory is given by [7]

$$M_{BPS}^2 = |Z_\infty|^2 = e^K|n_1 X^I - m^I F_I|^2, \hspace{1cm} (113)$$

$n_1$ and $m^I$ are integers proportional to electric and magnetic charges of the fields (24) gauging the group $U(1)^4$. Subscript $\infty$ indicates that the equality is valid at spatial infinity.

Equality of the ADM and BPS masses of a BPS-saturated solution can be seen from the Nester construction in which both masses are expressed via the asymptotics of the function (cf. [25, 26])

$$T_{\mu\nu} = e^{K/2}(F_I \mathcal{F}^I_{\mu\nu} - X^I G_{I\mu\nu}).$$

Asymptotics of the fields $\mathcal{F}^I$ and $G_I$ are proportional to quantized electric and magnetic charges, correspondingly, which results in the expression for the BPS mass (113).

In the the tree-level case, calculating $M_{BPS}^2$ in the holomorphic section (54), we have

$$M^2 = \frac{1}{8y_1 y_2 y_3} |n_0 + n_1 y_2 y_3 + n_2 i y_2 + n_3 i y_3 + m^0 (-i y_1 y_2 y_3) + m^1 (-i y_1) + m^2 y_1 y_3 + m^3 y_1 y_2|^2.$$  \hspace{1cm} (114)
For non-zero electric and magnetic numbers \(n_2, n_3, m^0, m^1\) we obtain

\[
M^2 = \frac{1}{8e^{2\gamma_0}} n_0 + n_1 e^{2\gamma_0} + m^2 e^{\gamma_0 - \sigma_0} + m^3 e^{\gamma_0 + \sigma_0} |^2
\]  

(115)

which coincides with the ADM mass of the dyon written as

\[
M_{ADM} = 2(P + Q) = \hat{P}^0 e^{\gamma_0} + \hat{P}^1 e^{-\gamma_0} + \hat{Q}^2 e^{\sigma_0} + \hat{Q}^3 e^{-\sigma_0},
\]

(116)

provided we take \((n_2, n_3, m^0, m^1) = \sqrt{8}(-\hat{Q}^2, \hat{Q}^3, \hat{P}^0, \hat{P}^1)\). Here we have taken into account that the dilaton is normalized at the infinity to unity: \(y_1|_\infty = 1\).

At the one-loop level, the BPS mass can be obtained either from \(r \to \infty\) asymptotics of the function \(T^+\), or by substituting in (113) the loop-corrected (exact, because in \(N = 2\) supersymmetric theory prepotential receives only one-loop correction) period vector

\[
(\hat{X}^I, \hat{F}_I) = (X^0, F_1, X^2, X^3; F_0 + c_{00}X^0 + c_{0a}X^a, -X^1 + d_{11}F_1, F_2 + c_{20}X^0 + c_{2a}X^a, F_3 + c_{30}X^0 + c_{3a}X^a).
\]

(117)

We have

\[
M^2_{BPS} = e^K \left[ \hat{P}^0 (y^3 + 2v + h_ay_a) + \hat{P}^1 y_1 + (\hat{Q}^a - c_{0a} \hat{P}^0)y_a \right]^2 + O(e^2)
\]

(118)

The ADM mass is obtained from the asymptotic form of the metric

\[
e^{2U}|_\infty = e^{2U_0}(1 + u_1)|_\infty.
\]

(119)

In the limit \(r \to \infty\) we obtain the asymptotics of the correction to the metric

\[
u_1|_{r \to \infty} = \frac{1}{r} \left( \frac{VQ}{4} - \frac{3VP}{4} - LP \right),
\]

(120)

and it is seen that BPS and ADM masses are equal to each other.

8 Equations for Axions

As it was discussed in Sect.3, to have a static metric, we must take a solution for which \(Im T^+_{0n} = 0\). This requires discussion of the equations for the real parts \(a_i\) of the moduli \(z_i\).

To be concrete, let us consider the first set of the gaugini Eqs.(52). Taking the real and imaginary parts of these equations written in the holomorphic section associated with the heterotic string compactification, we have

\[
\hat{\bar{F}}^I_{0n} = \partial_n \left( e^{K/2+U} Im \hat{X}^I \right) + e^{K/2+U} Re \hat{X}^I Im(\partial_y K \partial_n y_i),
\]

(121)

\[
\sqrt{-g} \hat{F}^I_{0n} = e^{2U} \partial_n \left( e^{K/2-U} Re \hat{X}^I \right) - e^{K/2+U} Im \hat{X}^I Im(\partial_y K \partial_n y_i).
\]

(122)

Let us take Eqs. (121) for \(I = 0, 1\) and (122) for \(I = 2, 3\). Keeping the terms of the first order in string coupling, we obtain

\[
\hat{F}^0_{0r} = e^{K/2+U} Im(\partial_r K \partial_r y_1),
\]

(123)

\[
\hat{F}^1_{0r} = \partial_r \left( e^{K/2+U} (a_2 y_3 + a_3 y_2) \right) + e^{K/2+U} y_2 y_3 Im(\partial_r K \partial_r y_i)
\]

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and

\[ \partial_r \left( e^{K/2+U} a_a \right) + e^{K/2+U} y_a \, \text{Im}(\partial_t K \partial_r y_i), \quad a = 2, 3. \]  

(124)

Here \( y_i \) are real parts of the moduli taken in the leading order,

\[ \text{Im}(\partial_t K \partial_r y_i) = \frac{1}{2} \left( \frac{\alpha_1'}{y_1} + \frac{\alpha_2'}{y_2} + \frac{\alpha_3'}{y_3} \right). \]

A solution of these equations is

\[ a_2 = a_3 = 0, \quad a_1 \hat{P}^1 - q_0 = 0, \quad a_1 \hat{P}^0 - p_1 = 0, \]  

(125)

where the charges are subject to the relation \( q_0 \hat{P}^0 = q_1 \hat{P}^1 \). With this solution, the imaginary part of the function \( S_{0n} \) in (78) vanishes ensuring that the solution is static. The electric fields (72) are equal to zero also.

Finally, let us consider the axionic equations in the general case which corresponds to a stationary, but non-static solution for the metric. It is convenient to start from the form (49). Convoluting the Eq.(49) with the metric stationary, but non-static solution for the metric, we have

\[ k_{ij} \partial_n (\text{Re} \, y_i + ia_i) + 4e^{-U} \hat{f}_j \text{Im} \, N_{IJ} \mathcal{F}_{0n}^{-J} = 0. \]  

(126)

Introducing \( T_{0r} \equiv \text{Im} \, N_{IJ} \mathcal{F}_{0r}^{-J} \) and separating the imaginary part of Eq.(126), we obtain

\[ k_{ij} a_i' + 4e^{-U} \text{Im} \, (\hat{f}_j T_I) = 0 \]  

(127)

Since \( a_i \) are of the first order in string coupling, \( k_{ij} \) should be taken in the leading order. With the required accuracy, using the expressions in the section with the prepotential, we calculate

\[ T_{00r} = \left[ -q_0 + iy^3 \hat{P}^0 \text{Im} \, N_{00} \right] \frac{1}{2\sqrt{\gamma}^e}, \]

\[ T_{10r} = \left[ \hat{P}^0 \text{Re} \, N_{10} - \hat{Q}_1 + i(\hat{P}^0 \text{Im} \, N_{10} - q_1 \text{Im} \, N_{11}) \right] \frac{1}{2\sqrt{\gamma}^e}, \]

\[ T_{20r} = \left[ \text{Re} \, N_{20} \hat{P}^0 - \hat{Q}_2 + i\text{Im} \, N_{20} \hat{P}^0 \right] \frac{1}{2\sqrt{\gamma}^e}, \]

\[ T_{30r} = \left[ \text{Re} \, N_{30} \hat{P}^0 - \hat{Q}_3 + i\text{Im} \, N_{30} \hat{P}^0 \right] \frac{1}{2\sqrt{\gamma}^e}. \]  

(128)

In the combination \( \text{Im}(\hat{f}_j T_I) = -\text{Im} \, f_j^I \text{Re} \, T_I + \text{Re} \, f_j^I \text{Im} \, T_I \) which enters the Eq.(127) the imaginary parts of the functions \( T_I \) with \( I = 1, 2, 3 \) and \( \text{Re} \, T_0 \) are of the first order in string coupling. Thus, we need the corresponding functions \( \text{Re} \, f_j^I \) and \( f_j^0 \) in the leading order in \( \epsilon \).

The functions \( f_j^I \) calculated in the leading order in \( \epsilon \) are

\[ f_j^0 = \frac{e^{K/2}}{2} \left( \frac{1}{y_1}, \frac{1}{y_2}, \frac{1}{y_3} \right), \quad f_j^1 = \frac{e^{K/2} y_1}{2} \left( \frac{1}{y_1}, \frac{1}{y_2}, -\frac{1}{y_3} \right), \]

\[ f_j^2 = \frac{e^{K/2} y_2}{2} \left( -\frac{1}{y_1}, \frac{1}{y_2}, -\frac{1}{y_3} \right), \quad f_j^3 = \frac{e^{K/2} y_3}{2} \left( -\frac{1}{y_1}, -\frac{1}{y_2}, \frac{1}{y_3} \right). \]  

(129)
Here and below $y_i$ are real tree-level moduli. Also we find $Imf_i^I = \frac{a_i}{2y_i} e^{K/2}$, where $I, i = 1, 2, 3$. Collecting the above expressions, we obtain the system of three equations for three unknown functions $a_i$

\[
\begin{align*}
  a_1' + \frac{4e^{U+K/2}y_1}{r^2} \left[ q_0 + q_1 y_2 y_3 + a_1 P^1 + a_2 \hat{Q}_a + \hat{P}^0 (-a_1 y_2 y_3 + a_2 y_1 y_3 + a_3 y_1 y_2) \right] &= 0 \\
  a_2' + \frac{4e^{U+K/2}y_2}{r^2} \left[ q_0 - q_1 y_2 y_3 + a_1 \hat{P}^1 + a_2 \hat{Q}_a + \hat{P}^0 (+a_1 y_2 y_3 - a_2 y_1 y_3 + a_3 y_1 y_2) \right] &= 0 \\
  a_3' + \frac{4e^{U+K/2}y_3}{r^2} \left[ q_0 - q_1 y_2 y_3 + a_1 \hat{P}^1 + a_2 \hat{Q}_a + \hat{P}^0 (+a_1 y_2 y_3 + a_2 y_1 y_3 - a_3 y_1 y_2) \right] &= 0
\end{align*}
\]

In particular, it is seen that solution (125) satisfies the system. General solution contains three arbitrary constants which, in particular, can be adjusted so that to make asymptotic ("physical") charges of the electric fields (72) vanishing.

9 Discussion

In this paper the loop-corrected dyonic solution was obtained by solving the system of the loop-corrected Maxwell and Killing spinor equations. Loop-corrected equations of motion were obtained in perturbative approach in the first order in string coupling constant from the loop-corrected $N = 2$ supersymmetric effective action which has only one-loop perturbative correction.

In perturbative approach, all the expressions which are of the first order in string coupling and depend on the moduli are calculated by substituting tree-level moduli. Considerable simplifications are achieved for constant tree-level moduli $T$ and $U$.

The fields in the basis with prepotential and in heterotic basis are connected by a symplectic transformation which form is constrained by the requirement that the gauge couplings in the heterotic basis have the same structure as the loop-corrected gauge couplings obtained by path-integral calculation in heterotic string theory (cf. [14, 15]).

Although the tree-level dyonic black hole has two electric and two magnetic fields, at the one-loop level we allowed for appearance of two extra electric charges of the first order in string coupling. The asymptotic physical charges of the corresponding electric field strengths depend on the axions, and by a suitable choice of free constants can be set to zero.

Except for special values of the charges $P$ and $Q$, the moduli are away from the enhanced symmetry points, where second derivatives of the prepotential have logarithmic singularities [10, 11].

The expressions for the loop corrections are valid for all $r$ for which the perturbation expansion in string coupling is valid. For the dyonic solution, the sufficient condition is $\epsilon \frac{P}{Q} < 1$. In magnetic case, both the tree-level and the loop-corrected solutions can be used in the range $\frac{r}{\epsilon} > \epsilon V$. Perturbative corrections to dyonic and purely electric black holes, have finite limits $r \to 0$, in magnetic case the limit $r \to 0$ is singular.

In this paper we considered the $N = 2$ supersymmetric $STU$ model as stemming from the suitably compactified heterotic string theory. However, different embeddings of the $STU$
model in the underlying string theory are possible. In a series of papers [27] and refs. therein general classical 4D BPS-saturated generating black-hole solution preserving 1/8 of $N = 8$ SUSY was constructed, and its $NS – NS$ and $R – R$ embeddings in the type IIA and IIB theories were studied. In the case of the $NS – NS$ non-symmetric embedding of the $STU$ model, it is possible to make the same identification of the moduli with the string-theoretical fields as in the heterotic string compactification. The moduli are expressed via the metric and antisymmetric tensor components with the indices referring to the untwisted torus $T^2$. In $R – R$ embeddings, the moduli are expressed via the radii of the internal tori and off-diagonal components of the 10D metric in the IIB case and antisymmetric tensor in the IIA case, and the vector fields are the components of higher $R – R$ forms. It seems that the $NS – NS$ embedding, while less suitable for calculation of the microscopic entropy, is more natural for the study of string-loop corrections. In this case, dilaton is a natural string-loop expansion parameter, and the axion-dilaton parametrize a separate factor of the scalar manifold of the $STU$ model.
Acknowledgments

I would like to thank the members of the theoretical seminars at the Skobeltsyn Nuclear
Physics Institute and Lebedev Physical Institute for discussions, M.Bertolini and M.Trigiante
for useful e-mail correspondence. This work was partially supported by the RFFR grant No
00-02-17679.

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