Some remarks about the Taylor method of classification of diagrams

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Abstract

The derivation of scattering equations connecting the amplitudes obtained from diagrammatic expansions is of interest in many branches of physics. One method for deriving such equations is the classification-of-diagrams technique of Taylor. However, as we shall explain in this paper, there are certain points of Taylor’s method which require clarification. Firstly, it is not clear whether Taylor’s original method is equivalent to a simpler classification-of-diagrams scheme used by Thomas, Rinat, Afnan and Blankleider (TRAB). Secondly, when the Taylor method is applied to certain problems in a time-dependent perturbation theory it leads to the over-counting of some diagrams. This paper restates Taylor’s method and derives conditions for the equivalence of the Taylor method to the simpler method used by TRAB. It then explains the origin of the double-counting in Taylor’s method and outlines a solution.
I. INTRODUCTION

The method of classification of diagrams, developed by Taylor, is a powerful technique by which equations connecting the amplitudes obtained from a field theory may be derived, without it being necessary to explicitly specify the Lagrangian of the theory in the derivation [1,2]. This model-independence has made the technique particularly useful in theories of mesons and baryons, where it is not practical to use the QCD Lagrangian and the best equivalent Lagrangian containing meson and baryon degrees of freedom is not yet known. Examples of the application of the technique to simple systems of nucleons and pions include the original work on equations for the $\pi\pi\pi$, $\pi\pi N$, $\pi NN$ and $NNN$ systems by Taylor himself [3]; the derivation of the $\pi N - \pi\pi N$ equations by Afnan and Pearce [4,5]; studies of pion photoproduction on both a single nucleon and the deuteron [6,7] and the derivation by Avishai and Mizutani [8,9], on the one hand, and Thomas, Rinat, Afnan and Blankleider on the other [10-12], of the $NN - \pi NN$ equations. This work on the $NN - \pi NN$ system raised at least two questions about the Taylor technique, both of which, despite the technique's widespread application, remain unanswered. Although these problems originally arose in the context of the $NN - \pi NN$ equations it should be clear that the questions themselves are quite general ones about the Taylor method and, as such, are relevant independent of the particular system and Lagrangian under consideration.

The first question arose because Taylor's original technique was somewhat modified and simplified by, first, Thomas and Rinat [11] and, second, Afnan and Blankleider [12,13], in order to make it more useful for time-ordered perturbation theory calculations. However, the equations obtained by Afnan and Blankleider for the $NN - \pi NN$ system [12] were exactly the same as those derived by Avishai and Mizutani\(^1\), who used Taylor's original technique and a *time-dependent* perturbation theory [9]. Is this pure coincidence, or did Thomas and Rinat and Afnan and Blankleider (TRAB) discover a simplification of Taylor's technique? This question was posed and left unanswered by Avishai and Mizutani [9]. In this paper we answer it by deriving the conditions under which TRAB's method is equivalent to Taylor's original technique.

The second problem is that Taylor's method can lead to the double-counting of certain diagrams when it is applied in a time-dependent perturbation theory, such as covariant perturbation theory. This problem was first pointed out by Kowalski, Siciliano and Thaler who showed that there was double-counting in some models of pion absorption on nuclei [14]. While Kowalski et al. did not refer specifically to Taylor's method, the double-counting problem certainly arises when one applies the classification-of-diagrams technique to the problem of summing all possible diagrams contributing to, say, pion absorption on the deuteron. If one applies the Taylor method, as described below, to pion absorption on the deuteron, one obtains contributions from both of the diagrams in Figure 1. In this figure

\(^1\)Note that Avishai and Mizutani included both a term to account for heavy-meson exchange and a three-body force in their calculation, whereas Afnan and Blankleider included neither of these effects. However, the heavy-meson exchanges and a three-body force can easily be included in Afnan and Blankleider's derivation, and when that is done the resulting equations are exactly those obtained by Avishai and Mizutani.
$t^{(1)}$ is the $\pi N$ t-matrix with the pole part removed, $T$ is the full $NN$ t-matrix (provided one assumes the absence of anti-nucleons in the deuteron) and the nature of the $\pi NN$ vertex is explained below. Kowalski et al. pointed out that the inclusion of the crossed term (depicted in Figure 2) in $t^{(1)}$ leads to double-counting, as follows. In a time-dependent perturbation theory the contribution of this part of $t^{(1)}$ to the diagram on the right of Figure 1 is the diagram shown in Figure 3. However, this diagram has already been included as distortion in the initial channel, via the diagram on the left of Figure 1. Note that in a time-ordered perturbation theory this double-counting problem does not arise, since the only contribution made by Figure 2 to the right-hand diagram of Figure 1 is the diagram shown in Figure 4. In a time-ordered approach this diagram is not included as distortion in the incoming $NN$ channel and so is not over-counted.

The existence of this problem raises two questions. Firstly, why does this erroneous double-counting occur in a method which Taylor claimed worked regardless of the perturbation scheme being used? Secondly, how can the over-counting be eliminated? Avishai and Mizutani attempted to provide answers to both of these questions in their 1983 paper [9]. They claimed that the double-counting problem occurred because the derivation of the $NN - \pi NN$ equations had only considered the s-channel structure of the amplitudes in question. (We are using the notation of Mandelstam here [15,16].) They suggested that examining the s, t and u-channel structure simultaneously would remove the double-counting. However, this proposal contradicts Taylor’s original work in which he derived the classification-of-diagrams technique so as to have no double-counting whichever channel or channels the amplitudes’ structure was examined in. He only proposed performing the structure examination in a number of channels simultaneously as part of an approximation he intended to use in order to close the set of equations obtained from his method. We shall see below that Avishai and Mizutani’s suggestion is partly right: the lack of specification of the $t$ and $u$-channel cut structure of the sub-amplitudes from which the full amplitude is constructed can be thought of as the cause of the double-counting problem. Consequently, if the double-counting is to be eliminated the cut structure of these amplitudes in channels other than the s-channel does need to be considered. But, to cut in all channels simultaneously is unnecessary. Furthermore, such a solution is impractical, as it results in highly non-linear equations.

Another “solution” suggested by Avishai and Mizutani is just to ignore the double-counting, since (they claim) “…compared with the important role played by the direct nucleon pole term in nuclear $\pi$ absorption, the possible overcounting of the crossed pole term should hardly affect the essential physics.” [9]. The accuracy of this statement is open to question and the validity of such an approach was never tested, since the numerical calculation based on Avishai and Mizutani’s original work used Blankenbecler-Sugar reduction [17] in order to reduce the four-dimensional equations, thus time-ordering them and eliminating the double-counting difficulty [18]. Therefore this “solution” is practical, but, since it amounts to ignoring the problem completely we question whether it really is a solution at all!

The example presented by Kowalski et al. [14] shows that double-counting can occur when the Taylor method is applied to a time-dependent perturbation theory of the $NN - \pi NN$ system. The presence of this double-counting indicates a fundamental flaw in the Taylor method, which must be resolved before the method can be used confidently in order to derive
four-dimensional equations for any system. In this paper we solve this double-counting problem in general, by first pointing out why the double counting arises in Taylor’s method, and then explaining how to eliminate it. Consequently, we answer both the questions which were posed about the double-counting problem above.

Therefore, this paper resolves two issues associated with Taylor’s method: the validity of TRAB’s simplification of Taylor’s original work and the double-counting problem. In order to do this we first recapitulate Taylor’s original argument, in Section II. Then, in Section III we compare Taylor’s result for an arbitrary amplitude with the result obtained from TRAB’s conceptually simpler approach. In Section IV the flaw in Taylor’s argument which leads to the double-counting problem is explained and two examples from the $NN - \pi NN$ system, including the example given by Kowalski et al. [14] and discussed above, are given. Finally, in Section V we outline a general solution to the double-counting problem.

II. THE CLASSIFICATION-OF-DIAGRAMS METHOD OF TAYLOR: A REVIEW

In order to place the two points made in this paper about the classification-of-diagrams technique in their proper context we first review the Taylor method, summarizing the arguments presented in Taylor’s original paper [1]. In this review we examine the method as applied in the $s$-channel, but with appropriate modifications the technique may be applied in any channel.

Taylor’s method is a topological procedure which allows the summation of a series of diagrams via the classification of these diagrams according to their irreducibility. The method does not assume that these diagrams have been generated by a field theory. The diagrams could, for example, be diagrams representing the perturbation series expansion for an interacting system of $N$ particles. However, in this paper we take the view that the diagrams under consideration are Feynman diagrams generated by some field theory. In this view, the Taylor method provides a means of deriving equations connecting the amplitudes obtained from this underlying field theory.

Therefore, we assume there exists some perturbation expansion of Feynman diagrams, which when summed give a set of $m \rightarrow n$ Green’s functions, which in momentum-space we represent by:

$$G_{n-m}(p'_1, p'_2, \ldots, p'_n; p_1, \ldots, p_m). \quad (2.1)$$

If, of the $m(n)$ particles in the initial (final) state, $j$ ($j'$) are nucleons and the rest are pions, LSZ reduction [19] may be used to obtain the amplitude corresponding to this Green’s function:

$$A_{n-m}(p'_1, p'_2, \ldots, p'_n; p_1, \ldots, p_m) d_N^{-1}(p'_1) \ldots d_N^{-1}(p'_j) d_\pi^{-1}(p'_{j'+1}) \ldots d_\pi^{-1}(p'_n)$$

$$G_{n-m}(p'_1, p'_2, \ldots, p'_n; p_1, \ldots, p_m) d_N^{-1}(p_1) \ldots d_N^{-1}(p_j) d_\pi^{-1}(p_{j+1}) \ldots d_\pi^{-1}(p_m). \quad (2.2)$$

(Note that the use of the terms “initial” and “final” state here, and in the ensuing argument, is slightly liberal, since in time-dependent perturbation theory there is nothing which restricts the times associated with the $m$-particles with reference to those associated with the $n$-particles. But, by “initial” state we mean the state with $m$ particles having momenta $p_1, \ldots, p_m$ and by “final” state we mean the state with $n$ particles, having momenta
$p'_1, \ldots, p'_n$) Taylor's method provides a way of classifying all the perturbation diagrams contributing to $A_{n-m}$ according to their topology.

However, Taylor's method works only if all the particles involved are fully dressed. In order, therefore, for us to be able to discuss the Taylor method, we need to assume that all particles are fully dressed. (For a discussion of how this renormalization might be achieved see [20].)

Furthermore, in order to simplify matters as much as possible, we consider only distinguishable particles. Equations for indistinguishable particles may then either be obtained by symmetrizing or anti-symmetrizing the equations for distinguishable particles in the usual way, or by making the necessary changes to the Taylor method in order for it to apply to indistinguishable particles. Taylor himself pursued the latter approach in his original work [1]. For examples of the former approach see Ref. [3] or the papers [12,21].

The classification-of-diagrams technique is then based on the following definitions, which apply to any perturbation diagram, regardless of the perturbation scheme used to construct the diagram. (Note that the definitions would have to be suitably modified if we intended to consider the structure of the amplitude in any channel other than the $s$-channel.)

**Definition (r-cut)** An $r$-cut is an arc which separates initial from final states and intersects exactly $r$-lines, at least one of which must be an internal line. If all of the $r$ lines cut are internal lines then the cut is called an internal $r$-cut.

**Definition (r-particle irreducibility)** A diagram is called $r$-particle irreducible if, for all integers $0 \leq k \leq r$, no $k$-cut may be made on it. An amplitude is called $r$-particle irreducible if all diagrams contributing to it are $r$-particle irreducible.

Using these two definitions any diagram contributing to the connected $(r-1)$-particle irreducible $m \rightarrow n$ amplitude, $A^{(r-1)}_{n-m}$, may be placed in one of five classes. The class of the diagram is determined by what $r$-cuts may be made on it; the criteria for selection in a particular class are as follows:

$C_1$: No $r$-cut may be made on the diagram, i.e. it is $r$-particle irreducible;

$C_2$: Only an internal $r$-cut may be made on the diagram;

$C_3$: Only an $r$-cut which intersects at least one line from the initial state, but no lines from the final state, may be made;

$C_4$: At least one $r$-cut which intersects at least one line from both the initial and final states may be made, but no cut intersecting at least one line from the final state and no lines from the initial state may be made;

$C_5$: At least one $r$-cut which intersects at least one line from the final state and no lines from the initial state may be made.

Because of the way the classes are defined, any perturbation diagram must belong to one and only one class. Therefore, we may sum each of $C_1$ to $C_5$ separately, and then express $A^{(r-1)}_{n-m}$ as the sum of the five expressions we thereby obtain.
Now, while class \( C_1 \) may be summed directly, the classes \( C_2 - C_5 \) must each be summed by exhibiting a unique latest \( r \)-cut in each diagram and so splitting the diagram into an \( r \)-particle irreducible part and an \((r - 1)\)-particle irreducible part. This is done via the following lemma, known as the Last Internal Cut Lemma.

**Lemma (Last Internal Cut)** Any \((r - 1)\)-particle irreducible diagram which admits an internal \( r \)-cut has a unique internal \( r \)-cut which is nearest to the final state.

We now rehearse Taylor’s proof of this result, since the structure of the proof will turn out to be important in understanding the double-counting problem. The proof is based on that given by Taylor [1], but has been slightly modified in order to (we hope!) make its structure clearer.

**Proof:** Consider any two internal \( r \)-cuts \( c_1 \) and \( c_2 \). We wish to find an internal \( r \)-cut as late or later than both of them. If the cuts do not intersect it is clear which of the two is earlier and which later, and the later of the two cuts is thus the internal \( r \)-cut we are looking for. If they do intersect we define \( c_1^- \) and \( c_2^- \) to be the portions of the two cuts nearest the initial state and \( c_1^+ \) and \( c_2^+ \) to be the portions of the two cuts nearest the final state. We then construct \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \). (Note that the use of set notation here corresponds to viewing these cuts as sets whose members are the lines they cut.) These definitions of \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \) do not, however, tell us in which of the two sets to place a line that is cut by both \( c_1 \) and \( c_2 \). In order to resolve this ambiguity we proceed as follows. The diagram under consideration may be distorted so that any line which is cut by both \( c_1 \) and \( c_2 \) is either intersected by both cuts while it is horizontal, or intersected by both cuts when it is vertical. Lines which fall into the first category are called horizontal in \( c_1 \cap c_2 \), and lines which fall into the second category are called vertical in \( c_1 \cap c_2 \). The sets \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \) are then defined to both contain any line which is horizontal in \( c_1 \cap c_2 \), and to both not contain any line which is vertical in \( c_1 \cap c_2 \). Figure 5 provides a pictorial example of these definitions.

Now, clearly \( c_1^+ \cup c_2^+ \) is nearer to the final state than either \( c_1 \) or \( c_2 \). It is also clearly an internal cut, since it is composed entirely of internal lines. But, is it an \( r \)-cut? Denote by \( N(c) \) the number of lines cut by an arc \( c \). Then:

\[
N(c_1) = r; \quad N(c_2) = r. \tag{2.3}
\]

Furthermore, since the diagram in question is \((r - 1)\)-particle irreducible and \( c_1^+ \cup c_2^+ \) and \( c_1^- \cup c_2^- \) both constitute cuts on it, we have:

\[
N(c_1^- \cup c_2^-) \geq r; \quad N(c_1^+ \cup c_2^+) \geq r. \tag{2.4}
\]

But, because of the way \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \) are defined:

\[
c_1 \cup c_2 \supseteq (c_1^- \cup c_2^-) \cup (c_1^+ \cup c_2^+); \tag{2.5}
\]

therefore,

\[
N(c_1) + N(c_2) - N(c_1 \cap c_2) \geq N(c_1^- \cup c_2^-) + N(c_1^+ \cup c_2^+) - N([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]). \tag{2.6}
\]

We wish to determine the value of \( N(c_1^- \cup c_2^-) + N(c_1^+ \cup c_2^+) \), and hence the values of \( N(c_1^- \cup c_2^-) \) and \( N(c_1^+ \cup c_2^+) \). In order to do this we note the following facts about \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \):

\[
...\]
1. Using the definitions of \( c_1^- \), \( c_2^- \), \( c_1^+ \) and \( c_2^+ \) it is easy to show that:

\[
[c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+] \subseteq c_1 \cap c_2.
\] (2.7)

Consequently the notion of horizontal and vertical lines set up in \( c_1 \cap c_2 \) may be imported into \([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]\). In the terminology developed here we say that every line in \([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]\) is either horizontal in \( c_1 \cap c_2 \) or vertical in \( c_1 \cap c_2 \).

2. However, by definition, neither \( c_1^- \cup c_2^- \) or \( c_1^+ \cup c_2^+ \) contains any vertical line which is vertical in \( c_1 \cap c_2 \). Therefore, all of the lines in \([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]\) must be horizontal in \( c_1 \cap c_2 \). Furthermore, it is clear from the definition of \( c_1^- \cup c_2^- \) and \( c_1^+ \cup c_2^+ \) that \([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]\) contains every line which is horizontal in \( c_1 \cap c_2 \).

3. Since every line in \( c_1 \cap c_2 \) is either horizontal in \( c_1 \cap c_2 \) or vertical in \( c_1 \cap c_2 \) this suggests that:

\[
N(c_1 \cap c_2) = N([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]) + v,
\] (2.8)

where \( v \geq 0 \) is the number of lines which are vertical in \( c_1 \cap c_2 \).

There are now two possibilities:

1. The set \( c_1 \cap c_2 \) is empty. In this case the relation (2.7) implies that \([c_1^- \cup c_2^-] \cap [c_1^+ \cup c_2^+]\) is empty too. Consequently, Eq. (2.6) becomes:

\[
2r \geq N(c_1^- \cup c_2^-) + N(c_1^+ \cup c_2^+),
\] (2.9)

which, with Eq. (2.4), gives:

\[
N(c_1^- \cup c_2^-) = N(c_1^+ \cup c_2^+). \quad (2.10)
\]

2. The set \( c_1 \cap c_2 \) is not empty. In this case we may simply substitute Eq. (2.8) straight into Eq. (2.6) in order to obtain:

\[
N(c_1^- \cup c_2^-) + N(c_1^+ \cup c_2^+) \leq 2r - v,
\] (2.11)

and therefore,

\[
N(c_1^- \cup c_2^-) + N(c_1^+ \cup c_2^+) \leq 2r. \quad (2.12)
\]

The only way this equation can be reconciled with Eq. (2.4) is to have:

\[
N(c_1^- \cup c_2^-) = N(c_1^+ \cup c_2^+) = r. \quad (2.13)
\]

In particular, note that if \( v > 0 \) Eqs. (2.11) and (2.4) are contradictory and so, given the above assumptions it is impossible that \( c_1 \) and \( c_2 \) both intersect an internal line which is vertical in \( c_1 \cap c_2 \).
From this discussion we conclude that the only possibility is:

\[ N(c_1^+ \cup c_2^+ \cup c_1^- \cup c_2^-) = N(c_1^+ \cup c_2^+) = r. \]  

Therefore, \( c_1^+ \cup c_2^+ \) is an internal \( r \)-cut, and so we have achieved our aim of finding an internal \( r \)-cut as late or later than both \( c_1 \) and \( c_2 \).

Applying the above procedure many times allows the construction of a unique latest \( r \)-cut, which is nearest to the final state. Note that this last cut lemma applies only to \textit{internal} \( r \)-cuts on \((r-1)\)-particle irreducible diagrams, which is why we must be careful to distinguish between class \( C_2 \), in which only an internal \( r \)-cut can be made, and classes \( C_3 \), \( C_4 \) and \( C_5 \), in which \( r \)-cuts that cut at least one external line are permitted.

Given the above argument, it is clear that we could easily also prove the following result:

**Lemma (First Internal Cut)** Any \((r-1)\)-particle irreducible diagram which admits an internal \( r \)-cut has a unique internal \( r \)-cut which is nearest to the initial state.

Armed with these two lemmas we now proceed to sum the five classes \( C_1-C_5 \).

**A. Class \( C_1 \)**

The sum of all \( r \)-particle irreducible diagrams contributing to \( A_{n-m}^{(r-1)} \) is clearly \( A_{n-m}^{(r)} \), the connected \( r \)-particle irreducible \( m \rightarrow n \) amplitude. This, therefore, is the sum of class \( C_1 \).

**B. Class \( C_2 \)**

It is apparent that by using the last internal cut lemma we may express any \((r-1)\)-particle irreducible diagram, \( a_{n-m}^{(r-1)} \), which belongs to \( C_2 \) as:

\[ a_{n-r}^{(r)} G^{(r)} a_{r-m}^{(r-1)} \]  

(2.15)

where \( G^{(r)} \) is the free propagator for \( r \) fully-dressed particles, and \( a_{n-r}^{(r-1)} \) and \( a_{r-m}^{(r)} \) are two diagrams which are \((r-1)\) and \( r \)-particle irreducible respectively. Summing over all diagrams which contribute to \( A_{n-m}^{(r-1)} \) and are in \( C_2 \) then involves summing over all \((r-1)\)-particle irreducible, \( m \rightarrow n \) Feynman diagrams with the structure given in Eq. (2.15). Consequently, the sum of \( C_2 \) is:

\[ C_2 = A_{n-r}^{(r)} G^{(r)} A_{r-m}^{(r-1)} \]  

(2.16)

where, because of the way \( C_2 \) is defined, the amplitudes \( A_{n-r}^{(r)} \) and \( A_{r-m}^{(r-1)} \) must both be connected. See Figure 6 for a pictorial representation of this sum of class \( C_2 \).

**C. Class \( C_3 \)**

Now consider class \( C_3 \). We wish to take any diagram in \( C_3 \) and find a unique \( r \)-cut nearest to the final state. However, in this case the situation is complicated by the fact that
only $r$-cuts which intersect at least one line from the initial state are possible. It is therefore necessary to eliminate the external lines from consideration before applying the last internal cut lemma.

Consider any diagram contributing to $A^{(r-1)}_{n-m}$ and in $C_3$. Construct the set of all $r$-cuts which may be made upon the diagram. For each $r$-cut, define $r_i$ to be the number of lines from the initial state which that cut intersects. Then, take the minimum of $r_i$ over all possible $r$-cuts and denote the result by $t_i$. This $t_i$ is then the minimum number of lines from the initial state cut by any $r$-cut possible in this particular diagram. We call any $r$-cut which satisfies:

$$r_i = t_i$$  \tag{2.17}

a minimal $r$-cut, and we denote the set of all minimal $r$-cuts by $M_{t_i}$. It is clear that if we can construct a unique latest $r$-cut out of this set $M_{t_i}$ then, because the set contains those cuts which cut as few lines from the initial state as possible, this constructed cut will be the latest of all possible $r$-cuts in the diagram. However, to construct such a cut is difficult because the cuts in the set $M_{t_i}$ must be divided as follows. Although all cuts in $M_{t_i}$ must cut the same number of lines from the initial state they do not necessarily cut the same set of initial lines. Different $r$-cuts within $M_{t_i}$ may cut different sets of external lines $t_i$, as long as each such set $t_i$ has $t_i$ members. Therefore, the minimal $r$-cuts must themselves be divided into subsets according to which group of lines $t_i$ they cut. To this end we construct subsets of $M_{t_i}, M_{t_j}$, with each $r$-cut in the subset $M_{t_i}$ intersecting a specific set of lines from the initial state, $t_i$. Note that each subset $M_{t_i}$ still may contain many $r$-cuts. However, within any such subset $M_{t_i}$ there is always a unique latest cut, constructed as follows. Consider any $r$-cut in $M_{t_i}$, and suppose the lines it intersects form a set $s$. Now remove the lines $t_i$ from each set $s$ in $M_{t_i}$. This turns all the cuts in $M_{t_i}$ into internal $(r - t_i)$-cuts in what may now be regarded as an $(r - t_i - 1)$-particle irreducible diagram. Consequently, by the last internal cut lemma, there exists a unique last internal cut, which cuts $(r - t_i)$ lines. By joining the set of lines $t_i$ to this cut we obtain a unique latest $r$-cut, out of all the cuts in this particular subset $M_{t_i}$.

However, in principle there are many sets $t_i$ and so many different “latest” $r$-cuts will be obtained when the above procedure is applied to the various subsets $M_{t_i}$. It is not immediately clear whether it is possible to construct an overall latest minimal $r$-cut from all these different “latest” minimal $r$-cuts. One might think that two latest minimal cuts from two different subsets of $M_{t_i}$ could be taken and a cut later than either of them constructed using the procedure outlined in the proof of the last internal cut lemma. The problem is that, in this case, since the $r$-cuts involved include lines from the initial state, $c_1^+ \cup c_2^-$ may consist entirely of lines from the initial state, and so may not be a true cut at all. If this happens then the cut $c_1^+ \cup c_2^-$ need not satisfy Eq. (2.4) and so the argument by which the last internal cut lemma was proved may break down. However, if:

$$m \geq r$$  \tag{2.18}

then, even if $c_1^+ \cup c_2^-$ consists entirely of external lines it must still have at least $r$ members, since it must cut all $m$ lines from the initial state in order to be an $s$-channel cut. Consequently, if condition (2.18) is satisfied Eq. (2.4) will still hold and the last internal cut lemma argument will not fail.
This necessity that $c_{(1)} \cup c_{(2)}$ cut all $m$ lines from the initial state also imposes another pre-condition for the last internal cut lemma argument to fail to produce a unique latest cut out of all the cuts in $M_i$. Since $c_{(1)} \cup c_{(2)}$ cuts at most $2t_i$ lines from the initial state it follows that a necessary condition for it to be impossible to construct a unique last cut via this approach is:

$$m \leq 2t_i.$$  \hspace{1cm} (2.19)

Since $t_i \leq r - 1$ it follows that if the condition:

$$m > 2(r - 1) \quad \text{or} \quad m \geq r.$$  \hspace{1cm} (2.20)

is satisfied then it will definitely be possible to define a unique latest cut out of all the cuts in the set $M_i$. Since $r$ and $m$ are always positive integers this condition reduces to Eq. (2.18). We now have two possibilities:

1. $m \geq r$, in which case each diagram may be split, in a unique fashion, into an $r$-particle irreducible and an $(r - 1)$-particle irreducible part, and so a sum for $C_3$ may be constructed.

2. $m < r$, in which case we must be more careful, since a unique latest cut cannot be constructed directly.

**Case 1 ($m \geq r$):** Suppose that $m \geq r$ and consider any diagram $a_{n-m}^{(r-1)}$ which belongs in $C_3$. For this diagram we may construct the set of minimum $r$-cuts $M_i$, each of which cuts precisely $t_i$ external lines. The above argument then guarantees the existence of a unique last $r$-cut among all those cuts in $M_i$. This cut will cut a certain set of initial lines $\tilde{t}_i$. Applying the procedure described above, of first removing the lines $\tilde{t}_i$ from consideration and then applying the last internal cut lemma, we find the diagram $a_{n-m}^{(r-1)}$ may be expressed uniquely as:

$$a_{n-m}^{(r)} G^{(r/\tilde{t}_i)} \tilde{a}_{(r-t_i)-1} ^{(r-t_i)-1} (m-t_i),$$  \hspace{1cm} (2.21)

where $G^{(r/\tilde{t}_i)}$ is the free propagator for $r$ fully-dressed particles, but with the particles in the set $\tilde{t}_i$ removed, and the amplitude $\tilde{a}_{(r-t_i)-1} ^{(r-t_i)-1} (m-t_i)$ may be disconnected, as long as no part of it represents a one-to-one process. We now sum over all diagrams in $C_3$ contributing to $A_{n-m}^{(r-1)}$, but, for the present, restrict the sum to those diagrams for which the minimum number of lines from the initial state which are cut by any possible $r$-cut is $t_i$. Since this procedure involves a sum over all topologically distinct diagrams of the form (2.21), it follows that the result is:

---

$^2$In some cases a stronger condition than this can actually be derived, but pursuing this point at this stage merely complicates the argument unnecessarily. All we really require is a condition under which it will definitely be all right to construct a unique latest cut using the last internal cut lemma procedure.
\[ C_3^{t_i} = \sum_{\text{All sets } t_i} A^{(r)}_{n-m} G^{(r/t_i)} \tilde{A}^{(r-t_i-1)}_{(r-t_i)(m-t_i)}, \]  

(2.22)

where the amplitude \( \tilde{A}^{(r-t_i-1)}_{(r-t_i)(m-t_i)} \) may contain disconnected pieces, provided these pieces only represent processes in which each particle interacts with at least one other particle.

We have denoted the sum of this sub-class of \( C_3 \), which includes all diagrams in which the minimum number of lines from the initial state cut by any possible \( r \)-cut is \( t_i \), by \( C_3^{t_i} \). Clearly then, the sum in Eq. (2.22) must be restricted to those sets \( t_i \) with \( t_i \) members. Note also that terms in the sum containing any one-to-one amplitude must always be eliminated, since, due to all particles being fully dressed, we set all one-to-one amplitudes to zero. Now, diagrams in \( C_3 \) contributing to \( A^{(r)}_{n-m} \) can have any number, \( t_i \), of initial lines cut by the \( r \)-cut in question, from a minimum of \( t_i = 1 \) up to a maximum of \( t_i = r - 1 \). Therefore, it follows that if condition (2.20) is satisfied:

\[ C_3 = \sum_{t_i=1}^{r-1} C_3^{t_i}. \]  

(2.23)

**Case 2 (\( m < r \))**: Our earlier discussion made it clear that in this case the existence of a unique latest \( r \)-cut could not be guaranteed. Consequently, if \( m < r \) the class \( C_3 \) cannot be summed using the methods developed above. Taylor claims that if \( m < r \) \( C_3 \) may be summed by splitting it into subclasses \( C_3^{t_i} \), where \( C_3^{t_i} \) is defined to be the set of all diagrams belonging to class \( C_3 \) in which the minimal set of initial lines cut is \( \hat{t}_i \). So, consider any diagram in \( C_3^{t_i} \). As we saw above, once a diagram and a minimal set of lines \( \hat{t}_i \) is chosen, there exists a unique latest \( r \)-cut in that diagram, which intersects the set of lines \( \hat{t}_i \). If the procedure described above is applied, of first removing the external lines \( \hat{t}_i \) from consideration and then applying the last internal cut lemma, we find this diagram may be written exactly as in Eq. (2.21). Consequently, when we sum over all diagrams in \( C_3^{t_i} \) we obtain:

\[ C_3^{t_i} = A^{(r)}_{n-r} G^{(r/t_i)} \tilde{A}^{(r-t_i-1)}_{(r-t_i)(m-t_i)}. \]  

(2.24)

Taylor claims that by summing over all possible minimal sets \( \hat{t}_i \) one obtains the sum of all diagrams in \( C_3 \). We shall see in Section IV that this mistaken claim is precisely the origin of the double-counting problem mentioned in the Introduction. However, if we, for the present, continue on the basis of this assumption, we find that:

\[ C_3^{t_i} = \sum_{\text{All sets } t_i} A^{(r)}_{n-r} G^{(r/t_i)} \tilde{A}^{(r-t_i-1)}_{(r-t_i)(m-t_i)}, \]  

(2.25)

where the sum is restricted to those sets \( \hat{t}_i \) with \( t_i \) members. We may now make the identification of this sum over all possible minimal sets \( \hat{t}_i \) with \( t_i \) members with a sum over all possible sets of initial lines with \( t_i \) members. This is possible because we are summing over all topologically distinct diagrams and considering distinguishable particles. Consequently, if the contribution from one set of initial lines \( \hat{t}_i \) is included, the contribution from all other possible sets of initial lines with \( t_i \) members must also be included. This identification shows that, given the assumption \( C_3^{t_i} = \sum C_3^{t_i} \) (where the sum is defined to run over all minimal
sets \( \hat{t}_i \) with \( t_i \) members), the sum of \( C_{3}^{t_i} \) is exactly the same in the case \( m < r \) as in the case \( m \geq r \).

If \( m < r \), \( t_i \) may take on values from 1 to \( m - 1 \). Therefore, it follows that, if condition (2.20) is not satisfied, the sum of class \( C_3 \) may be written as:

\[
C_3 = \sum_{t_i=1}^{m-1} C_{3}^{t_i},
\]

with the sum of \( C_{3}^{t_i} \) given by Eq. (2.25).

The sum of \( C_3 \) in both of the above cases is represented diagrammatically in Figure 7. Even though we have not been able to find a single unique latest cut in all diagrams in \( C_3 \), the argument given above appears to show that we may express \( C_3 \) as the sum of a number of terms, in each of which there is a different unique latest cut.

**D. Class \( C_4 \)**

The argument used above to construct the sum of class \( C_3 \) is very similar to that used to find the sum of \( C_4 \). Consider any diagram in \( C_4 \), and consider any \( r \)-cut which can be made on that diagram. For this \( r \)-cut, \( r_i \) is defined as above, and \( r_f \) is defined to be the number of lines from the final state which the cut intersects. Then, once more, \( t_i \) is defined to be the minimum of \( r_i \), with the minimum taken over all possible \( r \)-cuts, and \( s_f \) is defined to be the maximum of \( r_f \), with the maximum taken over all \( r \)-cuts satisfying \( r_i = t_i \). This defines a set of \( r \)-cuts, known as minimal/maximal \( r \)-cuts, which all obey the condition:

\[
\begin{align*}
    r_i &= t_i \\
    r_f &= s_f.
\end{align*}
\]  

We know that this set contains all the candidates for “latest \( r \)-cut”. But, among all the cuts in this set, is there a unique latest \( r \)-cut, and, if so, how is it to be constructed? Within this set of minimal/maximal \( r \)-cuts, different \( r \)-cuts may cut several different sets of external lines \( \hat{t}_i \) and \( \hat{s}_f \), as long as each such pair of sets \( \hat{t}_i \) and \( \hat{s}_f \) have \( t_i \) and \( s_f \) members respectively. Once we restrict the \( r \)-cuts under consideration to those intersecting a specific pair of sets \( \hat{t}_i \) and \( \hat{s}_f \), a unique latest \( r \)-cut may be constructed, by again using a similar argument to that used in proving the last internal cut lemma. But, if the type of argument used in the proof of the last internal cut lemma can break down it will not necessarily be possible to construct a unique overall latest \( r \)-cut from all of these latest minimal/maximal cuts. In the case under consideration here the argument used in the proof of the last internal cut lemma may break down in two ways:

1. If \( c_1^- \cup c_2^- \) contains only lines from the initial state then \( c_1^- \cup c_2^- \) need not satisfy condition (2.4).

2. If \( c_1^+ \cup c_2^+ \) contains only lines from the final state and so the constructed latest “cut” is not a cut at all!

We saw above that the first possibility did not lead to the breakdown of the argument used in the proof of the last internal cut lemma, provided that condition (2.20) was satisfied. Now, the second possibility definitely cannot arise if:
\[ n > 2(r - 1), \]

since \( c_1^+ \cup c_2^+ \) contains at most \( 2s_f \) lines from the final state, and \( s_f \leq r - 1 \). A further condition for the second possibility to occur may also be derived, as follows. We begin by replacing the equation \( N(c_1^- \cup c_2^-) \geq r \), used in the proof of the last internal cut lemma, by:

\[ N(c_1^- \cup c_2^-) \geq \begin{cases} m & \text{if } m < r \\ r & \text{if } m \geq r, \end{cases} \]

since \( c_1^- \cup c_2^- \) may now consist entirely of lines from the initial state. Combining this result with Eq. (2.12) then gives:

\[ N(c_1^+ \cup c_2^+ \geq 2r - \min\{m, r\}. \]

It follows that if \( c_1^+ \cup c_2^+ \) is going to consist entirely of lines from the final state, and so invalidate the use of the last internal cut lemma argument, the condition:

\[ n \leq 2r - \min\{m, r\}, \]

must be satisfied. Consequently the condition under which the second of the two above possibilities becomes forbidden is:

\[ n > 2(r - 1) \quad \text{or} \quad n + \min\{m, r\} > 2r. \]

Therefore the argument used in the proof of the last internal cut lemma may definitely be used to construct a unique latest cut out of all the minimal/maximal \( r \)-cuts if:

\[ m \geq r \quad \text{and} \quad n > r. \]

(\text{Note that here we have used the fact that } r \text{ must be a positive integer greater than one in order to determine that } r \leq 2(r - 1).\) Condition (2.33) is, in fact, a less stringent condition for the success of the last internal cut lemma argument than the condition which was used by Taylor:

\[ n, m > 2(r - 1). \]

We now consider two cases:

\textbf{Case 1 (}m \geq r \text{ and } n > r\): If condition (2.33) is satisfied then similar arguments to those used in the summation of \( C_5 \) show that the sum of all diagrams in \( C_4 \) in which the minimal/maximal \( r \)-cut intersects \( s_f \) lines from the final state and \( t_i \) lines from the initial state is:

\[ C_4^{s_f t_i} = \sum_{\text{All sets } \hat{s}_f \& \hat{t}_i} A_{(n-s_f)-(r-s_f)}^{(r-s_f)} \prod_{(r-t_i-1)}^{(r-t_i-1)} A_{(r-t_i-1)-(m-t_i)}. \]

where the sum is constrained to be only over those sets \( \hat{s}_f \) and \( \hat{t}_i \) containing \( s_f \) and \( t_i \) members respectively, and all one-to-one amplitudes are to be set to zero. Here, \( G_{(r/\hat{s}_f \& \hat{t}_i)}^{(r-t_i-1)} \) is the free propagator for \( r \) fully-dressed particles, but with any particle which is in either
of the two sets \( \tilde{t}_i \) and \( \tilde{s}_f \) removed. It follows that the sum of \( C_4 \) can be constructed by summing over all possible values of \( s_f \) and \( t_i \), yielding:

\[
C_4 = \sum_{s_f=1}^{r-1} \sum_{t_i=1}^{r-1} C_4^{s_f t_i},
\]

(2.36)

where the sum over \( s_f \) and \( t_i \) is restricted to those \( s_f \) and \( t_i \) which obey:

\[
s_f + t_i \leq r - 1.
\]

(2.37)

**Case 2** (\( n \leq r \) or \( m < r \)): On the other hand, if condition (2.33) is not satisfied, we may (Taylor claims) still sum \( C_4 \) by splitting it into sub-classes \( C_4^{\tilde{s}_f \tilde{t}_i} \). Here \( C_4^{\tilde{s}_f \tilde{t}_i} \) is defined to be the set of all diagrams belonging to \( C_4 \) for which the minimal/maximal \( r \)-cut intersects the lines \( \tilde{t}_i \) from the initial state and the lines \( \tilde{s}_f \) from the final state. Once again, similar arguments to the above show that when all contributions to \( C_4^{\tilde{s}_f \tilde{t}_i} \) are summed:

\[
C_4^{\tilde{s}_f \tilde{t}_i} = A_{(n-s_f)}^{(r-s_f)} G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} A_{(r-t_i)-(m-t_i)}^{(r-t_i-1)};
\]

(2.38)

from which Taylor obtains:

\[
C_4^{s_f t_i} = \sum_{\text{All sets } \tilde{s}_f \text{ & } \tilde{t}_i} A_{(n-s_f)}^{(r-s_f)} G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} A_{(r-t_i)-(m-t_i)}^{(r-t_i-1)};
\]

(2.39)

where the sum is restricted to those sets \( \tilde{s}_f \) and \( \tilde{t}_i \) which contain, respectively, \( s_f \) and \( t_i \) members. Again, we note that the following facts about this result:

1. We question Taylor’s moving from Eq. (2.38) to Eq. (2.39), for the reasons explained in Section IV.

2. The sum over \( \tilde{s}_f \) and \( \tilde{t}_i \) is restricted to those sets \( \tilde{s}_f \) and \( \tilde{t}_i \) with \( s_f \) and \( t_i \) members respectively. As above, it can be shown that under these circumstances the sum is, in fact, a sum over all possible subsets of the sets of initial and final lines, but restricted to those subsets which have, respectively, \( t_i \) and \( s_f \) members. Consequently, Eq. (2.39) is the same equation as Eq. (2.35).

3. All one-to-one amplitudes in the sum must be set to zero.

4. The amplitude \( A_{(r-t_i)-(m-t_i)}^{(r-t_i-1)} \) is understood to be disconnected in the same sense discussed above.

The possible values of \( s_f \) and \( t_i \) may then be summed over in order to yield:

\[
C_4 = \sum_{t_i=1}^{\min\{m,r\}-1} \sum_{s_f=1}^{\min\{n,r\}-1} C_4^{s_f t_i},
\]

(2.40)

where, once again, the sums are restricted to \( t_i + s_f \leq r - 1 \). This result in fact encompasses Eq. (2.36), which applies only to the case \( r \leq m \) and \( r < n \). For a diagrammatic representation of the sum of class \( C_4 \) see Figure 8.
E. Class $C_5$

The method for summing classes $C_3$ and $C_4$ is very similar to that used in order to sum class $C_5$. Consider any diagram in class $C_5$ and consider any particular $r$-cut which can be made on that diagram. Define $r_f$ to be the number of lines from the final state cut by that $r$-cut. The maximum of $r_f$ over all possible $r$-cuts is taken and is defined to be $s_f$. The set of $r$-cuts satisfying:

$$r_f = s_f,$$  \hspace{1cm} (2.41)

is defined to be $M_{s_f}$, the set of maximal $r$-cuts. Once again, a unique latest cut may be extracted from this set of $r$-cuts, provided that the argument used in the proof of the last internal cut lemma is applicable. In the previous section we explained two ways in which this argument might break down when applied to a diagram in class $C_4$. When the argument is applied to a diagram in class $C_5$ it cannot break down in the first of these two ways, since, in this case, it is certain that $c_1^+ \cup c_2^+$ contains only internal lines. Therefore, the only way the last internal cut lemma argument can fail when applied to a diagram in $C_5$ is if $c_1^+ \cup c_2^+$ contains only lines from the final state. Arguing as we did above shows that $c_1^+ \cup c_2^+$ cannot contain only lines from the final state if:

$$n > 2(r - 1) \quad \text{or} \quad n > r.$$ \hspace{1cm} (2.42)

Since $n$ and $r$ are positive integers and $r > 1$, the condition for there to definitely be a unique latest cut among all the cuts in the set $M_{s_f}$ is therefore found to be:

$$n > r.$$ \hspace{1cm} (2.43)

We note that condition (2.43) is slightly different from the condition used by Taylor. He stated that the condition for the generation of a unique last cut in the set $M_{s_f}$ was $n \leq r$. However, the above discussion shows that the argument used in the proof of the last internal cut lemma may well also fail to generate a unique last cut if $n = r$.

Once more, we now consider two cases:

**Case 1** ($n > r$): If condition (2.43) holds then similar arguments to those used above may be employed in order to show that:

$$C_5 = \sum_{s_f=1}^{r-1} C_5^{s_f};$$ \hspace{1cm} (2.44)

with:

$$C_5^{s_f} = \sum_{\text{All sets } \tilde{s}_f} A_{(r-s_f)}^{(r-s_f)} \cdot A_{(n-s_f)-(r-s_f)}^{(r-s_f)} \cdot A_{r-m}^{(r-1)},$$ \hspace{1cm} (2.45)

where the sum is restricted to those sets $\tilde{s}_f$ with $s_f$ members, all one-to-one amplitudes are to be set to zero and $A_{r-m}^{(r-1)}$ contains both a connected and disconnected part, but the disconnected part only includes diagrams in which each particle interacts with at least one other particle.
Case 2 \((n \leq r)\): If condition (2.43) is violated \(C_5\) is split into sub-classes, \(C_i^{\tilde{s}_f}\), each of which contains all those diagrams whose maximal \(r\)-cut intersects the set of lines \(\tilde{s}_f\) from the final state. When \(C_i^{\tilde{s}_f}\) is summed its sum is found to be:

\[
C_i^{\tilde{s}_f} = A_{(n-s_f)-r(\tilde{s}_f)}^{(r-s_f)} \times \left( A_{n-r}^{(r)} G_{r-m}^{(r)} \right) \times \mathcal{A}_{r-m}^{(r-1)}.
\] 

By summing over all possible sets \(\tilde{s}_f\) with \(s_f\) members Taylor obtains Eq. (2.45) for \(C_5^{\tilde{s}_f}\), with the same comments which applied to that result still applying here. Note that, as we did for \(C_3\) and \(C_4\), we question this last step for the reasons described below. However, if this step is accepted, summing over \(s_f\) gives:

\[
C_5 = \sum_{s_f=1}^{m-1} C_5^{\tilde{s}_f}.
\] 

For a diagrammatic representation of the sum of class \(C_5\), see Figure 9.

F. Overall result

This achieves our original aim of finding expressions for each of the classes \(C_1\) to \(C_5\). If we now sum the results of our summation of each of the individual classes we find:

\[
A_{n-m}^{(r-1)} = A_{n-m}^{(r)} + A_{n-r}^{(r)} G_{r-m}^{(r)} A_{r-m}^{(r-1)}
\]

\[
+ \sum_{t_i=1}^{\min(n,r)-1} \sum_{\text{all sets } t_i} A_{n-r}^{(r)} G_{r}^{(r)} \left[ G_{r-t_i}^{(r-t_i-1)} \mathcal{A}_{r-t_i}^{(r-t_i-1)} \right]
\]

\[
+ \sum_{s_f=1}^{\min(n,r)-1} \sum_{\text{all sets } \tilde{s}_f} \left\{ \left[ A_{(n-s_f)}^{(r-s_f)} \times G_{r-m}^{(r)} \right] \times \left[ G_{r-t_i}^{(r-t_i-1)} \mathcal{A}_{r-t_i}^{(r-t_i-1)} \right] \right\}^{(c)}
\]

\[
+ \sum_{s_f=1}^{\min(n,r)-1} \sum_{\text{all sets } \tilde{s}_f} \left\{ \left[ A_{(n-s_f)}^{(r-s_f)} \times G_{r-m}^{(r)} \right] \times \mathcal{A}_{r-m}^{(r-1)} \right\}^{(c)}
\] 

(2.48)

where the superscript \((c)\) on the fourth and fifth terms indicates that only connected diagrams may be formed. Note that the five terms in this equation are each generated by a different Taylor class, with the \(n\)th term generated by \(C_n\), where \(n = 1 \ldots 5\).

We observe that in Eq. (2.48) we have expressed \(A_{n-m}^{(r-1)}\) in terms of amplitudes of equal or greater irreducibility and amplitudes involving fewer particles. We have done this without making any assumption about the structure of the underlying field theory, other than the fact that the theory has a perturbation expansion in terms of Feynman diagrams. This model-independence is what makes the Taylor method so powerful and useful.

Indeed, as was mentioned above, the Taylor method is valid even if no underlying field theory exists at all. The only pre-requisite for an application of the Taylor method is the presence of a diagrammatic expansion. Therefore, the Taylor method may be applied to a system of \(N\) particles, in order to derive equations for the \(N\)-particle amplitudes in terms of the \(n\)-particle amplitudes, where \(n < N\). When used in a three-particle system such a procedure results in Faddeev-type equations, and when it is used in a four-particle
system this procedure will lead to Yakubovsky-type equations. Thus, not only is the Taylor method an extremely valuable model-independent technique within field theory, but it is also applicable outside field theory too.

III. THE RELATIONSHIP OF TAYLOR’S ORIGINAL METHOD TO THE TRAB SIMPLIFICATION

In the last section we reviewed the Taylor method and showed how it allows us to write the amplitude $A^{(r-1)}_{n-m}$ in terms of amplitudes of equal or greater irreducibility and amplitudes involving fewer particles. However, since 1979, a simpler version of the Taylor method has also been used. This simplification was first developed for a time-ordered perturbation theory by Thomas and Rinat [10,11]. It was then applied by Afnan and Blankleider to the $NN-\pi NN$ system [12,13], by Afnan and Pearce to the $\pi N - \pi \pi N$ system [4,5], and by Afnan and Araki to the problem of pion photoproduction on the nucleon and deuteron [6,7]. In this paper we refer to this simplified method as the TRAB method, and we begin this section by reviewing the method. It has been suggested, most notably by Avishai and Mizutani [9], that, when applied to a time-dependent perturbation theory, the TRAB method of classifying diagrams leads to results different from those obtained using Taylor’s original method. In order to establish the exact conditions under which the TRAB and Taylor methods are equivalent we examine the expression obtained for $A^{(r-1)}_{n-m}$ in the previous section and compare it to that obtained in this section from the TRAB method.

The TRAB method is a simplification of the Taylor method which was explicitly designed only to apply to a time-ordered perturbation theory without anti-nucleons [11]. In the TRAB method the definition of an $r$-cut and an $r$-particle irreducible diagram are exactly those given for Taylor’s method in the previous section, but with the restriction that, since TRAB deal only with time-ordered diagrams, cuts can only be vertical lines separating the initial and final states. Once cuts are restricted to vertical lines the last-cut lemma is trivial to prove, and many of the restrictions imposed in Taylor’s work in order to guarantee the existence of a unique last cut become unnecessary. The last-cut lemma in the TRAB method may therefore be stated as:

Lemma (TRAB last-cut) There exists a unique latest $r$-cut in any time-ordered perturbation theory diagram whose irreducibility $k$ is less than $r$.

Of course, strictly speaking, this version of the last-cut lemma is only valid in time-ordered perturbation theory, but this new last-cut lemma is much easier to use than the older, more general, Taylor version. In order to use the TRAB last-cut lemma to find an equation for the connected amplitude, $A^{(r-1)}_{n-m}$, we merely observe that all diagrams contributing to this amplitude must be either $r$-particle irreducible or $r$-particle reducible. The sum of the diagrams in the first group is clearly the fully-connected $r$-particle irreducible amplitude $A^{(r)}_{n-m}$. The sum of the diagrams in the second group is, by the last-cut lemma:

$$\left[ A^{(r)}_{n-m} G(r) \tilde{A}^{(r-1)}_{n-m} \right]^{(c)},$$

where the amplitude $\tilde{A}$ may contain both connected and disconnected pieces, and the superscript $(c)$ indicates that the overall diagram must be connected. Note that the disconnected
part of the amplitude $\tilde{A}$ may contain diagrams in which one or more particles merely propagate freely. Putting the sums of the two groups of diagrams together implies that the TRAB method gives the following equation for $A^{(r-1)}_{n-m}$:

$$A^{(r-1)}_{n-m} = A^{(r)}_{n-m} + \left[ A^{(r)}_{n-r} G^{(r)} A^{(r-1)}_{r-m} \right]^{(c)}.$$ 

That is:

$$A^{(r-1)}_{n-m} = A^{(r)}_{n-m} + \left\{ A^{(r)}_{n-r} + \tilde{A}^{(r)(d)} \right\} G^{(r)} \left[ A^{(r-1)}_{r-m} + \tilde{A}^{(r-1)(d)} \right]^{(c)}.$$ 

where the amplitudes $A$ are connected while the amplitudes $\tilde{A}^{(d)}$ are disconnected. As mentioned above, these disconnected amplitudes may contain terms in which one or more particles merely propagate freely while the others interact.

This technique was applied by Afnan and Blankleider to the $NN - \pi NN$ and $BB - \pi BB$ problems in what was, apparently, a covariant approach [12,13]. This would appear to be incorrect since the TRAB approach was originally designed to be applied only to a time-ordered perturbation theory without anti-nucleons. The remarkable thing is that Afnan and Blankleider’s application of the TRAB technique produced exactly the same equations for the $NN - \pi NN$ system as those obtained by Avishai and Mizutani using the full Taylor method [8,9]. Avishai and Mizutani suggested that this coincidence of equations required investigation. Here we explain that the TRAB method gives exactly the same results as the full Taylor calculation, provided that we consider only amplitudes $A^{(r-1)}_{n-m}$ with $n, m$ and $r$ less than or equal to three.

Eq. (3.3) is to be compared to the equivalent equation obtained from the full Taylor method, Eq. (2.48). Now, Eq. (2.48) may be simplified in order to obtain:

$$A^{(r-1)}_{n-m} = A^{(r)}_{n-m} + \left\{ A^{(r)}_{n-r} + \sum_{s_f=1}^{\min\{n,r\}-1} \sum_{\text{sets } \tilde{s}_f} A^{(r-s_f)}_{(n-s_f) - (r-s_f)} G^{(r)}_{\tilde{s}_f} \right\} G^{(r)} \left[ A^{(r-1)}_{r-m} + \sum_{t_i=1}^{\min\{m,r\}-1} \sum_{\text{sets } \tilde{t}_i} G^{(-1)}_{\tilde{t}_i} \tilde{A}^{(r-1)(d)}_{(r-t_i) - (m-t_i)} \right]^{(c)}$$

where, as discussed above, the amplitude $\tilde{A}$ has both a connected part, $A$, and a disconnected part, $\tilde{A}^{(d)}$, with this disconnected portion only containing those disconnected diagrams in which each particle interacts with at least one other external particle, i.e. containing no processes in which one or more particles propagate freely while the other particles interact.

However, we know, by definition, that:

$$\tilde{A}^{(r-1)(d)}_{r-m} = \tilde{A}^{(r-1)(d)}_{r-m} + \sum_{t_i=1}^{\min\{m,r\}-1} \sum_{\text{sets } \tilde{t}_i} G^{(-1)}_{\tilde{t}_i} A^{(r-t_i-1)}_{(r-t_i) - (m-t_i)}$$

and:

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\[ A_n^{(r)} = A_n^{(r)} + \sum_{r_j = 1}^{\min(n, r)} \sum_{\text{all } s \neq r_j} [A_{(n-s_j)}^{(r-s_j)} G_{s_j}^{-1}]. \]  

(3.6)

Therefore it follows that Eqs. (2.48) and (3.3) will give the same result for \( A_{n-m}^{(r-1)} \) if and only if:

\[ \tilde{A}_n^{(r)} = 0 \quad \text{and} \quad \tilde{A}_n^{(r-1)} = 0 \quad \text{for all } t_i = 0, 1, \ldots, \min\{m, r\} - 1. \]  

(3.7)

If \( m, r \) and \( n \) are all no bigger than three then there is no way in which a disconnected diagram can be produced without at least one particle undergoing free propagation, since if every particle interacts with at least one other particle a connected diagram must result. Consequently, if \( m, r \) and \( n \) are all less than or equal to three the condition (3.7) is automatically satisfied. If, on the other hand, any of \( m, r \) and \( n \) are bigger than three then at least one of the \( A^{(d)} \)'s in condition (3.7) may be non-zero, in which case the expressions for \( A_{n-m}^{(r-1)} \) gleaned from the two methods will not be equivalent. Therefore, condition (3.7) is satisfied independent of the model under consideration if and only if \( m, r \) and \( n \) are less than or equal to three. If any of \( m, r \) or \( n \) is greater than 3 then the TRAB method overcounts certain diagrams, due to its direct application of the last-cut lemma to time-dependent perturbation theory cuts which intersect external lines. In other words, condition (3.7) is satisfied automatically just for calculations in which only two and three-body unitarity are taken into account in deriving the equations.

This result verifies the original suspicions of Avishai and Mizutani: that the TRAB method may lead to erroneous conclusions in certain cases. However, in the \( NN - \pi NN \) problem \( m \) and \( n \) are clearly less than or equal to three, while Avishai and Mizutani's and Afnan and Blankleider's work only considered the two and three-particle unitarity cut structure of the amplitudes involved, so \( r \leq 3 \) too. Therefore, the above result also explains why the TRAB method gives the same \( NN - \pi NN \) equations as the full Taylor method, even when it is (erroneously) applied in time-dependent perturbation theory.

Consequently, in general, care must be exercised when applying the TRAB method to the classification of diagrams. But, for the special case of the derivation of the \( NN - \pi NN \) equations (and indeed in the case of the work done by Afnan and Pearce on the \( \pi N - \pi \pi N \) system), the only problem with the TRAB method is a general one with any method based on Taylor's original work: all such methods contain the possibility of double-counting, as mentioned in the Introduction. We now discuss the flaw in Taylor's method which allows this double-counting to occur.

**IV. THE DOUBLE-COUNTING PROBLEM IN THE TAYLOR METHOD**

Taylor's classification-of-diagrams technique was developed in an effort to sum the diagrammatic expansion obtained for some Green's function or amplitude in a perturbation theory, counting each topologically distinct diagram once and only once. In this section we first give two examples which show that, in a time-dependent perturbation theory, such as covariant perturbation theory, the method fails to achieve this aim. I.e., when the Taylor method is used to sum certain series of time-dependent perturbation theory diagrams some diagrams in the series are counted twice. The flaw in Taylor's argument which leads to this
double-counting is then explained, and it is shown how that mistake leads directly to the two examples of double-counting given.

A. Two examples of double-counting in the Taylor method

1. Double-counting in pion absorption on the deuteron

The first example we examine is the double-counting of certain covariant perturbation theory diagrams in theories of pion absorption on the deuteron, a problem first pointed out by Kowalski et al. [14]. Consider two distinguishable nucleons, which in the initial state are labeled $N_1$ and $N_2$ and in the final state are labeled $N_1'$ and $N_2'$, and suppose that also present in the initial state is a pion, which we label simply $\pi$. We call the $3 \to 2$ amplitude for this process $F$. Consider, in particular, the two-particle irreducible part of $F$, $F^{(2)}$. Applying Taylor’s method, or the TRAB simplification of it (for this amplitude the two methods are equivalent) to $F^{(2)}$ gives the following equation:

$$F^{(2)} = F^{(3)} + \left\{ \left[ F^{(3)} + \sum_{j=1,2} f^{(2)}(j)d_j^{-1}\right] d_1d_2d_\pi \left[ M^{(2)} + \sum_{i=1,2} t_{\pi N}^{(1)}(i)d_i^{-1} + T_{NN}^{(1)}d_\pi^{-1}\right] \right\}^{(c)},$$

where $M$ is the connected three-to-three amplitude, $t_{\pi N}(i)$ is the two-body $\pi N$ t-matrix with nucleon $i$ as a spectator, $T_{NN}$ is the two-body $NN$ t-matrix, $f(j)$ is the $\pi NN$ absorption vertex with particle $N_j'$ as a spectator and $d_1$, $d_2$ and $d_\pi$ are the single-particle fully-dressed free propagators for nucleon 1, nucleon 2 and the pion. Note that the irreducibilities of all amplitudes are indicated by the bracketed superscript. Note also the use of spectator notation to show the particles involved in the two-body interaction. Eq. (4.1) is presented pictorially in Figure 10, which includes an indication of the Taylor class or sub-class that produced each term.

Consider, for the moment, only the product of the two-body t-matrices and the $\pi NN$ vertices. These give a contribution:

$$\sum_{i=1,2} f^{(2)}(i)d_\pi f_{NN}^{(1)} + \sum_{i,j=1,2} \delta_{ij} f^{(2)}(j)d_\pi t_{\pi N}^{(1)}(i)$$

(4.2)

to $F^{(2)}$. Here $\tilde{i}$ is defined by:

$$\tilde{i} = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}.$$  

(4.3)

We shall see in the next example that the one-particle irreducible $NN$ t-matrix contains a term representing one-pion exchange with undressed vertices:

$$\sum_{ij} \tilde{\delta}_{ij} f^{(2)}(j)d_\pi t^{(2)}(i).$$

(4.4)

(This term is depicted in Figure 11.) It can also be shown that the one-particle irreducible $\pi N$ t-matrix $t_{\pi N}^{(1)}(i)$ contains a crossed term:

$$\sum_{ij} \tilde{\delta}_{ij} f^{(2)}(j)d_\pi t^{(2)}(i).$$
(See Figure 12.) If these portions of the \( \NN \) and \( \pi N \) t-matrices are substituted into the expression (4.2), and the results treated in a time-dependent perturbation theory, then both terms in Eq. (4.2) produce the diagram in Figure 13. That diagram is double-counted. So, Taylor’s method breaks down in this example: at least one diagram is counted twice.

Note that some care must be exercised if the cut-structure of diagrams in channels other than the \( s \)-channel is to be considered later. We should really indicate that at this stage we are only talking about the \( s \)-channel cut structure of the amplitudes involved. However, making the channel under consideration explicit does not change the overall result: the diagram in Figure 14 is still obtained from both of the terms in the expression (4.2). Therefore, this diagram is definitely double-counted in time-dependent perturbation theory calculations in the Taylor method.

Note also that in time-ordered perturbation theory there is no double-counting since the terms involving the \( \NN \) and \( \pi N \) t-matrix contribute to different time-orders (compare Figure 4, where the vertices are now known to be two-particle irreducible in the \( s \)-channel, with Figure 13). It is only in a time-dependent perturbation theory, such as covariant perturbation theory, that the diagram in Figure 4 becomes equal to the diagram of Figure 13, and the double-counting problem arises.

2. Double-counting in the calculation of one-pion exchange

Another example of double-counting is provided by the calculation of the one-pion exchange potential in the two-to-two amplitude. Again, suppose we have two distinguishable nucleons, which in the initial state are labeled \( N1 \) and \( N2 \) and in the final state are labeled \( N1' \) and \( N2' \). Both Taylor’s method and the TRAB method give the following result for the two-particle irreducible two-to-two amplitude, \( T^{(2)} \):

\[
T^{(2)} = T^{(3)} + \left\{ \left[ F^{(3)} + \sum_{j=1,2} f^{(2)}(j) d_j^{-1} \right] d_1 d_2 d_\pi \left[ F^{(2)\dagger} + \sum_{i=1,2} f^{(1)\dagger}(i) d_i^{-1} \right] \right\}^{(c)},
\]

where all symbols are as defined above, with \( T^{(r)} \) the \( r \)-particle irreducible two-to-two amplitude and \( F^{(2)\dagger} \) the two-particle irreducible two-to-three amplitude. All the terms are indicated diagrammatically in Figure 15, along with the class (or sub-class) which produced them in the Taylor approach.

Now consider only the terms involving a product of \( \pi NN \) vertices. These give a contribution:

\[
\sum_{i,j=1,2} \delta_{ij} f^{(2)}(j) d_\pi f^{(1)\dagger}(i)
\]

(4.7)

to \( T^{(2)} \). The relationship:

\[
f^{(1)\dagger} = f^{(2)\dagger} + t^{(1)} d_\pi d_N f^{(2)\dagger}
\]

(4.8)
shows that the sum in Eq. (4.7) contains both \( f^{(2)}(2) d_{\pi} f^{(2)^{\dagger}}(1) \) and \( f^{(2)}(1) d_{\pi} f^{(2)^{\dagger}}(2) \). But, in any time-dependent perturbation theory, and in particular in covariant perturbation theory, these two diagrams are identical.

Again, some caution is required here regarding precisely which channel the cuts under consideration are made in. If we include the fact that so far we have only considered s-channel cuts, we find that the two diagrams, \( f^{(2)}(2) d_{\pi} f^{(2)^{\dagger}}(1) \) and \( f^{(2)}(1) d_{\pi} f^{(2)^{\dagger}}(2) \), are, in fact, slightly different. (See Figure 16.) However, we observe that if we consider only the portion of the vertex which is two-particle irreducible in all channels, then it is clear that the diagram in Figure 11 is double-counted, if the vertices are now regarded as being two-particle irreducible in all channels. So, once again, Taylor's method leads to at least one instance of double-counting.

### B. Why does this double-counting occur?

These two examples show clearly that double-counting does arise when Taylor's method is applied to a time-dependent perturbation theory. The next question is: why does it arise?

We begin to answer this question by noting that, in both the cases discussed above, the double-counting problem occurs in Taylor's class \( C_4 \). Therefore we now examine the argument used by Taylor in his attempt to sum \( C_4 \).

Observe that if the condition:

\[
m < r \quad \text{or} \quad n \leq r
\]  

(4.9)

holds, as it does for each of the two examples above, then Taylor sums \( C_4 \) by constructing the sum of the sub-classes \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \) and applying:

\[
C_4 = \sum_{\text{All sets } \tilde{s}_f \text{ & } \tilde{i}_i} C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i}.
\]  

(4.10)

(Note that here, in contrast to Eq. (2.39), the sum over the sets \( \tilde{s}_f \) and \( \tilde{i}_i \) is unrestricted.) However, what Taylor should be trying to construct is the union of all sub-classes \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \), not the sum. That is, the correct formula is:

\[
C_4 = \bigcup_{\text{All sets } \tilde{s}_f \text{ & } \tilde{i}_i} C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i}.
\]  

(4.11)

The operation of summation used in Eq. (4.10) is different to that of set union, since a diagram which is a member of two different subclasses \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \) and \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \) is included twice in such a summation, whereas it is included only once in a set union. Consequently, the identification:

\[
\bigcup_{\text{All sets } \tilde{s}_f \text{ & } \tilde{i}_i} C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} = \sum_{\text{All sets } \tilde{s}_f \text{ & } \tilde{i}_i} C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i}
\]  

(4.12)

may be made if and only if all the sets \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \) are disjoint. This means that if condition (4.9) holds then Taylor's method will produce the correct result for \( C_4 \) if and only if all the sub-classes \( C_{\tilde{s}_f \tilde{i}_i}^{\tilde{s}_j \tilde{i}_i} \) are disjoint. If condition (4.9) holds and the sub-classes are not disjoint then
any diagram which is a member of more than one sub-class will be double-counted. Similar results hold for $C_3$ and $C_5$: if the condition (2.20) is violated then $C_3$ will not be summed correctly unless the sub-classes $C_3^{\tilde{s}_j}$ are disjoint, and if condition (2.42) is not satisfied and the sub-classes $C_5^{\tilde{s}_j}$ are not disjoint then $C_5$ will not be summed correctly.

Suppose then that condition (4.9) holds. What justification is there for assuming that these sub-classes are disjoint? Very little, since, as was mentioned above in Section II, one diagram may have many different latest minimal/maximal $r$-cuts which cut different minimal and maximal sets of external lines $\tilde{s}_f$ and $\tilde{t}_i$. Such a diagram will, by the definition of the sub-classes, belong to many of the sets $C_4^{\tilde{s}_f \tilde{t}_i}$. Therefore, the different $C_4^{\tilde{s}_f \tilde{t}_i}$ for a fixed $s_f$ and $t_i$ are not necessarily disjoint and so it is not necessarily true that:

$$C_4^{s_f t_i} = \sum_{\text{All sets } \tilde{s}_f \& \tilde{t}_i} C_4^{\tilde{s}_f \tilde{t}_i}, \quad (4.13)$$

where the sum is now restricted to those sets $\tilde{s}_f$ and $\tilde{t}_i$ with $s_f$ and $t_i$ members respectively.

However, it is true that the sub-classes $C_4^{s_f t_i}$ (which are defined to be the set of all $C_4$ diagrams whose minimal/maximal $r$-cut intersects $s_f$ final and $t_i$ initial lines) are disjoint, and so:

$$C_4 = \sum_{s_f t_i} C_4^{s_f t_i}, \quad (4.14)$$

is the right formula for constructing the sum of class $C_4$, once the correct sums of the sub-classes $C_4^{s_f t_i}$ are known.

The two double-counting problems above provide perfect examples of this difficulty. Consider the first example. For the diagram which is double-counted in this example we have $s_f = t_i = 1$ (see Figure 17). But, there are four possible pairs of sets $\tilde{s}_f$ and $\tilde{t}_i$. Two of these four pairs are:

$$\tilde{s}_f = \{N1\}; \tilde{t}_i = \{N2\}, \quad (4.15)$$

$$\tilde{s}_f = \{N2\}; \tilde{t}_i = \{\pi\}. \quad (4.16)$$

For each pair of sets we may construct a unique latest three-cut, as shown in Figure 17. However, when we attempt to construct the overall unique latest three-cut via the technique used in the last internal cut lemma, we fail because the constructed “latest” cut, $c_1^+ \cup c_2^+$, does not constitute a cut at all, since neither of the lines $N1'$ or $N2'$ is an internal line. As explained above, it was in order to circumvent this difficulty in the construction of a unique latest $r$-cut that Taylor constructed the sum of the individual sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$ and then summed over all possible sub-classes. However, diagrams such as Figure 17 belong to more than one sub-class (in this case $C_4^{\{N1\}\{N2\}}$ and $C_4^{\{N2\}\{\pi\}}$) and so are counted twice in such a summation over sub-classes. Consequently, in this case, Taylor’s method does not accurately sum class $C_4$.

Similarly in the second example Figure 18 shows that the double-counted diagram belongs to both $C_4^{\{N1\}\{N2\}}$ and $C_4^{\{N2\}\{N1\}}$ and so is double-counted when the sum of $C_4$ is constructed by the methods advocated by Taylor. Again, Figure 18 shows the impossibility of constructing a unique last three-cut in this situation, but the problem is not, as is claimed
by Taylor, solved by summing over the sub-classes $C_4^j i_i$, since that summation merely leads
to double-counting of this diagram, and, indeed, of any other diagram which belongs to more
than one sub-class.

V. SOLVING THE DOUBLE-COUNTING PROBLEM IN THE TAYLOR
METHOD

Having discovered this problem with the Taylor method we now attempt to solve it. In
this section we construct a systematic, topological, solution to the double-counting problem.

Recall that double-counting only occurs as a result of having to construct the sum of
individual non-disjoint sub-classes and then sum over all possible such sub-classes. Conse-
quently, as mentioned above, the problem can only occur if conditions (2.20) and/or (2.42)
are violated. Therefore, if condition (2.20) is satisfied there is no double-counting in class
$C_3$, if condition (2.42) is satisfied there is no double-counting in class $C_5$ and if both condi-
tions are obeyed then the Taylor method contains no double-counting for the amplitude in
question.

If, however, one or both of these conditions break down, then there will be double-
counting in one or more of the classes $C_3$, $C_4$ and $C_5$. This is to be eliminated as follows.
Consider any class $C$ containing double-counting. We know that the sum of any sub-class
$S$ of $C$, which consists of all diagrams in which the latest $r$-cut intersects a certain set
of external lines, is correctly constructed in the Taylor method. Double-counting arises not
because these sub-classes are incorrectly summed, but because certain diagrams are included
in more than one sub-class. Therefore, the key to solving double-counting is to discover
precisely which diagrams are included in more than one sub-class. Each such diagram may
then be removed from all but one of the sub-classes in which it is included. Consequently,
the sum of $C$ may be constructed by the following procedure:

1. Choose any one of the sub-classes of $C$ and place its sum in $C$. So far there can be no
double-counting, since the sum of any one sub-class is correct.

2. Choose another sub-class, $S$, and compare it with those already placed in $C$. If any
diagrams in this sub-class currently under consideration have already been placed in
$C$ they should be removed from $S$.

3. The remaining diagrams may then be placed in $C$ and the process repeated from Step
2 until all possible sub-classes have been dealt with.

The key to this procedure is obviously the discovery of which diagrams in the sub-class
under consideration have already been included in a sub-class whose sum has been placed in
$C$. However, these offending diagrams are, by the definition of the sub-classes, merely those
which admit two possible “unique” latest $r$-cuts, each cutting a different set of external
lines. (The two specific cases discussed above provided good examples of this.) Therefore,
we need to examine the sub-class $S$ and remove from it all diagrams which admit an $r$-cut
that makes them members of other sub-classes whose sum has already been placed in $C$.

Such a procedure involves examining the full $s$-channel cut structure of the amplitude
representing the sum of $S$. In time-ordered perturbation theory this cut structure is fully
determined by the s-channel cut structure of the sub-amplitudes making up the full amplitude. However, in any time-dependent perturbation theory the s-channel cut structure depends, not only on the s-channel cut structure of the sub-amplitudes, but also on their t and u-channel structure. Therefore, in order to eliminate double-counting it is necessary to examine the s, t and u-channel cut structure of the sub-amplitudes. Taylor’s original method only examines the s-channel cut structure of the sub-amplitudes, and consequently it double-counts certain diagrams.

VI. CONCLUSION

In this paper we have reexamined the Taylor method of classification-of-diagrams. A review of the classification-of-diagrams scheme has been given and two questions regarding this method have been answered.

The first question involved the simplification of the Taylor method developed by Thomas, Rinat, Afnan and Blankleider [11,12]. We found that this method, which was originally derived for use in time-ordered perturbation theory, is, in fact, equivalent to the Taylor method in time-dependent perturbation theory too, provided that only amplitudes involving three or less particles are considered, and only the two and three-cut structure of these amplitudes is examined. This explains how Afnan and Blankleider, who used the TRAB method in their work on the NN – πNN system [12] still managed to obtain the equations found by Avishai and Mizutani using the full Taylor method [9].

Secondly, we showed that the Taylor method double-counts certain diagrams when it is applied in a time-dependent perturbation theory. We found that this double-counting occurs because Taylor’s division of all the diagrams within a class into sub-classes places some diagrams in more than one sub-class. These diagrams are then double-counted when the sums of all the different sub-classes are added together. We outlined a general procedure by which this double-counting produced by the Taylor method can be eliminated. This procedure involves examining each sub-class in turn, first eliminating from it any diagram already included in the sum of the class, and only then placing the sum of the remaining diagrams in an overall sum of the class. Such a procedure is equivalent to examining the full s-channel cut structure of the amplitude in question and placing constraints on the t and u-channel cut structure of the sub-amplitudes contributing to this amplitude in order to eliminate the double-counting. By contrast, Taylor’s original method does not constrain the t and u-channel cut structure of the sub-amplitudes—it only constrains their s-channel cut-structure. From a topological point of view this lack of specification of the t and u-channel cut-structure is the reason why Taylor’s method leads to double-counting when it is applied in a time-dependent perturbation theory.

The modified Taylor method developed in this paper may now be used in order to derive double-counting-free integral equations for systems of mesons and baryons. In particular, these ideas will be applied to the derivation of equations for the amplitudes in a covariant theory of nucleons and pions in a forthcoming paper [22]. This results in equations for the NN – πNN system which are four-dimensional and free from the double-counting problems of previous four-dimensional equations.
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FIG. 1. Two diagrams, both of which give contributions to pion absorption on the deuteron in the Taylor method.

FIG. 2. The crossed term in the $\pi N$ t-matrix.

FIG. 3. The diagram which Kowalski et al. pointed out was double-counted in certain models of pion absorption on the deuteron.

FIG. 4. The diagram obtained when the crossed term is substituted into the right-hand diagram of Figure 1 in time-ordered perturbation theory. Note that, in time-ordered perturbation theory, this diagram is *not* included in the left-hand diagram of Figure 1.
FIG. 5. Two of the ways in which two cuts, $c_1$ and $c_2$, may intersect, and the resulting definitions of $c_1^+ \cup c_2^+$ and $c_1^- \cup c_2^-$ in each of the two cases.

FIG. 6. A diagrammatic representation of the sum of Taylor class $C_2$.

FIG. 7. A diagrammatic representation of the sum of Taylor class $C_3$.
FIG. 8. A diagrammatic representation of the sum of Taylor class $C_4$

FIG. 9. A diagrammatic representation of the sum of Taylor class $C_5$
FIG. 10. The equation for the two-particle irreducible $\pi NN$ to $NN$ amplitude, $F^{(2)}$, which is obtained from Taylor’s method, with the Taylor classes or sub-classes which produce each term indicated.
FIG. 11. Part of the \( NN \) t-matrix.

FIG. 12. The crossed term in the \( \pi N \) t-matrix, with the irreducibility of each vertex indicated.

FIG. 13. One diagram which is double-counted when the Taylor method is used to derive an equation for \( F^{(2)} \).

FIG. 14. The diagram which is double-counted when we make it explicit that we are considering only the s-channel cut structure of the amplitudes. The numbers indicate the irreducibility of the amplitude they are attached to, and the lines drawn thorough each amplitude indicate the channel to which these irreducibilities apply.
FIG. 15. The equation for the two-particle irreducible $NN$ to $NN$ amplitude, $T^{(2)}$, obtained from Taylor's method, with the classes and sub-classes which produce each term indicated.
FIG. 16. The two slightly different diagrams which the Taylor method produces for undressed one-pion exchange. Note that, despite the fact these two diagrams are different, there is double-counting here, since both contain Figure 11, if the vertex in that diagram is considered to be a vertex which is two-particle irreducible in all channels.

FIG. 17. The two possible “latest” cuts, \( c_1 \) and \( c_2 \), which lead to the double-counting of Figure 13, and the “cuts”, \( c_1^\dagger \cup c_2^\dagger \) and \( c_1^\dagger \cup c_2^\dagger \), which are obtained when we attempt to apply the argument used in the proof of the last internal cut lemma in order to construct an overall latest cut.

FIG. 18. The two possible “latest” cuts which lead to the double-counting of Figure 11, and the “cuts” which are obtained upon attempting to use the last internal cut lemma argument in order to construct an overall latest cut.