How important is the three-nucleon force?

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Abstract

By calculating the contribution of the $\pi - \pi$ three-body force to the three-nucleon binding energy in terms of the $\pi N$ amplitude using perturbation theory, we are able to determine the contribution of the different $\pi N$ partial waves to the three-nucleon force. The division of the $\pi N$ amplitude into a pole and non-pole gives a unique procedure for the determination of the $\pi N N$ form factor. The total contribution of the three-body force to the binding energy of the triton is found to be very small.
The discrepancy between the results of the exact calculation of the binding energy of the triton using a number of realistic nucleon-nucleon potentials and the experimental value of 8.48 MeV, has been an outstanding problem in nuclear physics for a number of years. A commonly accepted solution has been the introduction of a three-nucleon force that will bridge the gap between the calculated binding energy \[1,2\], based on two-body interaction, and the experimental binding energy. The origin of such a three-body force lies in the fact that the nucleons are treated as point particles interacting via a two-body potential that is often assumed to be local. This approximation neglects the contribution of the mechanisms whereby one of the nucleons emits a meson that scatters off a second nucleon and then gets absorbed on the third nucleon, see Fig. 1. Thus in a theory where the meson degrees of freedom are suppressed, one needs to include the contribution from the three-body force given in Fig. 1. Unfortunately, to date, there is no consistent way of deriving both the two- and three-body force from an underlying and complete meson-nucleon theory.

Over the past ten years there has developed two approaches to the determination of this three-nucleon interaction. The first such three-body force was developed by the Tucson-Melbourne (TM) group \[3\]. They determined the \(\pi N\) amplitude that goes into the calculation of the three-body force, by emphasizing the role of symmetry, and in particular the importance of current algebra and the soft pion theorems needed to fix the behavior of the \(\pi N\) t-matrix in the off-mass shell energy region. But in the actual calculation in Ref. \[1,2\], the cut-off mass in the \(\pi NN\) form factor is treated as a free parameter that is adjusted to reproduce the triton binding energy. The second approach, advocated predominantly by the Hanover group \[4,5\], emphasizes the fact that the \(\pi N\) amplitude is dominated at medium energies by the \(\Delta(1230)\), which is considered as the first excited state of the nucleon. This suggests that one should introduce coupling between the \(NN, N\Delta\) and \(\Delta\Delta\) channels, and solve the coupled channel problem for the \(BBB\) system, where \(B = N, \Delta\). In this coupled channel formulation, the \(\Delta\) contribution is suppressed due to a cancellation between the three-body force and the dispersion term which results from the pion being absorbed on the same nucleon from which it was emitted. It was also found that the \(\Delta\) contribution in the
TM potential is not dominant [6].

The problems which arise from both approaches are: (i) We do not know the contribution
of the different $\pi N$ partial waves to the three-body force. In particular, it is not clear that the
different $\pi N$ partial wave contributions are consistent with the $\pi N$ data. (ii) The energy
at which we need the $\pi N$ amplitude, in Fig. 1, in the actual triton calculation is in the
unphysical region corresponding to $\pi N$ center of mass energy of

$$E_\pi = m_N - E_T - E_{KE},$$

(1)

where $m_N$ is the nucleon mass, $E_T$ the binding energy of the triton, and $E_{KE}$ the kinetic
energy of the two spectator nucleons. Since $E_{KE}$ depends on the momenta of the spectator
nucleons, we will need to integrate over the kinetic energy of the two spectator nucleons
in the evaluation of the contribution of Fig. 1 to the binding energy. To that extent, the
energy at which we need to know the $\pi N$ amplitude is $E_\pi < (m_N - E_T)$ and in this energy
region it is not clear how important a role the $\Delta(1230)$ resonance plays in determining the
$\pi N$ amplitude. (iii) Finally, the present models are sensitive to the range of the $\pi NN$ form
factor needed at the vertices where the pion gets emitted and absorbed. More important is
the fact that this form factor is not in any way constrained by the $\pi N$ amplitude used to
calculate the three-body force.

We propose to examine the above questions within a model in which the $\pi N$ data is
parameterized in terms of separable potentials. In this way we can examine the contribution
from the different partial waves, and be able to include the energy dependence of the $\pi N$
amplitude and the $\pi NN$ form factor. Finally, the $\pi NN$ form factor is determined by the fact
that the $\pi N$ amplitude in the $P_{11}$ channel is divided into a pole and non-pole contribution.
The pole part of the amplitude is used to determine the $\pi NN$ form factor for the production
and absorption of the pion in Fig. 1, while the non-pole $\pi N$ amplitude determines the
contribution of $\pi N$ scattering to the three-body force. The exclusion of the pole part of
the amplitude is required to avoid double counting of the one pion exchange potential. The
choice of a separable potential is made at this stage to simplify the calculation. We have

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also limited the calculation to the evaluation of the diagram in Fig. 1. In a more definitive
calculation we will need to include other time ordered diagrams, and possibly use a πN
amplitude that is based on a chiral Lagrangian, and in this way satisfy the symmetries that
the MT approach emphasize.

In the present analysis we use the πN potential used by Thomas [7] for π − d scattering.
The parameters of this potential were adjusted to fit the πN phase shifts in all S- and
P-waves except the P_{11}. This potential gives a partial wave amplitude that is of the form

\[ t_\beta(k, k'; E) = g_\beta(k) \tau_\beta(E) g_\beta(k') \]  \hspace{1cm} (2)

For the P_{11} partial wave we have used the potential of McLeod and Afnan [8]. This potential
was developed for and used in π − d scattering and pion absorption on the deuteron. This
potential gives a πN scattering amplitude that is a sum of a pole term and a non-pole term, i.e.

\[ t(k, k'; E) = f^R(k, E) d^R(E) f^R(k', E) + g(k) \tau(E) g(k') \]  \hspace{1cm} (3)

where the second term in Eq. (3) is the background or non-pole amplitude, while the first
term is adjusted to give a pole at the nucleon mass, with the residue being the πNN coupling
constant. The total amplitude is required to fit the phase shifts below the pion production
threshold. The renormalized πNN form factor f^R(k, E), which is energy dependent, is given by

\[ f^R(k, E) = Z_{2}^{1/2}(E) [f_0(k) + g(k) \tau(E) \langle g | G(E) | f_0 \rangle] \]  \hspace{1cm} (4)

where Z_{2}(E) is the wave function renormalization, and f_0 is the bare πNN form factor. The
dressed nucleon propagator d^R(E) has a pole with unit residue at E = m_N, i.e., d^R(E) =
(E − m_N)^{-1}. The important feature of the πN P_{11} amplitude is that the background term
g(k) \tau(E) g(k'), is part of the amplitude that is included in the calculation of the three-
nucleon force, while the pole term, gives the πNN form factor for the pion production
and absorption vertices. In this case the πNN form factor is not free, but is constrained
by the experimental $\pi N$ phase shifts. To that extent there are no free parameters in this three-nucleon force.

The above $\pi N$ amplitude gives a fit to the $\pi N$ phase shifts. The only freedom we have is the choice of the form factors $g_\beta(k)$ in all partial waves and the bare $\pi NN$ form factor $f_0(k)$. These form factors determine the off-shell behavior of the $\pi N$ amplitude. Since we have no direct experimental information about this off-shell behavior, we have chosen form factors which seem to give good results for $\pi - d$ scattering and pion absorption [9]. This $\pi N$ potential can be used to calculate the contribution of the three-nucleon force, Fig. 1, to the binding energy of the triton in perturbation theory. This contribution is given, for the $\beta \pi N$ partial wave, by

$$W^\beta = \sum_{\alpha_1\alpha_2} \int dp_1 dq_1 dp_3 dq_3 \, P_1^2 \, q_1^2 \, q_3^2 \, \psi_\alpha(p_1, q_1; E) \, W_{\alpha_1\alpha_2}^\beta(p_1, q_1, p_3, q_3; E) \, \psi_{\alpha_3}(p_3, q_3; E),$$

where $\psi_\alpha(p, q; E)$ is the triton wave function which is a solution of the Faddeev equation for a given two-body interaction. The three-body force operator $W_{\alpha_1\alpha_2}^\beta(p_1, q_1, p_3, q_3; E)$ is given in terms of the amplitude for a given $\pi N$ partial wave by

$$W_{\alpha_1\alpha_2}^\beta(p_1, q_1, p_3, q_3; E) = \frac{1}{2} \sum_{L, \sigma, \alpha} \int dx (-)^{j_{0}+i_{\alpha}+j_{\alpha}+l_{3}+l_{2}} \, \frac{q_1^{2}}{p_1^{2}} \, \frac{q_3^{2}}{p_3^{2}} \, \rho_{L, \sigma, \alpha} A_{\alpha_1\alpha_2}^{L, \sigma, \alpha} \, P_L(x) \times \langle f^{R}(E_\pi)|Q_1\rangle \langle p_1 q_1|G_{\pi NN}(E_T)|p'_1 q'_1\rangle \langle Q'_1|g_\beta \rangle \times \tau_\beta(E_\pi) \langle g_\beta|Q_3'\rangle \langle p'_3 q'_3|G_{\pi NN}(E)|p_3 q_3\rangle \langle Q_3|f^{R}(E_\pi) \rangle.$$  

(6)

Here, the coefficients $A_{\alpha_1\alpha_2}^{L, \sigma, \alpha}$ in Eq. (6) are given by Eq.(B6) of Ref. [10]. The momenta $p_i$, $q_i$ and $Q_1$ are defined in Fig. 1 and $\rho$ is defined by,

$$\rho = \frac{m_N}{2m_N + m_\pi}.$$  

(7)

The form factors $g_\beta(k)$ and $f^{R}(k, E_\pi)$ are those defined in Eqs. (2) and (3) for the $\pi N$ potential given in Ref. [7,8]. The energy of the $\pi N$ system, e.g. $E_{\pi 2}$, is defined by

$$E_{\pi 2} = -E_T + m_N - \frac{1}{2} \left\{ \frac{1}{m_N} + \frac{1}{m_N + m_\pi} \right\} p_1^2 + \left( \frac{1}{m_N} + \frac{1}{2m_M + m_\pi} \right) q_3^2.$$  

(8)
In a similar manner $E_{\pi 3}$ ($E_{\pi 1}$) is given by replacing $p_1$ by $p_3$ ($p_1$) and $q_1$ by $q_3$ ($q_1$) in Eq.(8).

For the present calculation we have taken the triton wave function to be the solution to the 18 channel Faddeev equation for the PEST potential [11,12] which is a separable expansion to the Paris potential [13]. We have taken sufficient terms in the expansion, see Table I, to get a reasonable agreement in the binding energy, $S_-$, $S_0^-$, and $D$-state probability. In Table II, we compare the results of the PEST potential with those of the exact coordinate space calculation [2]. Clearly, the results of the separable expansion are in good agreement with the more exact solution of the Faddeev equation for the Paris potential.

In Tables III we present the contribution of the three-body force in KeV to the binding energy of the triton as given by Eq. (6) for different choices of the number of terms in the partial wave expansion of the total wave function. Here again we see that we will need 18 terms in the partial wave expansion of the total wave function. The $P_{11}$ $\pi N$ amplitude used in the above calculations is the potential PJ of McLeod and Afnan [8]. From these results we may conclude that: (i) The total contribution from all $\pi N$ partial waves is very small, less than 10 KeV, and not of any significance in explaining the discrepancy between the results of Faddeev calculations based on two-body $NN$ interaction and the experimental binding energy. (ii) The largest contributions are from the $P_{33}$ and $S_{31}$ partial waves. This justifies the fact that the $\Delta(1230)$ has a large role in the determination of the three-body force. But then the $S$-wave $\pi N$ amplitudes are also important. Here we should point out that the $S$-wave $\pi N$ interaction used have a scattering lengths such that $a_1 + 2a_3 = -0.011 m^{-1}$, consistent with the requirement of current algebra. (iii) There is a cancellation between the $P_{33}$ and $S_{31}$ contribution, making the over all contribution of the three-body force even smaller. (iv) Finally, there is a correlation between the sign of the contribution of the different partial waves and the corresponding phase shift, which is expected considering the fact that the sign of the phase shift tells us if the interaction in that partial wave is attractive or repulsive. Here we should note that the $P_{11}$ contribution, which is the non-pole part of the amplitude in this channel, is attractive, as is the case with the $P_{33}$.

To examine the possible reason for this small three-body force contribution to the triton
binding energy, and in particular the importance of the cut-off mass in the \( \pi NN \) form factors, we have repeated the calculation of the three-body force contribution for the potential M1 of McLeod and Afnian [8], which has a different form factor for the \( \pi NN \) vertex. In Table IV we compare the results for these two different \( P_{11} \) potentials. Although the final contribution to the binding energy is substantially larger, it is still too small to be of any significance. Here again we have a cancellation between the \( P_{33} \) and the \( S_{31} \), and both have increased in going from potential PJ to M1.

Although this analysis is based on perturbation, the fact that the results are so small suggest that a more exact treatment of the three-body force based on the present formalism is not warranted. The inclusion of the other time order \( \pi - \pi \) three-body force can at most increase the contribution by a factor of four, which is still too small to bridge the gap between the experimental result of 8.48 MeV and the value for the binding energy based on the Paris two-body interaction of 7.39 MeV. We need at this stage to carry out a more detailed study of the sensitivity of the total three-body force contribution to the range of the \( \pi NN \) form factor, since the previous calculations [1,2] of the contribution of the three-body force were sensitive to the cut-off-mass in the \( \pi NN \) form factor. This work is presently in progress. However, within the present formalism, this cut-off mass is not a free parameter, and is constrained by the \( \pi N \) phase shift and the fact that the bare form factor gets dressed by the background term as indicated in Eq. (4). As a result we don’t expect a great deal of variation in the form factor within a separable potential model with monopole or dipole form factors. The dispersion term which led to the suppression of the \( \Delta \) contribution to the three-body binding energy is in principle present for all \( \pi N \) partial waves. A proper treatment of this dispersion contribution will require a consistent treatment of the two- and three-body force if we are to avoid double counting. Finally, we should point out that the magnitude of this three-body force should be determined from a chiral Lagrangian that is consistent with QCD.

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REFERENCES


FIGURES

FIG. 1. The contribution to the three-nucleon force.
TABLES

TABLE I. The rank of the separable expansion to the Paris potential in the different \(NN\) partial waves.

<table>
<thead>
<tr>
<th></th>
<th>(1S_0)</th>
<th>(^2S_1-^2D_1)</th>
<th>(^3P_0,^1P_1,^3P_1)</th>
<th>(^3P_0-^3F_2)</th>
<th>(^1D_2,^3D_2)</th>
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<tr>
<td>rank</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
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TABLE II. The Triton binding energy, \(S\)-, \(^1S\)- and \(D\)-state probability for 18 channel calculations for PEST potential and exact coordinate space calculation.

<table>
<thead>
<tr>
<th>Model</th>
<th>B.E. (MeV)</th>
<th>(P(S))%</th>
<th>(P(S'))%</th>
<th>(P(D))%</th>
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<tr>
<td>PEST</td>
<td>7.318</td>
<td>90.111</td>
<td>1.430</td>
<td>8.393</td>
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<tr>
<td>Exact</td>
<td>7.388</td>
<td>90.130</td>
<td>1.395</td>
<td>8.409</td>
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TABLE III. The contribution to the binding energy of the triton from the different \(\pi N\) partial waves in KeV for 5, 10, 18, 26 and 34 terms in the expansion of the total wave function.

<table>
<thead>
<tr>
<th>wave function</th>
<th>(S_{11})</th>
<th>(S_{31})</th>
<th>(P_{11})</th>
<th>(P_{31})</th>
<th>(P_{13})</th>
<th>(P_{33})</th>
<th>TOTAL</th>
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<tbody>
<tr>
<td>5</td>
<td>-2.0</td>
<td>-27.0</td>
<td>-3.4</td>
<td>-12.3</td>
<td>7.8</td>
<td>5.5</td>
<td>-31.4</td>
</tr>
<tr>
<td>10</td>
<td>-5.2</td>
<td>37.1</td>
<td>-7.2</td>
<td>-1.1</td>
<td>6.4</td>
<td>-8.5</td>
<td>21.5</td>
</tr>
<tr>
<td>18</td>
<td>-4.8</td>
<td>26.4</td>
<td>-8.8</td>
<td>-3.6</td>
<td>4.5</td>
<td>-16.0</td>
<td>-2.3</td>
</tr>
<tr>
<td>26</td>
<td>-5.3</td>
<td>26.2</td>
<td>-9.5</td>
<td>-3.4</td>
<td>3.5</td>
<td>-16.1</td>
<td>-4.6</td>
</tr>
<tr>
<td>34</td>
<td>-5.3</td>
<td>25.7</td>
<td>-9.5</td>
<td>-3.6</td>
<td>3.5</td>
<td>-16.9</td>
<td>-6.1</td>
</tr>
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TABLE IV. Comparison of the results for the contribution of the three-body force to the triton binding energy in KeV for two of the \(P_{11} \pi N\) potentials of Ref.[8].

<table>
<thead>
<tr>
<th>(P_{11}) potential</th>
<th>(S_{11})</th>
<th>(S_{31})</th>
<th>(P_{11})</th>
<th>(P_{31})</th>
<th>(P_{13})</th>
<th>(P_{33})</th>
<th>TOTAL</th>
</tr>
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<tbody>
<tr>
<td>PJ</td>
<td>-4.8</td>
<td>26.4</td>
<td>-8.8</td>
<td>-3.6</td>
<td>4.5</td>
<td>-16.0</td>
<td>-2.3</td>
</tr>
<tr>
<td>MI</td>
<td>-5.0</td>
<td>28.9</td>
<td>-15.3</td>
<td>-2.1</td>
<td>6.2</td>
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