APPROXIMATE SELF-CONSISTENT MODELS FOR TIDALLY TRUNCATED

STAR CLUSTERS

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Abstract

This paper generalises King’s models for tidally truncated star clusters by including approximately the non-spherical symmetry of the tidal field and the resulting non-spherical distortion of the cluster.

Key words: Gravitation - Methods: numerical - Celestial mechanics, stellar dynamics - globular clusters: general - open clusters and associations: general - Galaxies: star clusters
1. Introduction

The problem of constructing a model of a stellar system such as a globular cluster has a long history. In the last twenty years or so, however, much the most widely used models, at least in this context, are King’s models (King 1966) and various extensions of them.

Several factors have led to the popularity of these models. First, they are very easy to compute. Second, they are successful. Even the single mass models are an excellent fit to the light distribution of most observed galactic globular clusters (e.g. Illingworth & Illingworth 1976). Third, they are easily extended (with suitable assumptions) to include a spectrum of masses or anisotropy in the distribution of velocities, and indeed such models are the standard means of interpreting radial velocity data (e.g. Gunn & Griffin 1979, Meylan 1988). Fourth, they approximately incorporate essential aspects of theoretical stellar dynamics: (i) Jeans’ theorem (ii) relaxation effects and (iii) tidal truncation.

It is the aim of this paper to remove some approximations involved in these models, especially with regard to (i) and (iii). In the construction of King’s models the (specific) energy of a star is taken as $v^2/2 + \Phi_1$ where $v$ is the speed and $\Phi_1$ is the potential due to the cluster. In fact, each star is moving in the potential field of both the cluster and the galaxy. Even so, the energy is not conserved, because $v$ is measured in an accelerating frame and centrifugal terms must be added.

In this paper, then, we shall construct a series of models analogous to King’s series, but approximately making allowance for the non-spherical symmetry of the tidal field and the non-conservation of energy. The method we use is an expansion of the potential in
spherical harmonics, though truncated at very low order. We also use the same ‘lowered’
Maxwellian distribution function as in King’s models, except that it is now a function of
the Jacobi integral, which is a conserved quantity when the potential is in uniform rotation.
Thus, we envisage a cluster whose barycentre rotates on a circular orbit in the plane of
symmetry of an axisymmetric galaxy. A shortened version of our results is presented in
Heggie & Ramamani 1993, and we would like to mention that a similar set of models is
discussed by Weinberg (1993).

The outline of this paper is as follows. In section 2, we present the necessary prelim-
inaries. Section 3 describes the numerical method. In section 4 results are discussed for
two cases (i) when the galaxy is modelled as a point mass and (ii) when the tidal field is
modelled using observational data from the vicinity of the sun.

2. Theory of a tidally perturbed model star cluster

2.1. Case of a point mass perturber

The cluster under consideration has mass $M_1$, core radius $r_c$ and a limiting radius of
order $r_c$. It contains stars of equal mass and these stars will have an isotropic velocity
distribution with respect to a rotating frame of reference which will be described shortly.
The cluster moves in a circular orbit of radius $D$ around the Galaxy, which is represented
by a point mass of mass $M_2$. The changes in the Galactic potential across the cluster
are small for $D \gg r_c$, and the field of the Galaxy is adequately represented in a linear
approximation.
We choose a coordinate system rotating with an angular velocity equal to that of the orbital motion of the Galaxy-cluster system, and choose its origin to be at the centre of mass of this system. The $x$-axis is chosen to be pointing towards the Galaxy, the $y$-axis perpendicular to the $x$-axis in the orbital plane in the sense of the orbital motion, and the $z$-axis perpendicular to the orbital plane in the right handed sense. We denote all attributes of the cluster and the Galaxy with the suffices 1 and 2, respectively, and the central values of the cluster with the suffix $c$.

A star in the cluster, at a point $r = (x, y, z)$, has an integral of motion $J$, the Jacobi constant, which is given by

$$J = \Phi_1 + \Phi_2 - \frac{1}{2} \omega^2 (x^2 + y^2) + \frac{1}{2} v^2,$$

(1)

where $\Phi_1, \Phi_2$ are the potentials at $r$ due to the cluster and the Galaxy, respectively, $\omega$ is the angular velocity of the frame of reference, and $v$ is the speed of the star in the rotating frame. Writing the kinetic energy per unit mass as $T = \frac{1}{2} v^2$, we have $J = U + T$, where

$$U = \Phi_1 + \Phi_2 - \frac{1}{2} \omega^2 (x^2 + y^2)$$

(2)

$$\simeq \Phi_1 - \frac{GM_2}{D} - GM_2/D^3 (x_1^2 - \frac{1}{2} y_1^2 - \frac{1}{2} z_1^2) - \frac{GM_2}{2D} - GM_2/2D^3 (x_1^2 + y_1^2)$$

(3)

where $x_1, y_1, z_1$ are the coordinates of the star with respect to the centre of mass of the cluster, with the directions of the coordinate axes chosen as before. Expression (3) is obtained on the assumption that $M_2 \gg M_1$, and $r_1 \ll D$. The constant terms here are irrelevant in what follows, by redefinition of the zero-point of the potential.

From Jeans’ theorem, the phase-space distribution function (mass-density) for the stars of the cluster can be taken as any function of $J$, and in order to make as close an
analogy as possible with King’s models we take

$$f(r, v) = K(\exp(-2j^2J) - \exp(-2j^2J_e)),$$  \(4\)

where \(J_e\) is the value of \(J\) at the limiting zero velocity surface, at the edge of the cluster, and \(K, j\) are constants. Then the density distribution is, as usual,

$$\rho = \int_{0}^{v_e} f(r, v)4\pi v^2 dv,$$  \(5\)

where \(v_e\) is the maximum speed at the point \(r = (x_1, y_1, z_1)\), which is given by

$$v_e^2 = -2(U - J_e).$$  \(6\)

We now exploit the arbitrariness in the zero-point of the potential by supposing that \(J_e = 0\). Then it is easy to see that the density takes the form

$$\rho = \rho_c R(j^2U),$$  \(7\)

where \(\rho_c\) is the central density and \(R\) is expressible in terms of the error function, exactly as in the usual theory of King models (King 1966). Where the theory differs is in the expression for \(U\), which may be written

$$U = \Phi_1 + Q,$$  \(8\)

where \(Q\) is the sum of the quadratic terms in expression (3). Now if these terms were neglected we would obtain King’s models, and \(\Phi_1\) would be the corresponding potential. If the tidal field, represented by \(Q\), were weak, then \(\Phi_1\) would differ slightly. We therefore write

$$\Phi_1 = \Phi_k + \delta\Phi,$$  \(9\)
where $\Phi_k$ is the potential of the corresponding King model, and $\delta \Phi$ is the change in the potential due to the distortion of the cluster by the tidal field. As usual, potential and density are related by Poisson’s equation:

$$\nabla^2 \Phi_1 = 4\pi G \rho(U). \quad (10)$$

Here is another difference between these models and King’s: $\rho$ depends on $U$, not on $\Phi_1$.

Now we take the important step of linearising with respect to the deviation from the King model, writing

$$\rho(U) = \rho(\Phi_k) + (\delta \Phi + Q) \frac{d\rho}{d\Phi_k} + \ldots. \quad (11)$$

Then from eqs. (9) - (11) it follows that

$$\nabla^2 \Phi_k = 4\pi G \rho(\Phi_k) \quad (12)$$

and

$$\nabla^2 \delta \Phi = 4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k} \{\delta \Phi + Q\}. \quad (13)$$

Eq. (12) is the same as eq.(12) in King (1966). We now write

$$F(r) = 4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k}, \quad (14)$$

$$\delta \Phi(r, \theta, \phi) = \sum_0^\infty \delta \Phi_n(r) S_n(\theta, \phi) \quad (15)$$

and

$$Q = \sum Q_n(r) S_n(\theta, \phi), \quad (16)$$

where $r, \theta, \phi$ are spherical polar coordinates with origin at the cluster centre, and the $S_n(\theta, \phi)$ are a complete set of surface spherical harmonics. Eq.(13) can then be written as

$$[\nabla^2 - F(r)] \sum \delta \Phi_n(r) S_n(\theta, \phi) = \sum F(r) Q_n(r) S_n(\theta, \phi),$$
\[ \nabla^2 \delta \Phi_n(r) - \frac{n(n+1)}{r^2} \delta \Phi_n(r) - F(r)\delta \Phi_n(r) = F(r)Q_n(r). \]  
(17)

In fact only a few terms of these series are needed. We have

\[ x_1 = r \sin \theta \cos \phi, \quad y_1 = r \sin \theta \sin \phi \]
and

\[ z_1 = r \cos \theta, \]
whence eqs. (3) and (8) give

\[ Q = -\frac{GM_2}{D^3} r^2 \left\{ \frac{1}{3} S_0(\theta, \phi) + S_2(\theta, \phi) \right\}, \]
(18)

where \( S_0(\theta, \phi) \equiv 1 \) and

\[ S_2(\theta, \phi) = -\frac{5}{6} P^0_2(\cos \theta) + \frac{1}{4} P^2_2(\cos \theta) \cos 2\phi. \]
(19)

Here \( P^m_n \) are Legendre’s associated polynomials of degree \( n \) and order \( m \).

Since \( Q \) has only zero- and second-order spherical harmonics, we get only two relevant equations, with \( n = 0 \) and \( n = 2 \), from eq.(17). They are

\[ \frac{d^2(\delta \Phi_0)}{dr^2} + \frac{2}{r} \frac{d(\delta \Phi_0)}{dr} - 4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k} \delta \Phi_0 = -4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k} \frac{GM_2}{D^3} \frac{r^2}{3} \]
(20)

and

\[ \frac{d^2(\delta \Phi_2)}{dr^2} + \frac{2}{r} \frac{d(\delta \Phi_2)}{dr} - \left\{ \frac{6}{r^2} + 4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k} \right\} \delta \Phi_2 = -4\pi G \frac{d\rho(\Phi_k)}{d\Phi_k} \frac{GM_2}{D^3} \frac{r^2}{3}. \]
(21)

Since \( S_0(\theta, \phi) = 1 \) the quantity \( \delta \Phi_0 \) is the spherically symmetric part of the change in the potential which arises due to the distortion of the cluster, and \( \delta \Phi_2 \) multiplied by \( S_2 \)
gives the non-spherical part. It is worth pausing here to note the shape of this distortion.

Indeed from eq.(19) we have

\[ S_2(\theta, \phi) = \frac{5}{12} + \frac{3}{4} \sin^2 \theta \cos 2\phi - \frac{5}{4} \cos^2 \theta. \]

Thus \( S_2 \) is largest along the \( x_1 \)-axis and smallest along the \( z_1 \)-axis. The relevant values are \( S_2(\pi/2, 0) = 7/6, S_2(\pi/2, \pi/2) = -1/3 \) and \( S_2(0, 0) = -5/6 \).

2.2 Jacobi integral for a galactic disk

Chandrasekhar (1942, p.217) gives the equations

\[ m(\ddot{\xi} - 2\omega_c \dot{\eta} + \alpha_1 \xi) = -\partial \Omega / \partial \xi, \]
\[ m(\ddot{\eta} + 2\omega_c \dot{\xi}) = -\partial \Omega / \partial \eta, \]
\[ m(\ddot{\zeta} + \alpha_3 \zeta) = -\partial \Omega / \partial \zeta, \]

for the motion of a star of mass \( m \) moving in the potential \( \Omega \) of a cluster. The coordinates \((\xi, \eta, \zeta)\) have the same meaning as \((x_1, y_1, z_1)\) in §2.1, \( \omega_c \) is \( \omega \), \( \alpha_1 \) and \( \alpha_3 \) are constants, and subscripts \( i \) have been dropped. If \( \Omega = \Omega(\xi, \eta, \zeta) \), i.e. we suppose the potential due to the cluster is time-independent in the rotating frame, then multiplying these equations by \( \dot{\xi}, \dot{\eta}, \dot{\zeta} \), respectively, and adding, gives the Jacobi integral

\[ J = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) + \frac{1}{2} \alpha_1 \xi^2 + \frac{1}{2} \alpha_3 \zeta^2 + \Omega/m. \] (22)

Denoting by \( B \) the galactic potential, Chandrasekhar (p.217 again) gives the definitions

\[ \alpha_1 = \frac{\partial^2 B}{\partial \omega^2} - \frac{1}{\omega} \frac{\partial B}{\partial \omega}, \] (23)
\[ \alpha_3 = \frac{\partial^2 B}{\partial z^2}, \quad (24) \]

where \( z \) is height above the galactic plane and \( \varpi \) is the distance from the symmetry axis of the galaxy. Using Poisson’s equation in cylindrical polars (eq. 4.528 in Chandrasekhar) we find that

\[ \alpha_3 = 4\pi G \rho_G - \frac{\partial^2 B}{\partial \varpi^2} - \frac{1}{\varpi} \frac{\partial B}{\partial \varpi}, \]

where \( \rho_G \) is the galactic density. From the first of eqs.(5.501) in Chandrasekhar we deduce that

\[ \alpha_3 = 4\pi G \rho_G - \frac{2\Theta}{\varpi} \frac{d\Theta}{d\varpi}, \]

where \( \Theta \) is the local speed of circular galactic motion. Hence, by eq.(1.429) in Chandrasekhar,

\[ \alpha_3 = 4\pi G \rho_G - 2\omega(A + B), \quad (25) \]

where \( A \) and \( B \) are Oort’s constants. Similarly

\[ \alpha_1 = 4A(B - A), \quad (26) \]

by Chandrasekhar’s eq.(5.615).

The Jacobi integral has exactly the same form as in the case of a point mass \( M_G \) at distance \( D \). In this case \( B = -GM_G/(\varpi^2 + z^2)^{1/2} \) and \( \rho_G = 0 \), whence \( \alpha_1 = -3GM_G/D^3 \) (by eq.(23), where the potential derivatives are evaluated at \( z = 0, \varpi = D \)) and \( \alpha_3 = GM_G/D^3 \). Thus the Jacobi integral is

\[ J = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) - \frac{3}{2} \frac{GM_G}{D^3} \xi^2 + \frac{1}{2} \frac{GM_G}{D^3} \zeta^2 + \Omega/m. \]
This is essentially the same result as eq.(3) of section 2.1.

Returning to the general case (eq.(22)) and defining $Q$ by analogy with eq.(8), we have

$$Q = \frac{1}{2} \alpha_1 \xi^2 + \frac{1}{2} \alpha_3 \zeta^2$$

where $\xi, \zeta$ have the same meaning as $x_1, z_1$ in §2.1. Hence

$$Q = \frac{1}{3} \alpha_1 r^2 \left( \frac{1}{3} S_0 + S_2 \right)$$

where now

$$S_0 = \frac{3}{2} \left( 1 + \frac{\alpha_3}{\alpha_1} \right) P_0^0$$

(27)

and

$$S_2 = \left( \frac{\alpha_3}{\alpha_1} - \frac{1}{2} \right) P_2^0 + \frac{1}{4} P_2^2 \cos 2\phi.$$ 

3. Numerical method

In order to construct models it is necessary to solve eqs.(12), (20) and (21) simultaneously throughout the cluster by numerical methods. Following King (1966), we choose scaled variables $W \equiv -2j^2\Phi_k$ and $R \equiv r/r_c$, where $r_c$ is the core radius defined below. Analogously the perturbations in the potential are scaled to give $y_3 = -2j^2\delta\Phi_0$ and $y_5 = -2j^2\delta\Phi_2$. For the sake of definiteness we consider the case of §2.1, and define

$$\rho_1 = \frac{M_2}{\frac{4}{3} \pi D^3},$$

which is a measure of the density of the perturbing Galaxy. Following King again, we take the core radius as satisfying the relation $8\pi G j^2 \rho_c r_c^2 = 9$. With $R$ as the independent
variable, the equations then give

\[
\frac{d^2 W}{dR^2} + 2 \frac{dW}{R dR} = -9 \frac{\rho(W)}{\rho_c},
\]

(28)

\[
\frac{d^2 y_3}{dR^2} + 2 \frac{dy_3}{R dR} + 9 \frac{d(\rho/\rho_c)}{dW} y_3 = -9 R^2 \frac{\rho_1}{\rho_c} \frac{d(\rho/\rho_c)}{dW},
\]

(29)

and

\[
\frac{d^2 y_5}{dR^2} + 2 \frac{dy_5}{R dR} - \frac{6}{R^2} y_5 + 9 \frac{d(\rho/\rho_c)}{dW} y_5 = -27 R^2 \frac{\rho_1}{\rho_c} \frac{d(\rho/\rho_c)}{dW},
\]

(30)

where, following King,

\[
\frac{\rho}{\rho_c} = \frac{\exp(W) \int_W^0 \exp(-\eta) \eta^{3/2} d\eta}{\exp(W_c) \int_W^{W_c} \exp(-\eta) \eta^{3/2} d\eta}
\]

(31)

and

\[
(\rho/\rho_c)’ = \frac{d(\rho/\rho_c)}{dW} = \frac{\rho}{\rho_c} + \frac{W^{3/2}}{\exp(W_c) \int_W^{W_c} \exp(-\eta) \eta^{3/2} d\eta}.
\]

(32)

Here, as usual, a subscript \(c\) denotes a core or central value.

Since \(\rho\) depends only on \(W\), eqs.(28) to (30) may be conveniently transformed with the choice of \(W\) as the independent variable. We also use \(\ln(1 + R^2)\) rather than \(R\) as the dependent variable in transforming eq.(28), because this varies nearly linearly with \(W\) both near \(R = 0\) and in an isothermal halo. Thus the equations to be integrated take the form

\[
\frac{dy_1}{dW} = y_2
\]

(33)

\[
\frac{dy_2}{dW} = \left(\frac{y_2}{2R}\right)^2 \left(6 + 2R^2 + 9 \frac{\rho}{\rho_c} (1 + R^2)^2 y_2\right)
\]

(34)

\[
\frac{dy_3}{dW} = y_4
\]

(35)

\[
\frac{dy_4}{dW} = \frac{9}{4} \frac{(1 + R^2)^2}{R^2} y_2 \left\{-R^2 \frac{\rho_1}{\rho_c} \frac{d(\rho/\rho_c)}{dW} - y_3 \frac{d(\rho/\rho_c)}{dW} + \frac{\rho}{\rho_c} y_4\right\}
\]

(36)
\[
\frac{dy_5}{dW} = y_6 \quad \text{and} \\
\frac{dy_6}{dW} = \frac{3}{4} \frac{(1 + R^2)^2}{R^2} y_2^2 \left\{ \frac{2y_5}{R^2} - \frac{9R^2 \rho_1 d(\rho/\rho_c)}{\rho_c dW} - 3y_5 \frac{d(\rho/\rho_c)}{dW} + 3 \frac{\rho}{\rho_c} y_6 \right\}
\]

where \( y_1 = \ln(1 + R^2) \) and the additional variables \( y_2, y_4 \) and \( y_6 \) are essentially defined by eqs. (33), (35) and (37). Thus the original set of three second-order equations has been transformed to a set of six equations of first order.

The numerical treatment of these equations is straightforward. The above system can be written as

\[
\frac{dy}{dx} = g(x, y)
\]

and can be discretised as

\[
\frac{y_{i+1} - y_i}{\Delta x} = g(\bar{x}, \bar{y}),
\]

where \( \bar{x} = (x_i + x_{i+1})/2 \), etc., and a subscript \( i \) indicates the value at the \( i \)-th mesh point.

The boundary conditions at the centre of the system can be obtained using the following developments. Let

\[
W - W_c = aR^2 + bR^4 + \ldots \\
y_3 = a_0 R^2 + b_0 R^4 + \ldots \\
y_5 = a_2 R^2 + b_2 R^4 + \ldots
\]
Substitution of eqs.(40) into eqs.(28 - 30) is found to give the results

\[ a = -\frac{3}{2}, \]
\[ b = \frac{27}{40} \left( \frac{\rho}{\rho_c} \right)', \]
\[ a_0 = 0, \]
\[ b_0 = -\frac{9}{20} \frac{\rho_1}{\rho_c} \left( \frac{\rho}{\rho_c} \right)', \]
\[ b_2 = -\frac{9}{14} \left( \frac{\rho}{\rho_c} \right)' \left( a_2 + 3 \frac{\rho_1}{\rho_c} \right), \]

where \( (\rho/\rho_c)' \) is evaluated at the centre, and \( a_2 \) is yet to be determined. (It must be solved for along with the solution of the differential equations.) From these results it is easily found that the boundary conditions at the centre are \( y_2 = -2/3 \), \( y_6 = -2a_2/3 \), and \( y_1 = y_3 = y_4 = y_5 = 0 \). At the outside of the cluster \( \delta \Phi_2 \) should join smoothly to the exterior (vacuum) solution proportional to \( R^{-3} \), whence

\[ y_6 = -\frac{3}{2} \frac{1 + R^2}{R^2} y_5 y_2. \]

Eqs.(39), along with the above boundary conditions, were solved numerically using the Newton-Raphson method.

As a check of the numerical method, at least as far as the equations for \( y_1 \) and \( y_2 \) are concerned, King’s models were regenerated using our code, and good agreement was obtained with the results of King (1966). This yields a one-parameter family of solutions, depending on \( W_c \). But the system of eqs.(33)-(38) also possesses another dimensionless parameter, viz. \( \rho_1/\rho_c \). In physical terms the number of parameters can be understood by considering the number of radii which characterise each solution. In fact there are three: the core radius, \( r_c \); the radius of the system, \( r_e \); and, finally, the radius of the last closed
zero-velocity surface, on which the Lagrangian points of the cluster-Galaxy potential lie. These three radii give two independent dimensionless ratios, i.e. two parameters.

A solution of the equations having been obtained for given values of the two parameters $W_c$ and $\rho_1/\rho_c$, the functions $\Phi_k, \delta\Phi_0$ and $\delta\Phi_2$ can be determined from the runs of $y_3$ and $y_5$ against $W$, and then $\Phi_1$ can be found from eqs.(9) and (15). The limiting surface is obtained by setting $U = 0$, where, as before, $U = \Phi_k + \delta\Phi_0 S_0 + \delta\Phi_2 S_2 + Q$.

Of greatest interest are those systems in which the edge of the cluster coincides with the last closed zero-velocity surface. For a fixed value of $W_c$ this can be found by iterating the value of $\rho_1/\rho_c$ until a point on the $x_1$-axis is found at which both $U$ and $\partial U/\partial x_1$ vanish. (This point is the Lagrangian point.)

We now discuss the special features of the problem discussed in §2.2. In the disk model of the Galaxy, the values of $\alpha_1$ and $\alpha_3$ were calculated at sun’s distance using $A = 0.0144$ km/sec/pc and $B = -0.012$ km/sec/pc with the density at the solar distance in the Galaxy

$$\rho_G = 0.11 M_\odot /pc^3,$$

(cf. Kuijken & Gilmore 1989). Using eqs.(25) and (26) we obtain

$$\frac{\alpha_3}{\alpha_1} = -3.99,$$

and so $S_0 = -4.49$, by eq.(27). Again $S_2$ is largest along the $x_1$-axis and smallest along the $z_1$-axis: the relevant values are $S_2(\pi/2, 0) = 3.00$, $S_2(\pi/2, \pi/2) = +1.50$ and $S_2(0, 0) = -4.49$. Comparison with the values of $S_2$ given for the point-mass model indicate that the effects will be more pronounced in the case of the disk model.
4. Results and Discussion

4.1 Description of the models

Space density profiles of the resulting models are shown in Figs. 1-3 and Figs. 6-8 for the point-mass and the disk model of the galaxy, respectively. The space density profiles along the $x_1$-axis are compared with the corresponding King models in Figs. 1 and 6. Figs. 4, 5, 9 and 10 show the surface densities of our model clusters along the $x_1$- and $z_1$-axes as seen from the $y_1$ direction. It can be seen from the results that:

1. The clusters are triaxial, with the longest axis in the direction to the galactic centre and the shortest in the direction at right angles to the orbital plane. Denoting by $a, b, c$ the semi diameters in the $x_1, y_1$ and $z_1$ directions, respectively, it is found that the axial ratios $a/b$ and $b/c$ decrease only slightly as the scaled central potential $W_c$ increases from 2.5 to 10. Typical values are $a/b = 1.5, b/c = 1.04$ for the point mass perturber and $a/b = 1.50, b/c = 1.34$ for the tidal field representing a disk galaxy. The strong flattening in the $z_1$ direction in the latter case is already known from $N$-body simulations (Terlevich 1987). The axial ratio will clearly depend on the values of $\alpha_3/\alpha_1$.

2. The ratio of the maximum extent of the cluster to that of the corresponding King model decreases slightly as $W_c$ increases and is almost the same in the case of a point mass perturber as in that of a disk galaxy: a representative value is 1.5, i.e. the approximate self-consistent model is 50 percent larger. Interestingly, a similar conclusion was reached by Spitzer (1987) for a model in which both the cluster and the galaxy are treated as point masses. This agreement suggests that departures of the cluster potential from spherical
3. The central regions of the clusters are almost unaffected; they are nearly spherical and differ little from the central regions of the corresponding King models.

4.2 Discussion

Our result (2 above) that our models are more extended than the corresponding King models may have significant consequences for the application of King models to observations of globular clusters. Since limiting radii have sometimes been inferred by fitting King models, it is clear that the resulting estimates are likely to be systematically too small. Furthermore, since the inferred strength of the tidal field or the inferred mass of the cluster (Freeman 1980, Kontizas & Kontizas 1983) depends on the cube of the limiting radius, still larger systematic errors are liable to be present in the estimates of these quantities. Nevertheless, it should not be forgotten that the assumptions underlying both King models and the models discussed in this paper are not strictly applicable to globular clusters at all, since their galactic orbits are unlikely to be circular.

From the theoretical point of view, our models may be useful for the construction of initial conditions for $N$-body models of tidally perturbed clusters. At present the initial conditions used in such simulations are spherically symmetric, and over the first few crossing times the system presumably settles into a quasi-equilibrium in the non-spherically symmetric tide.
In order to check these ideas a number of 1000-body simulations have been conducted using NBODY5 (Aarseth 1985) with three different sets of initial conditions and parameters: (i) one of the models described in this paper, with scaled central potential $W_c = 2.5$, and the tidal parameters given at the end of §3; (ii) a King model with the same central potential and tidal parameters; and (iii) the same King model without an external tide. All models have stars of equal mass, and units are standard (Heggie & Mathieu 1986), with the crossing time being $2\sqrt{2}$ units. Figs.(11)-(13) show, for these three models, the mean square values of $x_1$, $y_1$ and $z_1$, measured relative to the density centre (Casertano & Hut 1985).

These results indicate that the model constructed according to the theory of the present paper remains in approximate dynamic equilibrium for about the first 4 time units. (The dip in one of the curves in Fig.11 is no greater than that in one of the curves in Fig.13, indicating that it can be ascribed to finite-$N$ effects). After about $t \simeq 4$ there begins a gentle expansion (on a time scale of about 70 time units). The tidally perturbed King model, on the other hand, flattens in the $z_1$-direction over the same period of about 4 units and spreads in the other two directions. Indeed, at about $t = 4$ it quite closely resembles the model shown in Fig.11. Eventually this model shows the same tendency to expand, and it does so on a very similar time scale to the model shown in Fig.11. We interpret the expansion as being due to two-body relaxation, and this is confirmed by the isolated model, which shows no tendency to triaxiality, but does expand in the last part of the run.

In conclusion we remark that it would be desirable to improve the models by a higher-
order expansion, or use of a grid-based potential solver. Such a method has been used by Rix and White (1989) to discuss models for a binary galaxy. The models obtained by us can be regarded as a limiting and simplified case of the same problem.

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Figure Captions

Fig.1 Space density on the $x_1$-axis for nine of the models described in this paper and the corresponding King models, in the case in which the tidal field is due to a large distant point mass. The scaled central potentials $W_c$ (denoted $W_0$ on the figure) of the models are 2.5, 3, 4, 5, 6, 7, 8, 9 and 10, and the King models are distinguished by dotted curves. The scaled distance from the cluster centre is $R$, and the density is scaled by the central value.

Fig.2 As Fig.1, but for the $y_1$-axis. The King models are not shown.

Fig.3 As Fig.1, but for the $z_1$-axis.

Fig.4 The projected (surface) density of the nine models displayed in Figs.1-3. The surface density is scaled to the central value. The cluster is viewed along the $y_1$-axis and the profile along the $x_1$-axis is plotted.

Fig.5 As Fig.4, except that the profile along the $z_1$-axis is plotted.

Fig.6 As Fig.1, except that the tidal field is a model of that in the solar neighbourhood ("disk" case).

Fig.7 As Fig.2, but for the disk case.

Fig.8 As Fig.3, but for the disk case.

Fig.9 As Fig.4, but for the disk case.
Fig. 10 As Fig. 5, but for the disk case.

Fig. 11 Evolution of a 1000-body model with initial conditions obtained from one of the models described in this paper. The model had scaled central potential $W_c = 2.5$ and the tidal field is a model of the tide in the solar neighbourhood. The quantities plotted are the mean square values of the coordinates $x_1$, $y_1$ and $z_1$, but using as origin the “density centre”. The lowest curve corresponds to $z_1$, the uppermost to $x_1$.

Fig. 12 As Fig. 11, except the initial conditions are constructed using a standard (spherically symmetric) King model. Again the lowest curve corresponds to $z_1$, the uppermost to $x_1$.

Fig. 13 As Fig. 12, but without any tide. The dashed curve gives one third of the mean square distance from the density centre, $\langle x_1^2 + y_1^2 + z_1^2 \rangle / 3$. 