Boundary States, Extended Symmetry Algebra and Module Structure for certain Rational Torus Models

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Abstract

The massless bosonic field compactified on the circle of rational $R^2$ is reexamined in the presence of boundaries. A particular class of models corresponding to $R^2 = 1/2^k$ is distinguished by demanding the existence of a consistent set of Newmann boundary states. The boundary states are constructed explicitly for these models and the fusion rules are derived from them. These are the ones prescribed by the Verlinde formula from the S-matrix of the theory. In addition, the extended symmetry algebra of these theories is constructed which is responsible for the rationality of these theories. Finally, the chiral space of these models is shown to split into a direct sum of irreducible modules of the extended symmetry algebra.

1 Introduction

The massless bosonic field compactified on the circle has been particularly useful for studying the moduli space of $c=1$ theories [1] [2]. In this case, there are two continuous families of theories corresponding to the torus and the $\mathbb{Z}_2$ orbifold models. Theories in each family are connected by marginal operator deformations. In both cases there is a duality that identifies the model with radius $R$ with the model with radius $1/2R$. Also the theory corresponding to $R = \sqrt{2}$ for the torus models is identified with the $R = 1/\sqrt{2}$ theory for the orbifold models. For the self-dual radius $R = 1/\sqrt{2}$ the theory possesses an extended $SU(2) \times SU(2)$ symmetry. By dividing this symmetry by the three special discrete subgroups of $SU(2)$, the tetrahedral, octahedral and icosahedral, [1] constructed three more theories that are not connected to the others by marginal operator deformations. This list of $c = 1$ theories has been shown to be complete in the case the partition function is a linear combination of toroidal partition functions by [2]. However the partition functions of these theories contain an infinite sum of products of holomorphic times antiholomorphic Virasoro characters, as predicted by a theorem of [3]. In the particular case $R^2$ is rational, these products group into a finite sum of holomorphic times antiholomorphic blocks. Furthermore this is the complete list of $c=1$ theories possessing this property as proven in [4].

Next we consider conformal field theory on surfaces with boundaries [5]. On the finite cylinder it is possible to construct a partition function in two ways. Either through a closed string propagating between two boundary states or through an open string satisfying corresponding boundary conditions propagating around a loop. Compatibility between these two points of view gives two conditions on the boundary states. One is the Ishibashi condition [6] arising

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from the restriction of the conformal field theory to the upper half plane (necessary in the open string picture) and the other is the Cardy condition \([7]\) \([8]\) which arises from the representational compatibility of the two constructions. Cardy in particular showed that given the Verlinde formula and a complete set of Ishibashi states it is possible to construct a consistent set of boundary states for diagonal theories. In the particular case the representation space of the theory splits into a direct sum of irreducible representations of an extended \(\hat{su}(2)\) chiral algebra the Ishibashi states have been constructed in \([6]\). The case of the \(\mathbb{Z}_2\) orbifold model at radius 1 has been considered in \([9]\). In this case boundary states have been derived that do not correspond to bulk operators.

In this work we study the rational torus models from the boundary CFT point of view. We distinguish a particular class of models \((R = \frac{1}{\sqrt{2k}})\) which possesses a consistent set of Neumann boundary states (or dually Dirichlet boundary states) in the sense of \([7]\). These boundary states have been constructed explicitly. Next the extended symmetry algebra of these theories has been written down. Finally the infinite direct sum of Fock modules, that constitutes the chiral representation space of this particular set of torus models, is shown to decompose into a finite sum of irreducible extended algebra modules.

# Torus Models Revisited

These models correspond to a free massless real bosonic field compactified on a circle of radius \(R\). The action in this case is

\[
S = \frac{1}{2\pi} \int \partial X \bar{\partial} X
\]

(1)

The partition function for these models turns out to be

\[
Z(\beta) = \sum_{m,n} \frac{q^{\frac{1}{2} \left( \frac{m^2}{2R} + nR \right)^2}}{\eta(q)} \frac{\bar{q}^{\frac{1}{2} \left( \frac{m^2}{2R} - nR \right)^2}}{\eta(\bar{q})}
\]

(2)

Here \(q = e^{2\pi i \tau} = e^{-2\pi \beta}, \beta = -i\tau\), and \(\eta(q)\) is the Jacobi eta function. The representation space for these models can be read off from the partition function to be

\[
H = \bigoplus_{n,m \in \mathbb{Z}} F_{\alpha_{m,n}} \otimes F_{\bar{\alpha}_{m,n}}
\]

(3)

where the possible charges are \(\alpha_{m,n} = \frac{m}{2R} - nR\) and \(\bar{\alpha}_{m,n} = \frac{m}{2R} - nR\).

In the particular case \(R^2 = \frac{p}{p'}\) the partition function becomes a finite sum of holomorphic times antiholomorphic parts:

\[
Z(q) = \sum_{r=0}^{2p-1} \sum_{s=0}^{2p'-1} f_{r,s}(q) f_{r,-s}(\bar{q})
\]

(4)

where

\[
f_{r,s}(q) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2} \left( \frac{np^2}{2p'} + \frac{np}{2p'} + \frac{np'}{2p} \right)^2} \frac{1}{\eta(q)}
\]

(5)

Now consider the free field more closely. We can view our field as describing an open string propagating along a strip periodic in the horizontal direction but with boundary conditions
along the end point lines of the string. What we really have in this way is an open string propagating along the sides of a cylinder, while it obeys boundary conditions on the end point circles. The length of the circumference will be taken to be $\beta$, while the height of the cylinder will be $1/2$. This one loop open string diagram is equivalent to a closed string propagating along the cylinder from one boundary to the other. The boundaries of the closed string are to be interpreted as states. Following the closed string picture we will take time to be the vertical direction. The free field admits now the following expansion into oscillator modes:

$$\phi(\sigma, t) = \hat{x} + \frac{2\pi}{\beta} R \hat{w} \sigma + \frac{\pi}{\beta} \hat{p} t + \frac{1}{2} \sum_{n \neq 0} \left( \frac{a_n}{n} e^{-\frac{2\pi}{\beta} n(t+\sigma)} + \frac{\tilde{a}_n}{n} e^{-\frac{2\pi}{\beta} n(t-\sigma)} \right)$$  (6)

This splits into left and right modes according to

$$\phi(\sigma, t) = \phi_L(x^+) + \phi_R(x^-)$$  (7)

where $x^+ = t + \sigma$ and $x^- = t - \sigma$. Here

$$\phi_L(x^+) = \frac{\hat{x}}{2} + \frac{\pi}{\beta} a_0 x^+ + \frac{1}{2} \sum_{n \neq 0} \frac{a_n}{n} e^{-\frac{2\pi}{\beta} n x^+}$$

$$\phi_R(x^-) = \frac{\hat{x}}{2} + \frac{\pi}{\beta} \tilde{a}_0 x^- + \frac{1}{2} \sum_{n \neq 0} \frac{\tilde{a}_n}{n} e^{-\frac{2\pi}{\beta} n x^-}$$  (8)

where $a_0 = \hat{p}/2 + R \hat{w}$ and $\tilde{a}_0 = \hat{p}/2 - R \hat{w}$. The Hamiltonian in the closed string picture turns out to be

$$H_\beta = \frac{2\pi}{\beta} \left[ (R \hat{w})^2 + (\hat{p}/2)^2 + \sum_{n=1}^\infty a_{-n} a_n + \sum_{n=1}^\infty \tilde{a}_{-n} \tilde{a}_n - 1/12 \right]$$  (9)

Using the oscillator mode representation of the Virasoro generators

$$L_m = \frac{1}{2} \sum_{-\infty}^\infty : a_{m-n} a_n :$$

$$\bar{L}_m = \frac{1}{2} \sum_{-\infty}^\infty : \tilde{a}_{m-n} \tilde{a}_n :$$  (10)

we have that

$$H_\beta = \frac{2\pi}{\beta} (L_0 + \bar{L}_0 - 1/12)$$  (11)

The boundary states of the closed string have to satisfy the Ishibashi condition [6]:

$$(L_n - \bar{L}_{-n}) \mid B \rangle = 0$$  (12)

This is certainly satisfied if

$$(a_m \pm \tilde{a}_{-m}) \mid B \rangle = 0$$  (13)

The plus sign correspond to Newmann boundary conditions in the open string picture while the minus sign correspond to Dirichlet boundary conditions [9]. Solving the conditions (13) we get the following Ishibashi states:

$$\mid \iota_N \rangle = e^{- \sum_{n=1}^\infty \frac{a_{-n} \tilde{a}_{-n}}{n}} \mid p = 0, w = nR \rangle$$

$$\mid \iota_D \rangle = e^{\sum_{n=1}^\infty \frac{a_{-n} \tilde{a}_{-n}}{n}} \mid p = \frac{m}{2R}, w = 0 \rangle$$  (14)
Note that in the Newmann case the zero mode condition demands that there is no momentum while in the Dirichlet case it demands that there is no winding. Now we have the following lemma:

**Lemma 1** The Ishibashi states (14) give rise to the following inner products:

\[
\begin{align*}
\langle \iota_{nR}^N | e^{-H_\beta/2} | \iota_{mR}^N \rangle &= \frac{\tilde{q}^{n^2_m}}{\eta(\tilde{q})} \delta_{n,m} \\
\langle \iota_{n/2R}^D | e^{-H_\beta/2} | \iota_{m/2R}^D \rangle &= \frac{\tilde{q}^{n^2_m}}{\eta(\tilde{q})} \delta_{n,m} \\
\langle \iota_{nR}^N | e^{-H_\beta/2} | \iota_{m/2R}^D \rangle &= \tilde{q}^{-1} \frac{1}{2\pi} \prod_{n=1}^{\infty} \frac{1}{1 + \tilde{q}^n} \delta_{n,0} \delta_{m,0} = \frac{\sum_{n \in \mathbb{Z}} (-1)^n \tilde{q}^{n^2}}{\eta(\tilde{q})} \delta_{n,0} \delta_{m,0}
\end{align*}
\]

where \( \tilde{q} = e^{-2\pi i/\beta} \).

The proof of this lemma is a simple free field calculation. Note that it is possible to interchange the Newmann and Dirichlet Ishibashi states by interchanging the directions of \( t \) and \( \sigma \).

There is however another condition that must be satisfied by the boundary states [7]. This states that if \( |X_A> \) are the boundary states, then

\[
Z_{AB}(\beta) = \langle X_A | e^{-H_\beta/2} | X_B > = \sum_i n_{AB}^i \chi_i(q)
\]

for some integers \( n_{AB}^i \) where \( q = e^{-2\pi i/\beta} \). In particular there is a special vacuus state \( |X_0> \) for which \( n_{0j} = \delta_{i,j} \). This condition arises from the possibility to see \( Z_{AB}(\beta) \) as an one loop amplitude of an open string satisfying the boundary conditions A,B on the boundary circles.

Now lets restrict ourselves to the Newmann sector. Consider the case \( R^2 \) is rational, and lets demand that there is a Newmann vacuum boundary state

\[
|X_0^N> = \sum_{n \in \mathbb{Z}} C_n |\iota_{nR}^N>
\]

Of course demanding a Dirichlet vacuum boundary state is completely equivalent because of the \( t, \sigma \) interchange duality. Consider the partition function on the cylinder with two vacuum boundary states at the boundary circles:

\[
Z_{00}(\beta) = \langle X_0^N | e^{-H_\beta/2} | X_0^N > = \sum_{n \in \mathbb{Z}} |C_n|^2 \tilde{q}^{n^2} \frac{\eta(\tilde{q})}{\eta(\tilde{q})}
\]

According to the Cardy condition [7], \( Z_{00}(\beta) \) must be equal to the vacuum character. To change the variable from \( \tilde{q} \) to \( q \), we need to use the Poisson resummation formula, which in this case takes the form:

\[
\sum_{n \in \mathbb{Z}} \tilde{q}^{A(n+b)^2} \frac{1}{\eta(\tilde{q})} = \sqrt{2A} \sum_{n \in \mathbb{Z}} q^{2\pi i n b} e^{2\pi i n b} \frac{\eta(q)}{\eta(q)}
\]
In the case of the rational theories and in order to be able to get the identity character after use of formula (19) there must exist a minimum integer \( m \) such that the constants \( C_n \) in the classes \( n \equiv r \mod m \) are equal. This gives rise to the following partition function:

\[
Z_{00}(\beta) = \sum_{r=0}^{m-1} |C_r|^2 \sum_{n \in \mathbb{Z}} \frac{q^{\frac{n^2}{2m^2}(n+m)^2}}{\eta(q)} = \sum_{r=0}^{m-1} \frac{|C_r|^2}{mR} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{n^2}{2m^2}} e^{2\pi in\frac{r}{m}}}{\eta(q)}
\]  

(20)

Now letting \( n = mn' + s \) we get

\[
Z_{00}(\beta) = \sum_{r,s=0}^{m-1} \frac{|C_r|^2}{mR} e^{2\pi irs/m} \sum_{n \in \mathbb{Z}} \frac{q^{\frac{(n'+s/m)^2}{2m^2}}}{\eta(q)}
\]  

(21)

Since \( s/m < 1 \), the power \( q^{1/2R^2} \) appears only when \( s = 0 \), and taking into account that in the vacuum character only integer powers of \( q \) appear and that \( \sum_{r=0}^{m-1} \frac{|C_r|^2}{mR} \neq 0 \) we have that

\[
R^2 = \frac{1}{2k}.
\]

These are the values of the radious that we are going to analyse further. For these values the closed string partition function regroups to

\[
Z(\beta) = \sum_{s=0}^{2k-1} \chi_s(q) \chi_{-s}(\bar{q})
\]

(23)

where

\[
\chi_s(q) = \sum_{n \in \mathbb{Z}} q^{\frac{k(n+s)^2}{2}} / \eta(q)
\]

(24)

The vacuum character must be a partial sum of \( \chi_0(q) \). Comparing this with \( Z_{00}(\beta) \) we get that \( s = 0 \), which in turn means that \( \chi_0(q) \) itself is the vacuum character. Furthermore it is necessary that all \( |C_r| \) are equal, so that the value \( s = 0 \) is the only one that survives the sum over \( r \). So, without loss of generality we can assume that \( m = 1 \). In this case we have

\[
Z_{00}(\beta) = \frac{|C|^2}{R} \sum_{n \in \mathbb{Z}} \frac{q^{kn^2}}{\eta(q)}
\]

(25)

Equality with \( \chi_0(q) \) demands that \( |C| = \sqrt{R} = \frac{1}{\sqrt{2k}} \). So the corresponding vacuum state is

\[
|X_0^N> = \frac{1}{\sqrt{2k}} \sum_{n \in \mathbb{Z}} |\psi_{n/\sqrt{2k}}^N> = \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} \sum_{n \in \mathbb{Z}} |\psi_{\sqrt{2k}(n+l/2k)}^N>
\]

(26)

Now it is necessary (because of the Cardy condition) that

\[
Z_{0m}(\beta) = \chi_m(q) = \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} e^{\frac{2\pi ilm}{2k}} \chi_l(\bar{q})
\]

(27)

This demands that the other boundary states are

\[
|X_m^N> = \frac{1}{\sqrt{2k}} \sum_{l=0}^{2k-1} e^{\frac{2\pi ilm}{2k}} \sum_{n \in \mathbb{Z}} |\psi_{\sqrt{2k}(n+l/2k)}^N>
\]

(28)
It is not too difficult to check that
\[ Z_{m_1 m_2} = \frac{1}{2k} \sum_{l,m=0}^{2k-1} e^{2\pi i \frac{l(m_2 - m_1 + m)}{2k}} \chi_m(q) = \chi_{m_1 - m_2}(q) \] (29)

This is certainly an integer sum of characters so the Cardy condition is satisfied. It is worth mentioning that since the construction of the boundary states is based on the modular properties of the characters the Verlinde formula gives integer fusion rules as was necessary. In particular we can read off from (27) that
\[ S^i_j = \frac{1}{\sqrt{2k}} e^{2\pi ilj} \] and from the Verlinde formula
\[ \sum_{i=0}^{2k-1} S^i_j N^i_{k' l} = S^i_j S^j_i / S^0_j \] (30)

we can read the fusion rules
\[ N^i_{k' l} = \delta_{(i,k' + l) \mod 2k} \] (31)

3 Extended Symmetry Algebra

We will now try to find which extended symmetry gives rise to the characters \( \chi_m(q) \) for each \( k \). Consider first the identity character
\[ \chi_0(q) = \sum_{n \in \mathbb{Z}} q^{kn^2} = q^{-1/24} \frac{1 + 2q^k + 2q^{4k} + \cdots + 2q^{n^2 k} + \cdots}{(1 - q)(1 - q^2) \cdots (1 - q^k)(1 - q^{k+1}) \cdots} \] (32)

Expanding in \( q \) we get
\[ \chi_0(q) = q^{-1/24} \left( 1 + P(1)q + P(2)q^2 + \cdots + P(k-1)q^{k-1} \right) + \]
\[ + q^k \left[ [P(k) + 2] + [P(k+1) + 2P(1)]q + \cdots + [P(4k-1) + 2P(3k-1)]q^{3k-1} \right] + \cdots + \]
\[ + q^{n^2 k} \left[ [P(n^2 k) + \cdots + 2] + \cdots + [P((n+1)^2 k - 1) + \cdots + 2P((2n+1)k - 1)]q^{(2n+1)^2 k - 1} \right] + \]
\[ + \cdots \] (33)

Here \( P(n) \) is the number of partitions of \( n \). Now the Fock module built on the zero charge vacuum, \( F_0 \), has \( P(n) \) linearly independent states at level \( n \). From the above expansion we see that at level \( k \) we have two extra linearly independent states that are orthogonal to \( F_0 \), and since they are the first such states they have to be killed by all \( a_n \) for \( n \) positive. So they can be taken to be highest weight Fock states of charge \( \alpha nm = \pm \sqrt{2k} \) because the \( L_0 \) eigenvalue \( h_{nm} = k \). They are generated by the spin \( k \) currents
\[ J^\pm(z) = e^{\pm i \sqrt{2k} \phi(z)} \] (34)

Here we have taken \( \phi(z) = 2\phi_L(x^+) \) and \( z = e^{2\pi i x^+ / \beta} \). Clearly, \( J^\pm \) adds charge \( \pm \sqrt{2k} \) on the states it acts. There is of course another current, of spin 1, the \( U(1) \) current which we take to be
\[ J^0(z) = \frac{1}{\sqrt{2k}} i \partial \phi(z) \] (35)

Now we have the following lemma:
Lemma 2 The modes of the above currents generate the space $\bigoplus_{n \in \mathbb{Z}} F_{n\sqrt{2k}}$.

Proof:

To prove this consider the mode expansion of the $J^+(z)$ current acting on the zero charge vacuum:

$$\sum_{n \in \mathbb{Z}} J^+_n z^{-n-k}|0> = e^{\sqrt{2k} \sum_{n>0} \frac{a_{-n} z^n}{n}} |\sqrt{2k}> $$  \hspace{1cm} (36)

where by $|\sqrt{2k}>$ we denote the vacuum state of charge $\sqrt{2k}$. From this relation we see that

$$J_0^+|0> = 0, \quad J_1^+|0> = 0, \quad \cdots \quad J_{k+1}^+|0> = 0 \quad J_{k}^+|0> = |\sqrt{2k}>.$$  \hspace{1cm} (37)

Similarly we get that $J_{-k}^-|0> = |\sqrt{2k}>$. These are the extra states at level $k$. Of course since the modes of $J^0(z)$ can act on those states all the Fock descendents of the states $|\pm \sqrt{2k}>$ are generated by the current modes. If we move to level $k+r$ then the number of them is $2P(r)$. Together with the descendents of $|0>$ we have overall $P(k+r) + 2P(r)$ states. This takes account of the second bracket in (33). But at level $4k$ there are again two more states. To take account of them consider the product

$$J^+(z)J_-^+|0> = J^+(z)|\sqrt{2k}> = z^k e^{\sqrt{2k} \sum_{n>0} \frac{a_{-n} z^n}{n}} |\sqrt{2k}>$$  \hspace{1cm} (38)

This means that we have

$$J_0^+J_1^+|0> = 0, \quad J_1^-J_0^+|0> = 0, \quad \cdots \quad J_{3k+1}^-J_{3k}^+|0> = 0 \quad J_{3k}^-J_{3k}^+|0> = |2\sqrt{2k}>.$$  \hspace{1cm} (39)

Considering also $J_{-3k}^+J_{-k}^-|0>$ we get the extra two states at level $4k$. Again at level $4k+r$ we have all the Fock descendents of $|0>$, $J_{k}^+|0> = |\pm \sqrt{2k}>$, and $J_{-3k}^+J_{-k}^-|0> = |\pm 2\sqrt{2k}>$. The number of these states is $P(4k+r) + 2P(3k+r) + 2P(r)$. Continuing in this fashion we get two extra states appearing at every level of the form $n^2k$, as indicated by (32). They are of the form

$$J_{-(2n-1)k}^\pm \cdots J_{-3k}^\pm J_{-k}^\pm|0> = |n\sqrt{2k}>$$  \hspace{1cm} (40)

Of course all the Fock descendents of these states are created by the $J^0$ modes so we have that the current modes generate completely the direct sum $\bigoplus_{n \in \mathbb{Z}} F_{n\sqrt{2k}}$. This ends the proof of the lemma.

The next question is what are the OPE’s satisfied by these high spin currents. This is not too difficult to obtain. These are the following:

$$J^0(z)J^\pm(w) = \pm \frac{1}{z-w} J^\pm(w) + reg $$

$$J^+(z)J^-(w) = \frac{1}{(z-w)^2 k} e^{\phi(z) - \phi(w)} + reg $$  \hspace{1cm} (41)

The second operator product have to be expressed in terms of the currents. This is possible by expanding $\phi(z)$ near $w$:

$$J^+(z)J^-(w) = \frac{1}{(z-w)^2 k} e^{\phi_O(w)} e^{2k \left[ (z-w)J^0(w) + \cdots + \frac{(z-w)^{2k-1}}{(2k-1)!} \phi^{2k-2}(w) \right] + reg}$$  \hspace{1cm} (42)
Here by $\exp_{O(2k-1)}$ we mean expansion up to and including terms of order $(z-w)^{2k-1}$. At this point it is worth mentioning that our product $J^+(z)J^-(w)$ is a bilocal field in the sense of \cite{10}. There it was remarked that for $k=1/2$ (unaccepted value for us) this product is the generating function for the $W_{1+\infty}$ algebra. Here however we will restrict our attention to integer values of $k$. Now, the product (42) can be expanded in terms of Schur polynomials. In general the Schur polynomials are defined by

$$e^{\sum_{k=1}^{\infty} t_k z^k} = \sum_{N=0}^{\infty} z^N S_N(t_1, t_2, \ldots, t_k, \ldots)$$  \hspace{1cm} (43)$$

and they turn out to be

$$S_N(t_1, t_2, \ldots, t_k, \ldots) = \sum_{n_1, \ldots, n_k, \ldots} t_1^{n_1} \cdots t_k^{n_k} \frac{n_1! \cdots n_k!}{n_1! \cdots n_k!}$$  \hspace{1cm} (44)$$

Let us now define, following \cite{10}, the associated polynomials:

$$f^l(\partial^l J^0(z)) = (l)! : S_l \left( J^0(z), \frac{\partial J^0(z)}{2!}, \ldots, \frac{\partial^{l-1} J^0(z)}{l!} \right) :$$  \hspace{1cm} (45)$$

They satisfy the following recurrence relation:

$$f^{l+1}(\partial^l J^0(z)) = (J^0(z) + \partial) f^{l}(\partial^l J^0(z)) = \cdots = (J^0(z) + \partial)^l J^0(z)$$  \hspace{1cm} (46)$$

Using now these new definitions we have

$$J^+(z)J^-(w) = \frac{1}{(z-w)^{2k}} \sum_{N=0}^{2k-1} S_N \left( 2k J^0(w), \ldots, 2k \partial 2k-2 J^0(w)/(2k-1)! \right) (z-w)^N + \text{reg} =$$  \hspace{1cm} (47)$$

$$= \sum_{N=0}^{2k-1} \frac{f^N(2k \partial^k J^0(w))}{N!} (z-w)^{N-2k} + \text{reg}$$

The next question is what is the algebra satisfied by the modes of the currents $J^+(z)$, $J^-(z)$. This can be read of from the OPE’s:

$$[J^+_n, J^-_m] = \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^{n+k-1} w^{m+k-1} J^+(z)J^-(w)$$  \hspace{1cm} (48)$$

where the left integral is around the origin and the right is around $w$. Expanding the product of currents as above we get

$$[J^+_n, J^-_m] = \sum_{N=0}^{2k-1} \frac{\Gamma(n+k)}{\Gamma(2k-N)\Gamma(N+n-k+1)} \oint \frac{dw}{2\pi i} w^{n+m+N-1} f^N(2k \partial^k J^0(w))$$  \hspace{1cm} (49)$$

This suggests the following definition:

$$V_{m+n,N}^{k} = \oint \frac{dw}{2\pi i} w^{n+m+N-1} f^N(2k \partial^k J^0(w)) = \oint \frac{dw}{2\pi i} w^{n+m+N-1} (2k J^0(w) + \partial)^{N-1} 2k J^0(w)$$  \hspace{1cm} (50)$$

where we have made use of the recurrence relation (46). In this notation (49) becomes

$$[J^+_n, J^-_m] = \sum_{N=0}^{2k-1} \frac{\Gamma(n+k)}{\Gamma(2k-N)\Gamma(N+n-k+1)\Gamma(N+1)} V_{n+m,N}^{k}$$  \hspace{1cm} (51)$$
Following the same procedure for the remaining commutators we get

\[ [J^0_n, J^0_m] = \frac{n}{2k} \delta_{n+m} \quad [J^0_n, J^\pm_m] = \pm J^\pm_{m+n} \]  

(52)

while the other commutators are 0.

Note that in particular if \( k = 1 \) then \( V^1_{m+n,0} = \delta_{m+n} \) and \( V^1_{m+n,1} = 2J^0_{m+n} \), so in this case the commutator (51) becomes

\[ [J^+_n, J^-_m] = n \delta_{m+n} + 2J^0_{m+n} \]  

(53)

giving rise to the \( su(2) \) current algebra. Nevertheless it is only for \( k = 1 \) that this algebra closes so nicely. In general the operators \( V^k_{m+n} \) belong to the universal enveloping algebra of the U(1) current.

### 4 Primary Fields

Now that we have the extended symmetry algebra, the next question is what are the primary fields with respect to this algebra. To answer this we need to consider the characters \( \chi_m(q) \).

Under \( q \to e^{2\pi i}q \) we pick a phase \( e^{2\pi i \left( \frac{m^2}{4} - \frac{1}{4} \right)} \). So the conformal dimension of the corresponding field is \( h_m = \frac{m^2}{4k} \), \( 0 \leq m \leq 2k - 1 \). There are two Virasoro primary fields with this conformal dimension, the fields

\[ V^\pm_m(z) = e^{\pm i \frac{m}{\sqrt{2k}} \phi(z)} \]  

(54)

Observe that for \( m = 2k \) we get the two currents.

These primary fields satisfy the following operator product expansions:

\[ J^0(z)V^\pm_m(w) = \pm \frac{m}{2k} \frac{V^\pm_m(w)}{z-w} + \text{reg} \]

\[ J^+(z)V^-_m(w) = \frac{1}{(z-w)^m} : V^+_m(w)e^{2k(z-w)J^0(w)+2k \frac{(z-w)^2}{2!}\partial J^0(w)+\cdots+2k \frac{(z-w)^{m-1}}{(m-1)!}\partial^{m-2} J^0(w)} : + \text{reg} \]

\[ J^-(z)V^+_m(w) = \frac{1}{(z-w)^m} : V^-_m(w)e^{-2k(z-w)J^0(w)-2k \frac{(z-w)^2}{2!}\partial J^0(w)-\cdots-2k \frac{(z-w)^{m-1}}{(m-1)!}\partial^{m-2} J^0(w)} : + \text{reg} \]  

(55)

where the remaining OPE’s are trivial. Note that we can take the fields \( V^+_m(z) = e^{i \frac{m}{\sqrt{2k}} \phi(z)} \), \( 0 \leq m \leq 2k - 1 \) as a complete set of primary fields, since they are related to the \( V^-_m(z) \) through the last of the relations (55).

### 5 Module Structure

The primary fields \( V^+_m(z) \) add charge \( m/\sqrt{2k} \) to the vacuum so we have

\[ V^+_m(0)|0> = \frac{m}{\sqrt{2k}} |0> \]  

(56)
Recall now that the chiral Fock space associated with the partition function was $H = \bigoplus_{n,m \in \mathbb{Z}} F_{\alpha_{nm}}$, where $\alpha_{nm} = n\sqrt{2k} + \frac{m}{\sqrt{2k}}$. The currents now can only add charges that are multiples of $\sqrt{2k}$. So it is natural to decompose the space $H$ into the sum

$$H = \bigoplus_{m=0}^{2k-1} \mathcal{H}_m$$

where

$$\mathcal{H}_m = \bigoplus_{n \in \mathbb{Z}} F_{n\sqrt{2k} + m\sqrt{2k}}$$

is the space that is generated by the currents from $V_n^\pm(0)|0> = \frac{m}{\sqrt{2k}}$.

Let's examine now more closely what is the action of the currents on $\mathcal{H}_m$. Again by expanding the character $\chi_m(q)$ as a power series in $q$ we see that there is one extra state appearing at the levels $n^2k - nm + m^2/4k$ and $n^2k + nm + m^2/4k$ for all positive integers $n$. Considering the product $J^+(z)|m/\sqrt{2k}>$ we get

$$\sum_{n \in \mathbb{Z}} J^+_nz^{-n-k}|m/\sqrt{2k}> = \sum_{n \in \mathbb{Z}} J^+_n(3m+n)e^{\sqrt{2k}\sum_{n>0} \frac{a-n}{n}z^n}|\sqrt{2k} + m/\sqrt{2k}>$$

This implies that

$$J^+_{-(k+m)}|m/\sqrt{2k}> = |\sqrt{2k} + m/\sqrt{2k}>$$

and $J^+_r|m/\sqrt{2k}> = 0$ for all $r < k + m$. Applying $J^+(z)$ on this new state a number of times we get eventually that

$$J^+_{-(2n-1)k+m} \cdots J^+_{-(3k+m)} J^+_{-(k+m)}|m/\sqrt{2k}> = |n\sqrt{2k} + m/\sqrt{2k}>$$

These states account for the extra states at the levels $n^2k + nm + m^2/4k$. Applying $J^-(z)$ similarly a number of times on the state $m/\sqrt{2k}$ we get

$$J^-_{-(2n-1)k-m} \cdots J^-(3k-m) J^-_{-(k-m)}|m/\sqrt{2k}> = |n\sqrt{2k} + m/\sqrt{2k}>$$

and these states account for the extra states at levels $n^2k - nm + m^2/4k$.

Suppose now that the space $\mathcal{H}_m$ is reducible, as a module of our current algebra. Then there must be a state $|v_m>$ other than $|\frac{m}{\sqrt{2k}}>$. This forces us to consider highest weight Fock states. Such states in the module $\mathcal{H}_m$ are the states $|m/\sqrt{2k} + n\sqrt{2k}>$ for $n$ integer. Suppose now that $n > 0$. Applying $J^-_{-(z)}$ we get

$$\sum_{l \in \mathbb{Z}} J^-_lz^{-l-k}|m/\sqrt{2k} + n\sqrt{2k}> = z^{-(m+2nk)} e^{-\sqrt{2k}\sum_{l>0} \frac{a}{n}l^l}|m/\sqrt{2k} + (n-1)\sqrt{2k}>$$
and this in turn implies that

\[ J_{m+(2n-1)k}^- |m/\sqrt{2k} + n\sqrt{2k} > \neq 0 \]  

(66)

Since \( m + (2n-1)k \) is positive for positive \( n \), \( J_r^- |m/\sqrt{2k} + n\sqrt{2k} > \) cannot be 0 for all positive \( r \). If now \( n = -n' < 0 \) then we need to apply \( J^+(z) \). In this case we get

\[ J_{-m+(2n'-1)k}^+ |m/\sqrt{2k} - n'\sqrt{2k} > = |m/\sqrt{2k} - (n' - 1)\sqrt{2k} > \neq 0 \]  

(67)

Since \( -m + (2n'-1)k \) is positive for positive \( n' \) (negative \( n \)), \( J_r^+ |m/\sqrt{2k} + n\sqrt{2k} > \) cannot be 0 for all positive \( r \). So \( \mathcal{H}_m \) cannot be reducible. Hence we have the following theorem:

**Theorem 1** The space \( \mathcal{H} = \bigoplus_{n,m \in \mathbb{Z}} F_{\alpha_{nm}} \) where \( \alpha_{nm} = n\sqrt{2k} + m/\sqrt{2k} \) admits the decomposition \( \mathcal{H} = \bigoplus_{m=0}^{2k-1} \mathcal{H}_m \) into irreducible modules of the algebra generated by the modes of the spin \( k \) currents \( J^\pm(z) \) and the spin 1 current \( J^0(z) \). In terms of Fock modules we have \( \mathcal{H}_m = \bigoplus_{n \in \mathbb{Z}} F_{\frac{m}{\sqrt{2k}} + n\sqrt{2k}} \).

6 Conclusions

What we have achieved in this work is to single out a family of rational torus models by demanding the existence of a consistent set of Newmann (dually Dirichlet) boundary states. For this models we have written down the extended symmetry algebra which restricts the number of blocks to a finite number. Furthermore it is shown that the space of these torus models, which is an infinite sum of Fock modules, splits into a direct sum of a finite number of irreducible extended algebra modules.

The extended symmetry algebra that has appeared is a W type algebra since it contains high spin currents. It should be thought of as a generalization of the SU(2) current algebra at level one, a theory which corresponds to \( k = 1 \) in our list of theories. It is of interest to identify the algebras corresponding to general level, getting in this way theories that may not possess free field representations. Some work in this direction has been done in the context of W-algebras by [11]. A study of the representation theory of such algebras may give new examples of rational conformal field theories.

References


