Abstract

I review the equivalence between duality operators on two-forms and conformal structures in four dimensions, from a Clifford algebra point of view (due to Urbantke and Harnett). I also review an application, which leads to a set of "neighbours" of Einstein's equations. An attempt to formulate reality conditions for the "neighbours" is discussed.
There is a deep theory for how to solve the self-dual Yang-Mills equations

\[ *F_{\alpha\beta i} = g^{-1/2}g_{\alpha\gamma}g_{\beta\delta}\tilde{F}_{\gamma\delta} = 1/2g^{-1/2}g_{\alpha\gamma}g_{\beta\delta}\epsilon^{\gamma\delta\mu\nu}F_{\mu\nu i} = F_{\alpha\beta i} \quad (1) \]

where the duality operator is defined with respect to some fixed conformal structure, i.e. a metric up to a conformal factor (and some useful notation - the twiddle - has been introduced as well). Some time ago it occurred to Urbantke (1984) to pose this problem backwards: Given a field strength, with respect to which conformal structure is it self-dual? There is an elegant solution to this curious question, and an elegant proof - due to Urbantke and Harnett (1991) - based on the Clifford algebra of two-forms in four dimensional spaces. For the moment, let me state the result and then indicate how I want to use it. We need a triplet of two-forms, which is non-degenerate in the sense that it may serve as a basis in the three-dimensional space of self-dual two-forms. In particular, the index i ranges from one to three. Then

\[ g_{\alpha\beta} = -2/3\eta f_{ijk}F_{\alpha\gamma i}\tilde{F}_{\gamma\delta j}F_{\delta\beta k} \quad (2) \]

is Urbantke’s formula. It gives the metric with respect to which \( F_{\alpha\beta i} \) is automatically self-dual (the \( f_{ijk} \) are the structure constants of SO(3), and the conformal factor \( \eta \) is so far arbitrary).

Some work by Capovilla, Jacobson and Dell (1989) may be regarded as a more ambitious version of Urbantke’s formula. (See also Plebanski 1977, Capovilla, Dell, Jacobson and Mason 1991.) We may regard \( F_{\alpha\beta i} \) as the self-dual part of the Riemann tensor, considered - at the outset - as just an SO(3) field strength, with no connection to the metric. Then the question arises whether it is possible to formulate a set of differential equations, using the SO(3) connection (and the Levi-Civita tensor densities) alone, such that the above metric becomes Ricci flat. The answer turns out to be yes; more specifically, the answer is the field equations following from the action

\[ S = 1/8 \int \eta (Tr\Omega^2 - 1/2(Tr\Omega)^2) \quad (3) \]

where \( \eta \) is a Lagrange multiplier and

\[ \Omega_{ij} = \epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta i}F_{\gamma\delta j} \quad (4) \]
The existence of this action is closely related to Ashtekar’s (1987) formulation of the 3+1 version of Einstein’s theory - in fact the CDJ action is a natural Lagrangian formulation of Ashtekar’s variables. The action which leads to Einstein’s equations including a cosmological constant is less elegant.

The next question is: What happens if we use the above building blocks to write an arbitrary action

\[ S = \int \mathcal{L}(\eta; Tr\Omega, Tr\Omega^2, Tr\Omega^3), \]

where the only restriction on \( \mathcal{L} \) is that it has density weight one? (Due to the characteristic equation for three-by-three matrices, there are only three independent traces.) The action is certainly generally covariant. Suppose that we solve the field equations and use Urbantke’s formula to define a metric. Is that reasonable, and relevant for physics? What happens if we change the structure group from \( \text{SO}(3) \) to something else?

Now that you know where I am going, we return to prove Urbantke’s formula. For any four-dimensional vector space \( V \), the two-forms give a six-dimensional vector space \( W \), with a natural metric

\[ (\Sigma_1, \Sigma_2) = 1/2 \epsilon^{\alpha\beta\gamma\delta}\Sigma_1^{\alpha\beta}\Sigma_2^{\gamma\delta}. \]

There is a corresponding Clifford map to the space of endomorphisms on \( V \oplus V^* \):

\[ \gamma(\Sigma) = 2 \begin{pmatrix} 0 & \Sigma^\alpha_\beta \\ \Sigma^\alpha_\beta & 0 \end{pmatrix}; \quad \gamma(\Sigma)^2 = -(\Sigma, \Sigma) 1. \]

We see that the original vector space \( V \) now becomes the space of Weyl spinors for the Clifford algebra of two-forms.

Now we introduce a metric on \( V \), so that we can define the duality operator \( * \). \( W \) then splits into two orthogonal subspaces \( W^+ \) (self-dual forms) and \( W^- \) (anti-self-dual forms). We choose Euclidean signature, so that \( ** = 1 \), and without loss of essential generality we choose the determinant of the metric to equal one. Then

\[ \gamma(*\Sigma) = \gamma(Z)\gamma(\Sigma)\gamma(Z) \]

where
\( \gamma(Z) = \begin{pmatrix} 0 & g^{\alpha \beta} \\ g_{\alpha \beta} & 0 \end{pmatrix}. \) (9)

Using a well-known property of six-dimensional \( \gamma \)-matrices, and Swedish indices in \( \mathbf{W} \), we can find a totally anti-symmetric tensor \( Z^{\tilde{u} \tilde{a} \tilde{o}} \) such that

\[
g_{\alpha \beta} = Z^{\tilde{u} \tilde{a} \tilde{o}} \Sigma_{\alpha \gamma \tilde{u}} \Sigma_{\tilde{a} \gamma \delta} \Sigma_{\delta \tilde{o}}. \] (10)

This determines \( Z \) uniquely, and we observe that

\[
*\Sigma = Z \Sigma Z \quad \Rightarrow \quad Z \Sigma = \Sigma Z \quad (\Sigma \in W^+) \quad Z \Sigma = -\Sigma Z \quad (\Sigma \in W^-). \] (11)

We need a little bit more information about \( Z \).

To prove the result we are after, we will commit the atrocity of choosing a basis in \( \mathbf{W} \). First we choose an ON-basis in \( \mathbf{V} \), and then we set

\[
M_i = e_0 \wedge e_i \\
N_i = 1/2 f_{ijk} e_j \wedge e_k; \quad X_i = M_i - N_i \\
Y_i = M_i + N_i. \] (12)

Clearly, the \( X \)'s (\( Y \)'s) form a basis for \( \mathbf{W}^- \) (\( \mathbf{W}^+ \)), and

\[
(X_i, X_j) = -\delta_{ij} \quad (Y_i, Y_j) = \delta_{ij}. \] (13)

Looking back on eq. (11), we see that we can set

\[
Z = Y_1 Y_2 Y_3 \] (14)

- that is to say that \( Z \) is the unit volume element of \( \mathbf{W}^+ \). But, since \( \mathbf{W}^+ \) is three-dimensional, this is all we need. In terms of an arbitrary basis \( \Sigma_{\alpha \beta i} \) on \( \mathbf{W}^+ \), eq. (10) now becomes

\[
g_{\alpha \beta} \propto \epsilon^{ijk} \Sigma_{\alpha \gamma i} \Sigma_{\gamma j} \Sigma_{\delta \tilde{k}}. \] (15)

This is Urbantke’s formula.
When the metric on $V$ has neutral signature, the metric on $W^+$ becomes indefinite, but the discussion is similar, while it becomes slightly more subtle if the metric on $V$ is Lorentzian.

With this understanding of eq. (2), let us return to the action (5). Our main result so far (Capovilla 1992, Bengtsson and Peldán 1992, Bengtsson 1991, Peldán 1992) is that this action admits a 3+1 decomposition, and that the resulting formalism is a natural generalization of "Ashtekar’s variables" for gravity. As is well known, the constraint algebra of general relativity actually singles out the space-time metric by means of its structure functions. For the SO(3) case, it turns out that - up to some ambiguity concerning the conformal factor - the "Hamiltonian" metric is precisely the same as Urbantke’s. We refer to the models in this class as "neighbours of Einstein’s equations", since they all have the same number of degrees of freedom. I will not discuss the case of arbitrary structure groups here.

There are several holes that have to be filled before we can claim that we have really been able to generalize Einstein’s equations in an unsuspected way. For the case of Euclidean signatures, we have to show that the field equations derived from the action (5) ensure that the metric (2) is positive definite, rather than neutral. This can be done in specific cases. As an example, consider the action

$$S = \frac{1}{8} \int \eta (Tr\Omega^2 + \alpha (Tr\Omega)^2).$$

(16)

As is clear from the preceding discussion, there must be some property of the field equations that ensure that the matrix $\Omega_{ij}$ has definite signature. To see this, choose a gauge such that the matrix becomes diagonal. Then it is a straightforward exercise to show that the constraint that results when varying the action with respect to $\eta$ implies that the matrix $\Omega_{ij}$ has definite signature if and only if

$$\alpha \geq -1/2.$$ 

(17)

In particular, $\alpha = -1/2$, which leads to Einstein’s equations, is all right. (I owe this observation to Ted Jacobson.) Although it is not quite clear what a general statement is, it is clear that, in general, the requirement that the metric should have Euclidean signature will lead to some restrictions on the allowed actions.

A similar discussion can be given for neutral signature, provided that the definition of the traces in the action is appropriately changed.
Our understanding of the Lorentzian case is in much worse shape. It is necessary to show that propagation is causal with respect to the metric that we have defined. Moreover (since self-dual two-forms are necessarily complex in this case) the variables in the action are complex valued, and one must show how to impose restrictions that imply that the metric is real in any solution. I believe that the latter problem is the crucial one, and that the former property somehow follows from the latter. It will not come as a surprise if I state that the conformal structure is real if and only if

\[ (F_i, \overline{F}_j) = 0, \]  

where the bar denotes complex conjugation. However, this condition is not very helpful in itself. It is not difficult to write down solutions with real Lorentzian metrics - a small zoo of real solutions is already known, for various "neighbours" (generalizations of Schwarzschild, de Sitter, Kasner, ...). On the other hand, there will always be some solutions for which the metric is not real - also in the Einstein case. The correct formulation of the problem is presumably to require that the space of real solutions should be "reasonably" big - of the same order as the space of solutions of Einstein’s equations, say. It seems natural to switch to the Hamiltonian form of the equations, and to address the problem from an initial data point of view. Unfortunately, as soon as this is done, one discovers that the reality properties of the metric can be discussed easily (Ashtekar 1987) if and only if we deal with the Einstein case - for the more general models contained in the action (5), the calculations tend to b

Which is where the matter stands at the moment. It is perhaps appropriate to add that we have investigated, in a preliminary way, whether the "neighbours" can be used to explain any property of the real world. The preliminary answer was not very encouraging, but perhaps the final verdict is not in yet. Certainly the more difficult case of arbitrary structure groups (Peldán 1992), which was not discussed here, should be carefully studied in this regard.

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REFERENCES


