Are SP91 and COBE Inconsistent with Cold Dark Matter?

Emory Bunn, Martin White, Mark Srednicki* & Douglas Scott

Center for Particle Astrophysics
and Departments of Astronomy and Physics,
University of California, Berkeley, CA 94720

ABSTRACT

We present results on the consistency of standard cold dark matter (CDM) models with both the COBE normalization and the data from the 4th channel of the 9-point South Pole scan. We find that CDM models are consistent with both experiments, a conclusion which is at odds with some other analyses. This is partly due to our careful treatment of the temperature autocorrelation function, but also derives from a realization that the statistical conclusions depend strongly on assumptions about the prior distribution for the normalization $Q$.

* On leave from Department of Physics, University of California, Santa Barbara, CA 93106.

Subject headings: cosmic microwave background — cosmology: theory
1. Introduction

There has been some confusion in the literature concerning the implications for cold dark matter (CDM) models of the results of the COBE (Smoot et al. 1992) and SP91 (Gaier et al. 1992) experiments. In particular, there have been contradictory claims (e.g. Dodelso & Jubas 1993, Górski et al. 1993) about the fundamental question: how incompatible are standard CDM models with these anisotropy experiments? Our intention is to provide a definitive answer to this question, giving the details of our procedure.

We analyze in detail the data from the 4th (highest frequency) channel of the SP91 9-point scan (Gaier et al. 1992). This data set seems most likely to represent actual cosmic microwave background (CMB) fluctuations, and is the data set previously analyzed by Górski (1992), Dodelso & Jubas (1993) and Górski et al. (1993). We wish to stress three points about our analysis:

(1) We use the theoretical two-point autocorrelation function predicted for the SP91 measurement strategy by standard CDM models ($\Omega_0 = 1$, $h = 0.5$, $n = 1$, $0.01 < \Omega_n < 0.10$, $\Lambda = 0$, $T/S = 0$). We notice that points whose angular separation is near the peak-to-peak chopping angle have strong negative correlations. This result, explained below, is entirely due to chopping, and is completely independent of the underlying cosmological model. We compare the CDM autocorrelation function with the autocorrelation function assumed by Gaier et al. (1992) in their analysis.

(2) We quote results for an experiment-independent parameter, which we take to be $Q \equiv Q_{\text{rms-PS}} \equiv \langle Q_{\text{rms}}^2 \rangle^{0.5}$, the r.m.s. value of the quadrupole moment averaged over an ensemble of universes. In contrast, the usually quoted values of $C_0$ as a function of $\theta_c$ (see below) must be disentangled from the observing strategy to find information on the underlying spectrum of fluctuations. Deconvolving theory from experiment is essential in obtaining a meaningful comparison of experiments with different strategies measuring different parts of the sky. We remind the reader that the value of $Q$ measured by COBE (Smoot et al. 1992, Wright et al. 1993) from the overall normalization of the experimental two-point correlation function (assuming CDM with $n = 1$ and no gravity wave contribution) is $Q = 17 \pm 5 \mu$K. The quoted error includes the effects of cosmic and sample variance (e.g. White et al. 1993, Scott et al. 1993).

(3) We perform a maximum likelihood analysis, but investigate sensitivity to the “prior distribution”; that is, should we assume that, a priori, equal intervals in $Q$ are equally
likely (the usual assumption), or equal intervals in \(Q^2\) (which is simply proportional to the amplitude of the power spectrum), or some other choice? Different choices turn out to change the upper limit on \(Q\) from SP91 (at various confidence levels) by tens of per cent. Note that the choice of prior becomes less important in the limit that the data have great “discriminating power”. At present, sensitivity to the prior can be used as a measure of how constraining the 9-point scan is.

2. The Temperature Autocorrelation Function

We begin with the theoretical two-point autocorrelation function for CDM models. Let 
\[ T(\theta, \phi) = Q \sum_{\ell m} a_{\ell m} W_{\ell m} Y_{\ell m}(\theta, \phi), \]
where the \(W_{\ell m}\)’s represent the window function of the experiment, and the \(a_{\ell m}\)’s are random variables whose distribution must be specified by a specific cosmological model. In general, rotational invariance implies that
\[
\langle a_{\ell m} a_{\ell' m'}^* \rangle = C_\ell \delta_{\ell \ell'} \delta_{m m'},
\]
where the angular brackets denote an ensemble average over the probability distribution for the \(a_{\ell m}\)’s, and \(C_\ell\) is normalized so that \(C_2 = 4\pi/5\). For a pure Sachs–Wolfe, \(n = 1\) spectrum, \(C_\ell^{-1} \propto \ell(\ell + 1)\). We compute the \(C_\ell\)’s for CDM models using power spectra provided by Sugiyama (e.g. Sugiyama & Gouda 1992), which are essentially identical with those computed by Bond & Efstathiou (e.g. 1987).

The theoretical two-point autocorrelation function is given by
\[
C_{\text{th},ij} = Q^{-2} \langle T(\theta_i, \phi_i)T(\theta_j, \phi_j) \rangle = \sum_{\ell m} C_\ell |W_{\ell m}|^2 Y_{\ell m}(\theta_i, \phi_i) Y_{\ell m}^*(\theta_j, \phi_j),
\]
where we have divided by \(Q^2\) to make \(C_{\text{th}}\) dimensionless; there is then no \(Q\) dependence at all in \(C_{\text{th}}\). Note that \(C_{\text{th}}\) is calculated using an underlying theory (e.g. CDM) together with the experimental window function, and does not depend on any experimental data. Thus \(C_{\text{th}}\) is not the same thing as the experimental correlation function measured by the
COBE team, which is computed by taking a sky average of data without any theoretical input.

All the points in the 9-point scan of SP91 have the same zenith angle but different azimuth. In (3) we can write \( \theta_i = \theta = 27.75 \) and \( \phi_i = \Delta \phi \times i \) with \( \Delta \phi \sin \theta = 2.1 \) being the angular spacing of points on the sky. Furthermore the chopping is done in the \( \phi \) direction, which implies that

\[
W_{\ell m} = 2H_0(m\alpha) \exp[-(\ell + \frac{1}{2})^2 \sigma^2/2]
\]  

(Bond et al. 1991, Dodelson & Jubas 1993, White et al. 1993), where \( H_0 \) is the Struve function (Gradshteyn & Ryzhik 1980), \( \alpha \sin \theta = 1.5 \) is half the peak-to-peak chopping angle on the sky, and \( \sigma = 0.425 \times 1.4 \) is the Gaussian beam width of the antenna. Thus, equation (3) becomes

\[
C_{\text{th},ij} = \sum_{\ell=2}^{\infty} C_\ell \exp[-(\ell + \frac{1}{2})^2 \sigma^2] \sum_{m=-\ell}^{\ell} 4H_0^2(m\alpha)|Y_{\ell m}(\theta,0)|^2 \cos(m\phi_{ij}),
\]  

where \( \phi_{ij} = \Delta \phi \times |i - j| \). We find that \( C_{\text{th}} \) is sensitive to the value of \( \sigma \), and expect that departures from a Gaussian beam profile may also significantly affect the results.

In Figure 1, we show \( C_{\text{th}} \) as a function of \( \phi \sin \theta \), the separation angle on the sky, for \( \Omega_\text{m} = 0.03 \). We see that it becomes strongly negative for \( \phi \sin \theta \) near 3\(^\circ\), corresponding to the peak-to-peak chopping angle on the sky, \( 2\alpha \sin \theta \). This is easy to understand: the region halfway between the two negatively correlated points contributes positively to one point and negatively to the other, which will clearly produce an anticorrelation.

Results of small-scale CMB experiments are often quoted assuming an underlying Gaussian autocorrelation function (GACF); this corresponds to taking

\[
Q^2 C_\ell = 2\pi \theta_c^2 C_0 \exp[-(\ell + \frac{1}{2})^2 \theta_c^2/2],
\]  

where \( \theta_c \) and \( C_0 \) are parameters. In Figure 1, we show the corresponding \( C_{\text{th}} \) as a function of \( \phi \sin \theta \) for \( \theta_c = 1.2 \) and \( C_0/Q^2 = 7.9 \). The choice of \( \theta_c = 1.2 \) is the “best-fit” value of Gaier et al. (1992), which matches the peak of equation (6) to the peak of the SP91 window function in multipole space. The choice of \( C_0/Q^2 = 7.9 \) results in \( C_{\text{th}}(0) = 4.9 \), the same value as in the CDM model with \( \Omega_\text{m} = 0.03 \). Notice that for these particular choices of \( \theta_c \) and \( C_0 \), the anti-correlation at the nearest-neighbor separation of \( \Delta \phi \sin \theta = 2.1 \) is well reproduced. This agreement would not hold for general values of \( \theta_c \) and \( C_0 \).
The signal-to-noise of the experiment is currently such that experimental errors contribute large diagonal entries to the correlation matrix [see equation (8) below], reducing the effect of off-diagonal entries on the fit. If the signal-to-noise were to improve significantly, then the off-diagonal entries would play a larger role. As an example, with the current data we find that a diagonal $C_{th,ij}$ produces a very similar limit to that obtained in Gaier et al. (1992), showing that the assumed off-diagonal terms are playing little role in the analysis. This can also be seen in Table 1, where the upper limits on $Q$ for a given choice of prior distribution (see below) simply scale like $C_{th}^{-1/2}(0)$ as $\Omega_m$ is varied.

3. Maximum Likelihood Analysis

We now turn to the limits which can be placed on $Q$. In accord with standard CDM models we consider underlying cosmological fluctuations with a Gaussian probability distribution (not to be confused with fluctuations having a Gaussian power spectrum, or GACF). We assume that the experimental errors $\sigma_i$ for each $T_i$ are uncorrelated and Gaussian distributed. Then the unnormalized likelihood function for $Q$ is given by

$$\mathcal{L}(Q) \propto \frac{1}{\sqrt{\det K}} \exp\left[-\frac{1}{2} T_i (K^{-1})_{ij} T_j\right], \quad (7)$$

where the matrix $K$ is

$$K_{ij} = Q^2 C_{th,ij} + \sigma_i^2 \delta_{ij}. \quad (8)$$

Equation (7) assumes that the temperatures have no systematic errors. In fact, SP91 has a possible, unknown, systematic offset and linear gradient, which are removed from the $T_i$. This requires in turn a modification of $K$. To impose the constraint that the quoted temperatures have zero offset and linear gradient we write (following Schuster 1993) $\bar{T}_a = R_{aj} T_j$ with $R_{aj} = (a + 1/2) P_a(\psi_j)$ and $\psi_j = (j - 5)/4$. Here $a = 2, \ldots, 8$, $j = 1, \ldots, 9$ and $P_a$ is a Legendre polynomial. Instead of equation (7) the unnormalized likelihood function is now given by

$$\mathcal{L}(Q) \propto \frac{1}{\sqrt{\det M}} \exp\left[-\frac{1}{2} \bar{T}_a (M^{-1})_{ab} \bar{T}_b\right], \quad (9)$$

where $M_{ab} = R_{ai} K_{ij} R^T_{jb}$. This procedure is equivalent to that used by Bond et al. (1991).

We can now use equation (9) to set confidence levels on $Q$. We must first choose a prior distribution for $Q$; that is, we must decide whether equal intervals of $Q$, $Q^2$, or some
other monotonic function \( f(Q) \), are equally likely a priori. Given a choice of \( f(Q) \), the upper limit on \( Q \) at a confidence level of \( c \) is the solution \( Q_{\text{max}} \) to the equation

\[
c = \frac{\int_0^{Q_{\text{max}}} \mathcal{L}(Q) \, df(Q)}{\int_0^\infty \mathcal{L}(Q) \, df(Q)}.
\]  

(10)

Due to the large measurement errors in the data set, assuming \( f(Q) = \log Q \), as recommended for an overall scale factor, can be problematic (see e.g. Readhead et al. 1989). One is thus led to consider prior distributions which are uniform in the scaling variable. The usual choice (Bond et al. 1991, Dodelson & Jubas 1993, Górski et al. 1993) is \( f(Q) = Q \) (the “bias” parameter \( b_\rho \) is proportional to \( Q^{-1} \), so that \( dQ = db_\rho/b_\rho^2 \)). However, there is no compelling reason to assume a prior distribution that is uniform in \( Q \). It is just as natural, for example, to assume that the prior distribution is uniform in the power spectrum normalization \( Q^2 \), or to make use of the COBE measurement of \( Q \). The choice of \( f(Q) \) is important because it strongly affects the resulting values of \( Q_{\text{max}} \). Note that for “good” data, the choice of the prior distribution should make very little difference, so “prior dependence” gives us a handle on the constraining power of the data. This is illustrated in Figure 2, where we show the cumulative likelihood (i.e. \( c \) vs. \( Q_{\text{max}} \)) for four different choices of \( \Omega_b \) and for \( f(Q) = Q \) and \( f(Q) = Q^2 \). Clearly the effect of the prior distribution is significant. This sensitivity may be understood as being due to limited sky coverage (see Scott et al. 1993).

In Figure 2 we also show a band corresponding to the COBE measurement of \( Q = 17 \pm 5 \mu K \) (1\( \sigma \) errors). We feel that this plot makes it clear that CDM models cannot be ruled out at high levels of confidence from a combination of the COBE and SP91 9-point data. This is also shown in Table 1, where we list the upper limits on \( Q \) at the 95% confidence level for the eight combinations of \( \Omega_b \) and choice of prior distribution \( f(Q) = Q \) and \( f(Q) = Q^2 \).

4. Conclusions

A reanalysis of the SP91 data from the 9-point scan shows no conflict with standard CDM models for any value of \( \Omega_b \) and for \( Q \) within the one-sigma range specified by the COBE results, \( Q = 17 \pm 5 \mu K \), in contradiction with the conclusions of some earlier analyses. We find that the window function for SP91 is sensitive to the beam size assumed
and suspect that deviations from a Gaussian beam pattern may significantly affect the results.

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Table 1: The 95% confidence level upper limits, $Q_{\text{max}}$ ($\mu$K), from the 9-point scan for CDM with a range of $\Omega_b$ and for prior distributions $dQ$ and $d(Q^2)$. Also listed is the value of $C_{\text{th}}(0)$ for each $\Omega_b$.

References


**Figure Captions**

Fig. 1. The theoretical temperature autocorrelation function, $C_{th}$, as a function of scan angle on the sky, $\phi \sin \theta$ (\(\phi\) is the azimuthal angle and \(\theta\) the zenith angle). The solid line is the autocorrelation function assuming the CDM power spectrum for $\Omega_m = 0.03$ and $h = 0.5$. The dashed line is the traditional Gaussian autocorrelation function (GACF) approximation, with a correlation angle $\theta_c = 1^\circ 2$, processed through the experimental observing strategy and matched to the CDM prediction at $\phi = 0$. For comparison, the corresponding power spectra, $\ell (\ell + 1) C_\ell$, for both theories are shown in the inset.

Fig. 2. The cumulative likelihood as a function of power spectrum normalization, $Q$ ($\mu$K), for CDM models with $\Omega_m = 0.01, 0.03, 0.06, 0.10$ ($\Omega_m$ increases to the left). Curves are labelled by the assumed prior distribution. The hatched band is the $\pm 1\sigma$ allowed range of $Q$ from COBE.