D=3: Singularities in gravitational scattering of scalar waves.

C. Klimčík

Nuclear Centre, Charles University, Prague
Czechoslovakia

and

P. Kolník

Department of Theoretical Physics, Charles University, Prague
Czechoslovakia

ICP classification number: 0430.

Abstract. Family of exact spacetimes of D=3 Einstein gravity interacting with massless scalar field is obtained by suitable dimensional reduction of a class of D=4 plane-symmetric Einstein vacua. These D=3 spacetimes describe collisions of line-fronted asymptotically null excitations and are generically singular to the future. The solution for the scalar field can be decomposed into the Fourier-Bessel modes around the background solitons. The criteria of regularity of incoming waves are found. It is shown that the appearance of the scalar curvature singularities need not stem from singularities of incoming waves. Moreover, in distinction to D=4 case, for all solutions with regular incoming waves the final singularities are inevitable.
§1. Introduction

Motivations for studying the Einstein gravity coupled to source field in D=3 are natural and were discussed many times in literature since the interest in the subject had appeared [1, 2]. Clearly, the absence of dynamical degrees of freedom for the pure Einstein gravity, as the crucial property of spacetimes with dimension 3, makes possible to identify all degrees of freedom of interacting systems with those of the matter field [1, 2]. This feature is obviously attractive from the point of view of quantum theory since one may identify the matter field degrees of freedom and attempt to quantize them. Such D=3 quantum gravity should elucidate a lot of intriguing questions arising in attempts to quantize gravity in D=4. One expects, in particular, that quantum corrections to the classical dynamics will be substantial near the ubiquitous singularities of classical general relativity. What, then, is the status of singularities in the quantum theory?

Consulting the literature on D=3 gravity it may seem that the curvature singularities are somewhat less ubiquitous and, in general, milder than their counterparts in D=4. Indeed, much effort was devoted to the study of spacetimes describing gravitating (moving) point masses [2, 3], in which only mild quasiregular singularities appear, located at the world lines of the particles. Such spacetimes are constructed by appropriate glueing of flat pieces in order to satisfy the Einstein equations with the point-like sources and they are distinguished rather by their global topological properties than by their local structure. The interaction of massive particles was actually studied by various methods in series of papers even at the quantum level [3–8], however, the problem of singularities was not the central one in these considerations.

We feel, however, that the local structure of matter solutions may itself represent an interesting problem to pursue. Indeed, the stronger singularities (i.e. not quasiregular) are not alien to D=3 general relativity, e.g. as the “collapsing dust” solution of [1] shows. Such curvature singularities, however, then automatically imply the necessary singularities of the matter stress tensor in distinction to D=4 case. This is not, to our point, a substantial flaw of D=3 general relativity, since the singularities of a matter distribution may well develop dynamically from a smooth nonsingular distribution of the matter in the past. Moreover, degrees of freedom responsible for the creation of the singularity are exclusively those of the matter field, hence their quan-
tization should reveal much about a quantum status of singularities. For a quantitative study of such project we need a tractable example of classical formation of the curvature singularity. The singularities occurring in classical scattering processes seem to be the best starting point for considering their quantum picture, since, the scattering processes are those best addressed in quantum field theory.

As is well-known, there exist a lot of solutions in D=4 general relativity describing collisions of gravitational plane-waves in which, generically, the scalar curvature singularities are created in the future [9–15]. These singularities can be physically interpreted as arising due to mutual focusing of colliding waves. This interpretation stems from the behaviour of a congruence of null geodesics hitting transversally the wave-front of a single D=4 gravitational plane-wave. The congruence gets focused to the line after crossing the wave [16]. The location of this focal line corresponds to the location of curvature singularities of the full spacetime describing the collision of the waves. In D=3 we observe a similar behaviour of geodesics hitting the wave-front of a single line-wave (see §2), hence a creation of curvature singularities in collisions of D=3 scalar field line-wave excitations may be expected. We shall see, in what follows, that it indeed happens. Moreover, the field equations get effectively linearized and one can obtain a mode decomposition of the matter scalar field around some (solitonic) background configuration. This feature seems to be particularly attractive from the point of view of further quantization.

From the technical point of view, we obtain D=3 solutions by suitable dimensional reduction of D=4 vacua describing collisions of collinear plane-waves [15]. The field equation for one component of D=4 metric turns out to be identical to that for D=3 massless scalar field\(^1\). Similarly, suitable combination of remaining D=4 metric components fulfil the same equations as the components of D=3 metric. In this way, we may readily translate D=4 vacua into D=3 matter solutions. What differs substantially, however, are conditions for regularity of incoming waves and a criterion for avoiding the final singularity. We shall show that D=3 “incoming” regularity conditions are more restrictive. This restriction is so strong that it excludes, in distinction to D=4 case, solutions avoiding the final singularity.

The paper is organized as follows. In §2 the properties of single line-wave

\(^1\)Some matter solution in D=4 were also constructed by reduction of D=5 vacua [13].
solutions in $D=3$ are discussed. The $D=3$ field equations for the colliding waves are obtained and compared to $D=4$ equations in §3 and the class of $D=4$ vacuum solutions is dimensionally reduced to give $D=3$ matter solutions in §4. In §5 the criteria for avoiding an incoming parallelly propagated non-scalar curvature singularities are found and solutions fulfilling these criteria are identified. The scalar curvature singularities are studied in §6 and in §7 we furnish conclusions and outlook.

§2. Line-waves in $D=3$.

Plane-wave spacetimes in general relativity in arbitrary dimension $D$ are metrics of the form

$$ds^2 = -dUdV + h_{ij}(U)X^iX^j dU^2 + dX^i dX_i$$  \hspace{1cm} (2.1)

where $U, V$ and $X^i (i=1,...,D-2)$ are so-called Brinkmann coordinates\(^2\). The exact form of the amplitude $h_{ij}(U)$ is given by Einstein equations, e.g. for the vacuum case one obtains the condition

$$\sum_{i=1}^{D-2} h_{ii}(U) = 0.$$  \hspace{1cm} (2.2)

It is obvious that for $D=3$ (where we use the term “line-waves” instead of “plane-waves”) (2.2) gives $h(U)=0$, i.e. the flat metric, as it should.

We shall not consider, for the moment, the dynamically allowed form of $h(U)$ in $D=3$ but, rather, we turn to the behaviour of geodesics in the metric (2.1). The most effective way to do it is to change coordinates putting the metric in the so-called Rosen form

$$ds^2 = -du dv + F^2(u) \, dx^2$$  \hspace{1cm} (2.3)

where, from now on, we shall consider the case $D=3$. The transformation from Brinkmann to Rosen coordinates is given by

$$U = u, \quad V = v + \frac{x^2}{2} F(u) F'(u), \quad X = \frac{xF(u)}{\sqrt{2}};$$  \hspace{1cm} (2.4)

\(^2\)The metric (2.1) describes propagation of disturbance of geometry in $Z(\equiv U - V)$ direction with velocity of light.
thus
\[ h(U) = \frac{F''(u)}{F(u)}. \] (2.5)

The symmetries of the line-wave metric (2.1) are clearly more explicit in the Rosen form (2.3), where the \( x \)-dependence drops out. There is a drawback of Rosen coordinates, however, that they do not cover all manifold by a simple chart as the Brinkmann coordinates do. This fact can be shown to follow easily from Einstein equations for scalar field coupled to D=3 gravity. One supposes that the metric (2.3) is flat for all \( u<u_w \) (\( w \equiv \) wavefront) and shows that for some \( u_0 > u_w \) the transformation (2.4) breaks down as \( F(u_0)=0 \).

It is easy to find a congruence of null geodesics crossing the wave front transversely. It contains lines
\[ v = v_c, \quad x = x_c. \] (2.6)

In Brinkmann coordinates, however, the congruence looks as follows
\[ V(u) = v_c + \frac{x_c^2}{2} F(u)F'(u), \quad X(u) = \frac{x_c F(u)}{\sqrt{2}} \] (2.7)

where \( u \) may be taken as the affine parameter. Clearly, at \( u_0=0 \), the family gets focused to the point \( V=v_c, X=0 \).

Assume that we consider a scattering of two line-waves described by \( F(u) \) and \( G(v) \) respectively and propagating in the opposite direction with sharp wave-fronts, i.e. \( F(u)=G(v)=\sqrt{2} \) for \( u<u_w, v<v_w \); \( G(v) \) plays role of \( F(u) \) in (2.3) for the other wave. The wave-front \( W_v \) of \( v \)-wave will be modelled in \( u \)-wave Rosen coordinates (2.3) by null geodesics with \( v_c=v_w \) and \( x_c \) varying over the \( x \)-axis. We draw the wave-front \( W_v \) in both Rosen (Fig.1a) and Brinkmann (Fig.1b) coordinates of \( u \)-wave. One may draw Fig.1a with \( x \) coordinate suppressed as in Fig.2. There we can see that the crossing wave-front \( v_w=0 \) cuts out the part of Rosen spacetime indicated by hatch. As Fig.1b shows, we cannot continue the remaining part over \( u > u_0 \) without crossing wave-front \( v_w=0^3 \). The apparent half plane \( u = u_0, v < 0 \) in Fig.2 is in fact the half line in the Brinkmann coordinates, playing the role of a seam

\[ ^3 \text{This continuation for single line-wave is possible showing that the } u_0 \text{ is only a coordinate singularity. In the case of colliding waves, however, } u_0 \text{ are truly singular points of the spacetime.} \]
in the crossing wave-front. Therefore, we expect on the basis of presented considerations, that the spacetime describing the colliding waves will look in the Rosen form as in Fig.3a where \( u = u_0 \) and \( v = v_0 \) are seams on the wave-fronts of colliding waves and represent themselves singularities of the spacetime. The character of these singularities depends on the amplitudes \( h(U) \) and \( \tilde{h}(V) \) of the incoming waves, as we shall see later.

We feel, however, that the waves with sharp wave-fronts are not very physically realistic. Therefore, we expect the colliding solutions as drawn in Fig.3b, which may be considered as the asymptotic version of Fig.3a. The jagged lines in the null infinities are singular points of the spacetime.

§3. Colliding waves spacetimes.

As the analysis of §2 suggests, we shall seek for a metric for colliding waves in the Rosen coordinates starting with a natural ansatz

\[
ds^2 = -e^{-K(u,v)} du \, dv + e^{-N(u,v)} dx^2.
\] (3.1)

This ansatz comes from analogy with higher dimensional case and we restrict ourselves to the metrics with one spacelike Killing vector.

The field equations with scalar field as a source then read

\[
N_{uu} - \frac{1}{2} N_u^2 + N_u K_u = 2\kappa \phi_u \phi_u
\] (3.2a)

\[
N_{vv} - \frac{1}{2} N_v^2 + N_v K_v = 2\kappa \phi_v \phi_v
\] (3.2b)

\[
K_{uv} = \kappa \phi_u \phi_v
\] (3.2c)

\[
N_{uv} = \frac{1}{2} N_u N_v
\] (3.3)

\[
4\phi_{uv} - N_u \phi_v - N_v \phi_u = 0
\] (3.4)

where

\[
\kappa = 8\pi G.
\] (3.5)

The equation (3.3) can easily be integrated to give

\[
N = -2 \ln(1 - f(u) - g(v))
\] (3.6)
where \( f(u) \) and \( g(v) \) are arbitrary functions.

It can be shown that for solutions of (3.4) with \( N \) given by (3.6), the equations (3.2abc) are automatically integrable.

Solutions of (3.2), (3.3) and (3.4) describing the collisions of line-waves have to satisfy certain boundary conditions. Two types of such boundary conditions are discussed in the literature about colliding gravitational waves in D=4. First one was introduced at the birth of the subject [9, 10] and corresponds to spacetimes with flat initial region I \((u < 0, v < 0)\), two regions II and III \((0 < u < u_0, v < 0)\) resp. \(0 < v < v_0, u < 0)\) with metrics of the single line-wave solution and an interacting region IV where the waves interact (Fig.3a). The complete solution \(K, N, \phi\) then has to fulfil prescribed junction conditions on the wave-fronts \(u=0\) and \(v=0\) respectively.

The second type of boundary conditions was introduced recently by Hayward [15] where the semiinfinite flat piece I of spacetime is excluded and the waves are supposed to interact for all times with prescribed asymptotic behaviour in the past timelike and null infinities (Fig.3b). Such a prescription obviously avoids problems with junction conditions on the wave-fronts and may be considered even physically more realistic. We shall discuss solutions of the second type in D=3. All functions \(K, N, \phi\) are supposed to be smooth and to satisfy following asymptotic conditions

\[
(K, N, \phi)(u \to -\infty, v \to -\infty) = 0 \tag{3.7a}
\]
\[
(K, N, \phi)(u \to -\infty, v) = (K, N, \phi)(v) \tag{3.7b}
\]
\[
(K, N, \phi)(u, v \to -\infty) = (K, N, \phi)(u) \tag{3.7c}
\]

Rescaling \(u\) and \(v\) by arbitrary functions of \(u\) and \(v\) respectively, \(N\) and \(\phi\) remain unchanged and \(K\) changes additively. Hence coordinate freedom is encoded in \(K\) and (3.7a) fixes the coordinates. Then (3.7b) and (3.7c) subject to constraints (3.2ab) play the role of initial data.

Nontrivial solutions of equations (3.2), (3.3) and (3.4) can be found easily by suitable dimensional reduction of vacuum solutions describing the collisions of gravitational waves. Indeed, taking D=4 ansatz \(^4\) [15]

\[
ds^2 = -2e^{-M}du \, dv + e^{-P}(e^{Q}dx^2 + e^{-Q}dy^2) \tag{3.8}
\]

\(^4\)We consider the case of colliding waves with aligned polarisations, i.e. without the term proportional to \(dzdy\).
the D=4 Einstein equations read [15]

\[2P_{uu} - P_u^2 + 2M_u P_u - Q_u^2 = 0\]  
(3.9a)

\[2P_{vv} - P_v^2 + 2M_v P_v - Q_v^2 = 0\]  
(3.9b)

\[2M_{uv} + P_u P_v - Q_u Q_v = 0\]  
(3.9c)

\[P_{uv} - P_u P_v = 0\]  
(3.10)

\[2Q_{uv} - P_u Q_v - Q_u P_v = 0\]  
(3.11)

It is easy to see that the equations (3.9), (3.10) and (3.11) give exactly equations (3.2), (3.3) and (3.4) respectively after taking

\[P = \frac{N}{2}, \quad M = K - \frac{N}{4}, \quad Q = \sqrt{2\kappa} \phi\]  
(3.12)

Moreover the D=4 asymptotic conditions [15]

\[(M, P, Q)(u \to -\infty, v \to -\infty) = 0\]  
(3.13a)

\[(M, P, Q)(u \to -\infty, v) = (M, P, Q)(v)\]  
(3.13b)

\[(M, P, Q)(u, v \to -\infty) = (M, P, Q)(u)\]  
(3.13c)

give, after the substitution (3.12), the D=3 asymptotic conditions (3.7).

Hayward [15] has shown how to obtain easily solutions of the Einstein equations (3.9), (3.10) and (3.11) fulfilling the asymptotic conditions (3.13) knowing the solutions satisfying “standard” boundary conditions (i.e. with semiinfinite flat wedge to the past). We feel that he made an important contribution carefully examining criteria for regularity of incoming waves. Without this information one could not exclude the possibility that the generic curvature singularities are caused simply by singular incoming waves. He has shown, in fact, that the creation of the curvature singularities to the future is generic even for the regular incoming waves \(^5\), though, in some special cases, the formation of the singularity can be avoided. In our paper we shall study

---

\(^5\) The criteria of the “incoming” regularity in D=4 turned out to be rather restrictive, e.g. all solutions given by Szekeres [11] have singular incoming waves, except the single Khan-Penrose solution describing the collisions of the impulsive waves. The incoming singularities for the Szekeres’ family of solutions were studied by Konkowski and Helliwell [17].
the same set of solutions from the D=3 point of view. In particular, we shall identify the criteria for the “incoming” regularity which turns out to be even more restrictive than D=4 criteria. We evaluate also D=3 conditions for avoiding the formation of the final singularities and show that for colliding regular incoming waves the final singularity is inevitable.

§4. Solutions of field equations.

The general solution for $\phi$ fulfilling (3.4) can be written in the form [14, 15]

$$
\phi = k \ln(1 - f - g) + p \cosh^{-1}\frac{1 + f - g}{1 - f - g} + q \cosh^{-1}\frac{1 - f + g}{1 - f - g}
+ \int_{0}^{\infty} [A(\omega)J_{0}(\omega(1 - f - g)) + B(\omega)N_{0}(\omega(1 - f - g))] \sin(\omega f - g) \, d\omega
$$

(4.1)

where the functions $f(u)$ and $g(v)$ were introduced in (3.6), $k, p, q$ are real numbers, the amplitudes $A(\omega), B(\omega)$ may be integrable functions or distributions and $J_{0}$ and $N_{0}$ are the zero-order Bessel and Neumann function respectively. The different terms in (4.1) possess different behaviour when $f + g \rightarrow 1^{-}$. The first term is obviously singular, so are the second and third terms which are usually referred to as gravitational solitons. The fourth term is regular but the fifth one again contributes to the divergence due to the behaviour of the Neumann functions near zero. We have to elucidate what restrictions on the general solution (4.1) come from the asymptotic conditions (3.7).

We choose, obviously, the functions $f(u)$ and $g(v)$ such that

$$
f(u = -\infty) = g(v = -\infty) = 0.
$$

(4.2)

Consider now Eq. (3.2a) at $v = -\infty$, i.e.

$$
\frac{f_{uu}}{1 - f} + \frac{f_{u}}{1 - f}K_{u} = \kappa \phi_{u} \phi_{u}
$$

(4.3)

hence

$$
\frac{f_{uu}}{f_{u}^{2}} = \kappa \phi_{f} \phi_{f}(1 - f) - K_{f}.
$$

(4.4)
We assume that the function \( f(u) \) (\( g(v) \)) is strictly monotonic. The asymptotic behaviour of \( \phi_f \) for \( f \sim 0 \) (and \( g=0 \)) reads [15]

\[
\phi_f \sim \frac{p}{\sqrt{f}} - C + p\sqrt{f} + ... \tag{4.5}
\]

where

\[
C = k - \int_0^\infty \omega [A(\omega)J_0(\omega) + B(\omega)N_0(\omega)]d\omega. \tag{4.6}
\]

Setting

\[
\frac{f_{uu}}{f_u^2} = H(f) \tag{4.7}
\]

where \( H \) is an appropriate function, we may integrate Eq. (4.4) to obtain the behaviour of \( K \) for \( f \sim 0 \), i.e. integrate

\[
K_f = -H(f) + \kappa \phi_f \phi_f (1 - f). \tag{4.8}
\]

We have for \( f \sim 0 \)

\[
\kappa \phi_f \phi_f (1 - f) \sim \kappa \left[ \frac{p^2}{f} + C^2 + p^2 - \frac{2pC}{\sqrt{f}} + ... \right] \tag{4.9}
\]

To ensure regular behaviour of \( K \) (or to compensate the term \( \frac{\kappa p^2}{f} \)) we must have

\[
H(f) = \tilde{H}(f) + \frac{\kappa p^2}{f} \tag{4.10}
\]

where

\[
\lim_{f \to 0} f \tilde{H}(f) = 0. \tag{4.11}
\]

Starting with \( \tilde{H}(f)=0 \), we have to solve the equation

\[
\frac{f_{uu}}{f_u^2} = \frac{\kappa p^2}{f} \tag{4.12}
\]

with the boundary condition \( f(-\infty)=0 \).

The latter condition excludes the cases \( \kappa p^2 < 1 \); hence, we have the following class of solutions\(^6\)

\[
f(u) = [-a (u - u_S)]^{1/(1-\kappa p^2)}, \text{ for } \kappa p^2 > 1 \tag{4.13a}
\]

\(^6\)We may also understand the equations (4.13) and (4.14) as a pure choice of coordinates, equivalent to fixing \( K \) on \( g = 0 \) and \( f = 0 \).
\[ f(u) = \exp[a (u - u_S)], \text{ for } \kappa p^2 = 1 \quad (4.13b) \]

where \( u_S \) is arbitrary and \( a \) is a positive number.

In an analogous way we obtain\(^7\)

\[ g(v) = [-b (v - v_S)]^{1/(1 - \kappa q^2)}, \text{ for } \kappa q^2 > 1 \quad (4.14a) \]

\[ g(v) = \exp[b (v - v_S)], \text{ for } \kappa q^2 = 1. \quad (4.14b) \]

We can conclude that for the choices of \( f(u) \) and \( g(v) \) (4.13) and (4.14) respectively, the presence of the solitonic terms in the general solution (4.1) is inevitable in order to fulfil the asymptotic conditions (3.7).

As is well-known, the solitonic terms are necessary in D=4 for standard colliding waves metrics in order to satisfy the junction conditions on the wavefronts \( u=0 \) and \( v=0 \) [14]. In our case, however, we do not need to cope with the junction conditions. One may therefore wonder whether the solitonic terms are really necessary in order to satisfy only the asymptotic conditions (3.7). Though we did not attempt to solve the problem in full generality, being satisfied with large classes (4.13) and (4.14) of allowed \( f(u) \) a \( g(v) \), we give, nevertheless, an indication that the solitonic terms are needed. Let us put \( p^2=0 \) in (4.9) and (4.10). Thus

\[ K_f = -\tilde{H}(f) + \kappa C^2 + ... \quad (4.15) \]

where \( \tilde{H}(f) \) satisfies condition (4.11). We choose for concreteness a power-like behaviour \((\alpha > 0)\)

\[ \tilde{H}(f) = f^{-1+\alpha} \quad (4.16) \]

and show that the condition \( f(-\infty)=0 \) is necessarily violated. We write

\[ \frac{f_{uu}}{f_u^2} = f^{-1+\alpha} \quad (4.17) \]

hence

\[ f_u = D \exp\left(\frac{1}{\alpha} f^\alpha\right). \quad (4.18) \]

Making a substitution

\[ r = \frac{1}{\alpha} f^\alpha \quad (4.19) \]

\(^7\)The choices (4.13b) and (4.14b) slightly enlarge the possible coordinate fixing of D=4 solutions obtained by Hayward [15].
we arrive at
\[ r_u = D(\alpha r)^{1 - \frac{1}{\alpha}} e^r \] (4.20)

thus
\[ D \alpha^{1 - \frac{1}{\alpha}} \int du = \int dr \, r^{\frac{1}{\alpha} - 1} e^{-r}. \] (4.21)

If \( f(-\infty) = 0 \), we have
\[ D \alpha^{1 - \frac{1}{\alpha}} \int_{-\infty}^u du' = \int_0^r dr' \, (r')^{\frac{1}{\alpha} - 1} e^{-r'}. \] (4.22)

Clearly the incomplete \( \Gamma \)-function is finite while the left-hand-side is infinite. Thus \( f(-\infty) \) cannot be equal to zero.

We complete our discussion by identifying the remaining unknown function \( K(u, v) \). Taking the general solution (4.1), the integration of constraints (3.2) cannot in general be performed explicitly, nevertheless, the processes of singularity formation can be studied (see §6). In particular case, however, when the Fourier-Bessel modes are absent, we may give \( K(u, v) \) even explicitly. It reads
\[ K = -\kappa (k^2 - (p + q)^2) \ln(1 - f - g) + \kappa q^2 \ln(1 - f) \]
\[ + \kappa p^2 \ln(1 - g) + 4\kappa pq \ln(\sqrt{fg} + \sqrt{1 - f} \sqrt{1 - g}) \] (4.23)
\[ - 2\kappa kp \ln\left(\frac{\sqrt{1 - g} + \sqrt{f}}{\sqrt{1 - g} - \sqrt{f}}\right) - 2\kappa kq \ln\left(\frac{\sqrt{1 - f} + \sqrt{g}}{\sqrt{1 - f} - \sqrt{g}}\right) \]

where \( f(u) \) and \( g(v) \) are given by (4.13) and (4.14) respectively. By construction this expression satisfies the asymptotic conditions (3.7).

§5. Criteria for “incoming” regularity.

We have identified in the previous sections a large class of solutions of the Einstein equations (3.2-4) in \( \text{D}=3 \) satisfying the asymptotic conditions (3.7). As we have already mentioned the asymptotic colliding line-waves interacting for all times seem better to correspond to physical reality than colliding waves with sharp wave-fronts. A question remains, however, whether all such
solutions (4.1) with \( f \) and \( g \) given by (4.13) and (4.14) are “enough” physically realistic. Though \( D=3 \) is itself not very physical dimension it makes sense to attempt a formulation of some criteria ensuring an “acceptability” of the process from \( D=3 \) point of view.

In the \( D=4 \) case Hayward [15] has postulated a criterion that parallelly propagated curvature singularities in the asymptotic caustics \( f=1, g=0 \) and \( f=0, g=1 \) should be absent. He has also shown that such a criterion amounts to boundedness of the amplitudes of the incoming gravitational waves in Brinkmann coordinates in the asymptotic caustics. We shall also claim in \( D=3 \) that parallelly propagated curvature singularities should be absent and show that this again amounts to the boundedness of the metric in Brinkmann coordinates. We note, nevertheless, that points in past null infinities marked by jagged lines in Fig.3b are in this case quasiregular but still singular points of the manifold.

We first identify the criterion for boundedness of the metric components in Brinkmann coordinates and pick up, for concreteness, the case \( g=0 \). In this null infinity the metric looks as follows

\[
\text{d}s^2 = -e^{-K(u,v=-\infty)}\text{d}udv + e^{-N(u,v=-\infty)}\text{d}x^2. \tag{5.1}
\]

We make the transformation of the coordinate \( u \) setting

\[
\frac{\text{d}u'}{\text{d}u} = e^{-K(u)}. \tag{5.2}
\]

Thus

\[
\text{d}s^2 = -\text{d}u'\text{d}v + e^{-N(u')}\text{d}x^2 \tag{5.3}
\]

and, following (2.5),

\[
h(u') = e^{\frac{N(u')}{2}}\frac{d^2}{du'^2}e^{-\frac{N(u')}{2}}. \tag{5.4}
\]

We rewrite (5.4) in the original coordinates

\[
h(u) = \frac{e^{2K}}{2}\left(\frac{N^2}{2} - N_{uu} - N_uK_u\right) = -e^{2K}\kappa\phi_u\phi_u \tag{5.5}
\]

where the last equality follows from the Einstein equation (3.2a).

We have to study behaviour of \( h(u) \) at \( f(u)=1 \). We first recall the asymptotic behaviour of the Bessel and Neumann functions for \( w \to 0^+ \)

\[
J_0(w) \sim 1 - \frac{w^2}{4} + ...
\]

13
\[ J'_0(w) \sim -\frac{w}{2} + \ldots \]
\[ N_0(w) \sim (1 - \frac{w^2}{4}) \ln w + \ldots \]  
\[ (5.6) \]
\[ N'_0(w) \sim \frac{1}{w} - \frac{w}{2} \ln w + \ldots \]

The scalar field solution (4.1) gives for \( g = 0 \)
\[ \phi = k \ln(1 - f) + p \ln \frac{1 + \sqrt{f}}{1 - \sqrt{f}} \]
\[ + \int_0^\infty [A(\omega)J_0(\omega(1 - f)) + B(\omega)N_0(\omega(1 - f))] \sin(\omega f) \, d\omega \]  
\[ (5.7) \]

Then it follows from (5.6) and (5.7) the behaviour of \( \phi_u \) for \( f \sim 1 \)
\[ \phi_u \sim \frac{cf_u}{1 - f} + d f_u \ln(1 - f) + ef_u + hf_u(1 - f) \ln(1 - f) + \ldots \]  
\[ (5.8) \]

where
\[ c = p - k - \int_0^\infty B(\omega) \sin(\omega) \, d\omega \]
\[ d = \int_0^\infty \omega B(\omega) \cos(\omega) \, d\omega \]
\[ e = \frac{p}{2} + \int_0^\infty \omega [A(\omega) + B(\omega) \ln \omega] \cos(\omega) \, d\omega \]
\[ h = \frac{1}{2} \int_0^\infty \omega^2 B(\omega) \sin(\omega) \, d\omega \]

Integrating the constraint (3.2a) for \( f \sim 1 \) we have (for the choice (4.13))
\[ K_u = \frac{\kappa c^2 f_u}{1 - f} + 2\kappa c d f_u \ln(1 - f) + \ldots \]  
\[ (5.9) \]

hence
\[ K = -\kappa c^2 \ln(1 - f) + \text{bounded} \]  
\[ (5.10) \]

If \( c \neq 0 \) then \( h(u) \) given by (5.5) is obviously singular. We have, therefore, the first necessary condition of boundedness of the metric in Brinkmann coordinates. It reads
\[ c = p - k - \int_0^\infty B(\omega) \sin(\omega) \, d\omega = 0. \]
With $c=0$ we have

$$h(u) = -\kappa \text{ const } f_u^2 [d^2 \ln^2(1 - f) + 2de \ln(1 - f) + \text{ bounded}]. \quad (5.11)$$

Thus conditions for boundedness of $h(u)$ read

$$c = d = 0. \quad (5.12a)$$

In the asymptotic caustic $f=0, g=1$, we require analogously

$$c' = d = 0 \quad (5.12b)$$

where

$$c' = q - k + \int_0^\infty B(\omega) \sin(\omega) \, d\omega.$$

We turn now to criteria of absence of parallelly propagated singularities in the asymptotic caustics $f=0, g=1$ and $f=1, g=0$. We shall work with the case $g=0$ and we choose the coordinates given by (5.2). The following is the parallelly propagated orthonormal frame for geodesics respecting $x$-symmetry

$$m = (1, 1, 0), \ n = (1, -1, 0), \ l = (0, 0, e^{N(u')^2}). \quad (5.13)$$

The only nonzero components of the Riemann tensor in this frame are given by

$$R_{tmot} = R_{tnln} = R_{tlnn} = \kappa \phi_u' \phi_u'. \quad (5.14)$$

Turning back to the original coordinates we have the condition for absence of the parallelly propagated singularity as follows:

$$\kappa e^{2K} \phi_u' \phi_u = \text{ bounded}. \quad (5.15)$$

We observe that this is the already encountered condition of boundedness of the metric in Brinkmann coordinates, therefore, the conditions (5.12) are also conditions for absence of the parallelly propagated singularities in the asymptotic caustics. We note that the conditions $c=c'=0$ imply boundedness of the scalar field itself as follows from (5.7) and from the asymptotic behaviour of the Bessel and Neumann functions given by (5.6). Since $\phi$ is the scalar field, its boundedness should have been expected.
We note also that in our particular case we may give a sufficient condition for boundedness of all covariant derivatives of the Riemann tensor in the frame (5.13). Indeed, we use the formula valid in D=3

\[ R_{\alpha\beta\gamma\delta} = \kappa [(g_{\alpha\gamma}T_{\beta\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\beta\gamma} - g_{\beta\gamma}T_{\alpha\delta}) + T(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})] \]  

(5.16)

where \(T_{\alpha\beta}\) is the matter stress tensor.

Taking into account that the only nonzero component of the \(T_{\alpha\beta}\) in our case is

\[ T_{u'u'} = \phi_{u'}\phi_{u'} \]  

(5.17)

and the only nonzero Christoffel symbols are

\[ \Gamma_{xx}^x = -\frac{1}{2} N_{u'}, \Gamma_x^v = -N_{u'} e^{-N} \]  

(5.18)

we arrive after some work at the sufficient condition of boundedness of all covariant derivatives of the Riemann tensor in the frame (5.13) up to the order \(J\).

It reads

\[ \frac{d^j \phi(u')}{du'^j} = \text{bounded}, \text{ for all } j \leq J + 1. \]  

(5.19)

Written in the original coordinates

\[ \left(e^K \frac{d}{du}\right)^j \phi(u) = \text{bounded}, \text{ for all } j \leq J + 1. \]  

(5.20)

If \(J=0\) we recover

\[ e^K \phi_u = \text{bounded} \]  

(5.21)

which clearly implies (5.15).

We shall not present the general discussion of the conditions (5.19) for the solutions (4.1). We note, however, that the background solution with \(B(\omega)=A(\omega)=0\) and \(p=q=k\) gives the bounded all orders covariant derivatives of the Riemann tensor evaluated in the parallelly propagated orthonormal frame (5.13).

Conditions (5.12) are obviously satisfied by huge number of solutions (4.1) (e.g. if \(B=0\) and \(p = k = q, A(\omega)\) can be arbitrary). In the next paragraph
we shall study which of them lead to the formation of the final curvature singularities.


Scalar curvature $R$ for the metric (3.1) reads

$$R = -4e^K K_{uv} \quad (6.1)$$

or

$$R = -4ke^K f_u g_v \phi_f \phi_g \quad (6.2)$$

where (6.2) follows from (6.1) and Einstein equation (3.2c). We wish to investigate scalar curvature $R$ near the caustic

$$1 - f(u) - g(v) = 0. \quad (6.3)$$

For this purpose we introduce coordinates

$$t = 1 - f(u) - g(v) \quad (6.4)$$
$$z = f(u) - g(v) \quad (6.5)$$

thus

$$R \sim e^K (\phi_t^2 - \phi_z^2) \quad (6.6)$$

since $f_u$ and $g_v$ are bounded at $t=0$.

From the form of the general solution (4.1) and asymptotic behaviour of the Bessel and Neumann functions (5.6) it follows for $t \sim 0$ (see also [15])

$$\phi_t \sim \frac{E}{t} - \frac{Ft}{2} + ... \quad (6.7)$$
$$\phi_z \sim G \ln t + I + ... \quad (6.8)$$

where

$$E = k - p - q + \int_0^\infty B(\omega) \sin(\omega z) \, d\omega \quad (6.9a)$$
$$F = p (1 + z)^{-2} + q (1 - z)^{-2} + \frac{1}{2} \int_0^\infty \omega^2 [2A(\omega) - B(\omega)] \sin(\omega z) \, d\omega \quad (6.9b)$$
\[ G = \int_0^\infty \omega B(\omega) \cos(\omega z) \, d\omega \quad (6.9c) \]

\[ I = p \left( 1 + z \right)^{-1} - q \left( 1 - z \right)^{-1} + \int_0^\infty \omega [A(\omega) + B(\omega) \ln \omega] \cos(\omega z) \, d\omega. \quad (6.9d) \]

The asymptotic behaviour of \( K \) for \( t \sim 0 \) can be obtained by integrating the Einstein equations (3.2)

\[ K_f = \kappa t \phi_f^2 - \frac{f_{uu}}{f_u^2} \quad (6.10a) \]

\[ K_g = \kappa t \phi_g^2 - \frac{g_{vv}}{g_v^2} \quad (6.10b) \]

hence

\[ K_t = -\kappa t (\phi_t^2 + \phi_z^2) + \frac{1}{2} \left( \frac{f_{uu}}{f_u^2} + \frac{g_{vv}}{g_v^2} \right) \quad (6.11a) \]

\[ K_z = -2\kappa t \phi_t \phi_z + \frac{1}{2} \left( \frac{g_{vv}}{g_v^2} - \frac{f_{uu}}{f_u^2} \right) \quad (6.11b) \]

The second terms in (6.11) for the choice (4.13) have no influence on divergence of the scalar curvature. Using (6.7) and (6.8) we arrive at

\[ K = -\kappa E^2 \ln t + ... \quad (6.12) \]

Collecting (6.7), (6.8) and (6.12) we have

\[ R \sim t^{-\kappa E^2} \left[ \frac{E^2}{t^2} - G^2 \ln^2 t + ... \right] \quad (6.13) \]

We conclude that \( R \) can be nonsingular on the caustic \( t=0 \) only if \( E=G=0 \). But both \( E \) and \( G \) have nontrivial dependence on \( z \) unless \( B(\omega)=0 \). In this case \( G=0 \) automatically and \( E=0 \) means \( k-p-q=0 \). Thus conditions for avoiding the scalar curvature singularities on the caustic \( t=0 \) (\( z \neq \pm 1 \)) read

\[ B(\omega) = 0, \ k-p-q = 0. \quad (6.14) \]

Comparing with conditions of “incoming” regularity (5.12) we see that for regular incoming waves the scalar curvature singularities at the caustic \( t=0 \) are inevitable.
§7. Conclusions and outlook.

We have obtained the large class of solutions of equations of motion for self-gravitating scalar field in D=3. They describe scattering of excitations and corresponding spacetimes are inevitably singular to the future for all incoming waves free of parallelly propagated singularities. The incoming waves do not have sharp wave-fronts, they interact for all times, the spacetime being flat to the past only asymptotically. In distinction to the case of D=4 vacua, by dimensional reduction of which our family of spacetimes was constructed, there are no exceptions from the final singularity formation.

The class of considered solutions possesses an interesting property. The solutions are given as the Fourier-Bessel mode decompositions around the background soliton. The asymptotic flatness gives restriction on the amplitudes $p, q$ of the solitary waves. Namely

$$p^2 \kappa \geq 1, \quad q^2 \kappa \geq 1$$

(7.1)

It means that we cannot switch off gravitational interaction by setting $\kappa=0$ without violating the conditions (7.1) and, consequently, the proper asymptotic behaviour. We see also that the phase space of scalar field in D=3 is enlarged due to proper self-gravitation.

The existence of Fourier-Bessel decomposition of general solution is more interesting in D=3 than in D=4. Indeed, an existence of a mode decomposition of scalar field solutions is more appealing than a mode decomposition of a metric component. As we have already mentioned in the Introduction, the matter field degrees of freedom are the only degrees of freedom in D=3 and we succeeded to extract them rather explicitly. The next logical step would rely on quantizing the matter degrees of freedom. We did not attempt to do it in this paper but we provide yet two more comments on the subject. First of all, we restricted ourselves to solutions with a line symmetry. It may well happen that the singularity formations come from high symmetry of the problem and they would disappear when dealing with waves with a finite transverse size [8, 18, 19]. This problem, to our knowledge, remains open also in D=4 and in D=3 it would probably influence the proper quantum treatment. Second comment relies on the fact that the background solution, around which the mode decomposition is made, is itself singular; hence,
unless something surprising happens, one should not probably expect the smearing of singularities in the quantum picture. Nevertheless, the quantum information about the role and behaviour of the scattering singularities is certainly of an interest and we hope to return to the problem elsewhere.

Acknowledgment.

We are grateful to Jiří Bičák for comments on the manuscript. We wish to thank also H Boušková and G Hanáková for drawing the figures.
References.

[7] Lancaster D Kyoto preprint KUNS 1016, HE(TH) 90/06 (May 1990)
[13] Ferrari V and Ibañez J 1989 Class. Quantum Grav. 6 1805
[15] Hayward S A 1990 Class. Quantum Grav. 7 1117
[17] Konkowski D A and Helliwell T M 1989 Class. Quantum Grav. 6 1847
Figure Captions.

Fig.1a: Wave-front $v_w(=0)$ in the $u$-wave Rosen coordinates with generating geodesics.

Fig.1b: Wave-front $v_w(=0)$ in the $u$-wave Brinkmann coordinates. The wave-front of the $u$-wave is also given by $u_w=0$.

Fig.2: The $u$-wave in Rosen coordinates with a part, indicated by hatch, cut out by the crossing wave-front. The jagged line indicates the coordinate singularity $u=u_0$. The $x$-coordinate is suppressed.

Fig.3a: Expected shape of the spacetime describing the collision of two waves with sharp wave-fronts $u_w=0$ and $v_w=0$. The jagged lines indicate the singular points of spacetime.

Fig.3b: Expected shape of the spacetime describing the collision of two waves without sharp wave-fronts. The jagged lines indicate (asymptotically) the singular points of the spacetime.