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ABSTRACT

We compute the group element of SO(2,2) associated with the spinning black hole found by Bañados, Teitelboim and Zanelli in (2+1)-dimensional anti-de Sitter space-time. We show that their metric is built with SO(2,2) gauge invariant quantities and satisfies Einstein’s equations with negative cosmological constant everywhere except at $r = 0$. Moreover, although the metric is singular on the horizons, the group element is continuous and possesses a kink there.

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The desire to have an interesting and mathematically tractable setting for studying quantum gravity can in part be fulfilled by investigating the properties of lower dimensional black holes [1]. Recently, interest in this subject has increased because of the discovery by Bañados, Teitelboim, and Zanelli [2] (BTZ) of a spinning black hole solution in \((2 + 1)\) dimensions. The aim of this note is to associate an element of the anti-de Sitter group with this solution and to discuss its physical characteristics.

We use a Chern-Simons formulation of gravity [3] with the gauge field

\[
A = e^a P_a + \omega^a J_a
\]  

(1)

decomposed in a basis of the Lie algebra \(so(2,2)\)

\[
[J_a, J_b] = \epsilon_{ab}^c J_c , \quad [J_a, P_b] = \epsilon_{ab}^c P_c , \quad [P_a, P_b] = \frac{1}{l^2} \epsilon_{ab}^c J_c
\]  

(2)

\([a, b, c = 0, 1, 2; \epsilon^{012} = 1\) and indices are lowered with the metric \(\eta_{ab} = \text{diag}(-1, 1, 1)\). In the absence of matter, the equations of motion

\[
\mathcal{F} = dA + A \wedge A = 0
\]  

(3)

imply the Einstein’s equations with negative cosmological constant \(\Lambda = -l^{-2}\) provided we identify \(e^a_\mu\) with the Dreibein \((g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab})\) and \(\omega^a_\mu\) with the spin-connection \([de^a + \epsilon_{abc} \omega^b \wedge e^c = 0\) is obtained from Eq. (3)]. When Eq.(3) holds, the gauge field is pure gauge

\[
A = U^{-1} dU
\]  

(4)

given by an element \(U\) of the SO(2,2) group.

Here we shall obtain the group element \(U\) associated with the spinning black hole metric of Ref. [2]

\[
ds^2_{\text{BTZ}} = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2
\]  

(5)

with lapse and angular shift functions

\[ N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} = \frac{1}{l^2} \left( \frac{r^2 - r_+^2}{r^2} \right) \left( r^2 - (Jr_+)^2 \right) \]

\[ N^\phi(r) = -\frac{J}{2r^2} = -Jl \left( \frac{r_+}{l} \right)^2 \frac{1}{r^2} . \] (6)

where \( J = \frac{H}{2r_+} \) and \( r_+^2 = \frac{Ml^2}{2} \left\{ 1 + \left[ 1 - \left( \frac{J}{Ml} \right)^2 \right]^{1/2} \right\} \). Here, \( r_+ \) and \( r_- = |J|r_+ \) are the two zeroes for the lapse function \( N(r) \). Of these, \( r_+ \) is the outer horizon, which exists only if the requirements \( M > 0, |J| \leq Ml \) (equivalent to \( |J| < 1 \)) hold; this is henceforth assumed.

We shall see that the metric (5) is locally anti-de Sitter i.e., it satisfies locally the Einstein’s equation with negative cosmological constant. In fact, using the following change of complex variables

\[ t' = ir_+ \left( \frac{t}{l} - J\phi \right) , \quad r' = -il \sqrt{\nu^2(r)} , \quad \phi' = -\frac{r_+}{l} \left( \frac{J}{l} \frac{t}{l} - \phi \right) \] (7)

with \( \nu^2(r) = 1 + \frac{(r/r_+)^2 - 1}{1 - J^2} \) and taking real values for \((t', r', \phi')\), the above metric is globally transformed in the anti-de Sitter form

\[ ds_{\text{ads}}^2 = -\left[ 1 + \left( \frac{r'}{l} \right)^2 \right] dt'^2 + \left[ 1 + \left( \frac{r'}{l} \right)^2 \right]^{-1} dr'^2 + r'^2 d\phi'^2 \] (8)

with a wedge removed [4], \((t', r', \phi') \equiv (t' - 2\pi Jr_+, r', \phi' + 2\pi r_+/l)\). [Note that the Jacobian of this transformation is singular at \( r = 0 \) and \( r = r_- \).] We shall use this correspondence to find the BTZ group element from the anti-de Sitter one.

In the anti-de Sitter case (8), the Dreibein and the spin-connection may be chosen (up to a Lorentz transformation) as

\[ e^0 = \sqrt{1 + (r'/l)^2} \ dt' \]

\[ e^1 = \frac{1}{\sqrt{1 + (r'/l)^2}} \ dr' \]

\[ e^2 = r' \ d\phi' \]

\[ \omega^0 = -\sqrt{1 + (r'/l)^2} \ d\phi' \]

\[ \omega^1 = 0 \]

\[ \omega^2 = -r'/l^2 \ dt' . \] (9)
To find $U$, we have to solve Eq. (4); explicitly

$$\sqrt{1+(r'/l)^2} P_0 - r'/l^2 J_2 = U^{-1} \partial_{r'} U$$

$$\frac{1}{\sqrt{1+(r'/l)^2}} P_1 = U^{-1} \partial_r U$$

$$r' P_2 - \sqrt{1+(r'/l)^2} J_0 = U^{-1} \partial_{\phi'} U$$

(10)

Employing the Ansatz (inspired by the Poincaré case)

$$U = e^{\Theta_0} e^{\Omega_0} e^{\Omega_1} P_0 e^{\Omega_2} P_1 e^{\Omega_2} P_2$$

(11)

where $\Theta^0, \Omega^a$ are functions of $(t', r', \phi')$, and, making use of the commutation relations (2), the right hand side of Eq. (10) can be put in closed form yielding eighteen differential equations. These have the solution

$$U_{\text{ads}}(t', r', \phi') = \exp\{-\phi' J_0\} \exp\{t' P_0\} \exp\{l \sinh^{-1}(r'/l) P_1\}$$

(12)

which is unique up to a left-multiplication by an invertible constant matrix, reflecting the fact that we are dealing with first order differential equations.

Since Eq. (4) is covariant under coordinate transformations, performing the analytic change of variables (7) on the Dreibein, the spin-connection and the group element of the anti-de Sitter case will give us a new version of these quantities that still obey Eq. (4) and a new metric, which coincides with the spinning black hole metric (5). We have to distinguish the three regions $r > r_+, r_- < r < r_+$ and $0 < r < r_-$.  

Let us first consider $r > r_+$. We need to specify how the function $\sqrt{1+(r'/l)^2}$ is defined when $r'/l \to -i \nu(r)$; this function appears explicitly in (9) and implicitly in (12) [since $\sinh^{-1}(r'/l) = \log((r'/l) + \sqrt{1+(r'/l)^2})$]. Our specification is

$$\sqrt{1+(r'/l)^2} \to -i \sqrt{\nu^2(r) - 1} \quad \text{for} \quad r > r_+$$

(13)
Then from Eq. (9), we get \([\nu^2(r) > 1\) and \(\nu(r) = \sqrt{\nu^2(r)}\) for \(r > r_+\)]

\[
\begin{align*}
e^0_{\text{BTZ}} &= r_+ \sqrt{\nu^2(r) - 1} \left( \frac{dt}{l} - \mathcal{J} d\phi \right) \\
e^1_{\text{BTZ}} &= \frac{l}{r_+^2 (1 - \mathcal{J}^2)} \frac{1}{\nu(r) \sqrt{\nu^2(r) - 1}} \frac{dr}{r} \\
e^2_{\text{BTZ}} &= r_+ \nu(r) \left( d\phi - \mathcal{J} \frac{dt}{l} \right)
\end{align*}
\]

\[
\omega^0_{\text{BTZ}} = -\frac{r_+}{l} \sqrt{\nu^2(r) - 1} \left( d\phi - \mathcal{J} \frac{dt}{l} \right) \\
\omega^1_{\text{BTZ}} = 0 \\
\omega^2_{\text{BTZ}} = -\frac{r_+}{l} \nu(r) \left( \frac{dt}{l} - \mathcal{J} d\phi \right)
\]

(14)

and

\[
\begin{align*}
U(t, r>r_+, \phi) &= U_{\text{adS}}(\{t'(t, r, \phi), r'(t, r, \phi), \phi'(t, r, \phi)\}) \\
&= \exp \left\{ i \frac{r_+}{l} \left( \mathcal{J} \frac{t}{l} - \phi \right) J_0 \right\} \exp \left\{ i r_+ \left( \frac{t}{l} - \mathcal{J} \phi \right) P_0 \right\} \exp \left\{ l \log (-i\nu(r) - i\sqrt{\nu^2(r) - 1}) P_1 \right\} .
\end{align*}
\]

(15)

One can check that Eq. (14) reproduces the metric (5) and that Eq. (15) is indeed a solution of Eq. (4) with the components of \(A\) given by (14). However, it does not belong to the group but to its complexification. This problem is easily dealt with by left-multiplying the element (15) by \(\exp \left( i \frac{\pi}{2} l P_1 \right)\), since a solution of Eq. (4) is determined up to a left multiplication by a constant matrix,

\[
U_{\text{BTZ}}(t, r>r_+, \phi) = \exp \left\{ i \frac{\pi}{2} l P_1 \right\} U(t, r>r_+, \phi)
\]

\[
= \exp \left\{ -\frac{r_+}{l} \left( \frac{t}{l} - \mathcal{J} \phi \right) J_2 \right\} \exp \left\{ - r_+ \left( \mathcal{J} \frac{t}{l} - \phi \right) P_2 \right\} \exp \left\{ l \cosh^{-1}\nu(r) P_1 \right\}
\]

(16)

which now belongs to the group. [We choose the positive branch of \(\cosh^{-1}\).]

Let us now turn to the second region, taking \(r_- < r < r_+\). In this case the lapse function \(N^2(r)\) changes sign, characteristic of an horizon, which interchanges the role of two coordinates. Thus, we get a real Dreibein and a real spin-connection from the anti-de Sitter case by interchanging \((e^0, \omega^0)\) and \((e^1, \omega^1)\) and we obtain \([0 < \nu^2(r) < 1\) and \(\nu(r) = \sqrt{\nu^2(r)}\)
As before we get from Eq. (12) a solution to Eq. (4)

\[ U_{\text{BTZ}}(t, r_-, r_+, \phi) = \exp \left\{ -r_+ \left( \frac{t}{l} - J\phi \right) J_2 \right\} \exp \left\{ -r_+ \left( J\frac{t}{l} - \phi \right) P_2 \right\} \exp \left\{ -l \cos^{-1} \nu(r) P_0 \right\} . \] (18)

At the horizon, the group elements \( U_{\text{BTZ}} \) given by Eqs. (16) and (18) coincide, but the gauge field \( A \) has a discontinuity at \( r = r_+ \) as can be seen from Eqs. (14) and (17). Moreover, the Dreibein is degenerate on the horizon since one of its components is zero and another one diverges.

Finally, for \( 0 < r < r_- \), the lapse function changes sign again, yielding a metric signature similar to that outside of the outer horizon. The transformation (7) of the gauge field components (9), after a cyclic permutation, gives \( \nu^2(r) < 0 \) and \( |\nu(r)| = \sqrt{-\nu^2(r)} \) for \( 0 < r < r_- \)

\[ e^0_{\text{BTZ}} = -r_+ |\nu(r)| \left( d\phi - J\frac{dt}{l} \right) \]  
\[ e^1_{\text{BTZ}} = -r_+ \sqrt{1 - \nu^2(r)} \left( \frac{dt}{l} - Jd\phi \right) \]  
\[ e^2_{\text{BTZ}} = \frac{l}{r_+^2 (1 - J^2)} |\nu(r)| \frac{r}{\sqrt{1 - \nu^2(r)}} dr \]  
\[ \omega^0_{\text{BTZ}} = 0 \]  
\[ \omega^1_{\text{BTZ}} = \frac{r_+}{l} \sqrt{1 - \nu^2(r)} \left( d\phi - J\frac{dt}{l} \right) \]  
\[ \omega^2_{\text{BTZ}} = 0 \]  
\[ \text{(19)} \]

which satisfy \( g_{\mu\nu} = e^a_{\mu} \xi^b_{\nu} \eta_{ab} \) and \( de^a + \epsilon^{abc} \omega^b \wedge e^c = 0 \) for this region. We get the corresponding group element from Eq. (12) and left multiplying by \( \exp \left\{ -\frac{\pi}{2} l P_0 \right\} \)

\[ U_{\text{BTZ}}(t, 0 < r < r_-, \phi) = \exp \left\{ -r_+ \left( \frac{t}{l} - J\phi \right) J_2 \right\} \exp \left\{ -r_+ \left( J\frac{t}{l} - \phi \right) P_2 \right\} \times \exp \left\{ -\frac{\pi}{2} l P_0 \right\} \exp \left\{ -l \sinh^{-1} |\nu(r)| P_2 \right\} . \] (20)
As before this solution and the previous one, Eq. (18), match at \( r = r_- \) although \( A \) does not.

With Eqs. (16), (18) and (20), the general solution is

\[
U(x) = C U_{\text{BTZ}}(x) \exp\{\lambda^a(x)J_a\} \tag{21}
\]

where \( C \) is a matrix of integration constants, which is in the group since \( U(x) \) must be in the group, and the right multiplication by \( \exp\{\lambda^a(x)J_a\} \) arises because the Dreibein and the spin-connection are defined up to a local Lorentz transformation. This solution is continuous in all space-time. The existence of \( U(x) \) insures that \( \mathcal{F} = 0 \) (i.e., the metric (5) fulfills the anti-de Sitter Einstein equations) almost everywhere, except maybe at \( r = 0, r = r_- \) and \( r = r_+ \) where \( U(x) \) is not differentiable. A convenient way to see a nonvanishing \( \mathcal{F} \) is to use the Wilson loops. At the same time, this enables us to compute the gauge invariant quantities.

Let us look at the Wilson loop enclosing the origin at fixed time

\[
W = \mathcal{P} \exp \oint A_\mu dx^\mu \tag{22}
\]

which is related to \( U \) by

\[
W = U^{-1}(t, r, \phi = 0) U(t, r, \phi = 2\pi) \equiv e^w \tag{23}
\]

A gauge transformation is equivalent to a conjugation of \( W \) by a group element \( V \)

\[
W \rightarrow V W V^{-1} = e^{VwV^{-1}} \tag{24}
\]

It is known how to extract quantities invariant under (24). Consider the Casimir of the adjoint representation of \( \text{SO}(2,2) \)

\[
\mathcal{C} = c_1 \left( J_a J^a + l^2 P_a P^a \right) + c_2 \left( J_a P^a + P_a J^a \right) \tag{25}
\]
where \(c_1, c_2\) are arbitrary constants. If \(w = \theta^a J_a + \xi_a P_a\) in Eq. (24), it follows from the Casimir that the two quantities

\[
m = \frac{1}{2\pi^2} \left( \theta_a \theta^a + \frac{1}{l^2} \xi_a \xi^a \right)
\]

\[
j = \frac{1}{2\pi^2} \theta_a \xi^a
\]

are invariant under (24).

The solution (21) gives to the Wilson loops (22) located in one of the three regions considered a unique value up to a conjugation by a group element depending on the position of the loop

\[
W = V(r)^{-1} \exp \left\{ 2\pi \frac{r_+}{l} (J_1 J_2 + l P_2) \right\} V(r). \tag{27}
\]

Eq. (26) reveals that the first invariant quantity coincides with \(M\) and the second one with \(J\), the constants parametrizing the black hole metric (5). Thus \(M\) and \(J\) are physically relevant, and according to [5] correspond to the mass and spin of the black hole.

A small Wilson loop around the origin enables us to compute \(F\) at this point. Namely for small \(\varepsilon\)

\[
W_{r=\varepsilon} = 1 + \int_0^\varepsilon dr \int_0^{2\pi} d\phi F_{r \phi} + \mathcal{O}(\varepsilon^2)
\]

\[
\log W_{r=\varepsilon} = 2\pi \frac{r_+}{l} V(\varepsilon)^{-1} (J_1 J_2 + l P_2) V(\varepsilon) \tag{28}
\]

where the second equality is obtained from Eq. (27). Since \(F = 0\) outside \(r = 0\), it must be the distribution (we use the explicit expression of \(V(0)\), which is read off Eq. (20))

\[
F_{r \phi} = \delta(r) \frac{r_+}{l} \frac{1}{\sqrt{1-J^2}} \left[ J_1 - J^2 J_0 + J l (P_1 - P_0) \right]. \tag{29}
\]
In (2+1)-dimensions, the $J_a$ components of $F$ are used to build the Einstein tensor with cosmological constant. For the Dreibein of Eq. (19), Einstein’s equations are modified at the origin by the presence of the localized spinning black hole

$$\sqrt{-g} \left( G_{tt} + \frac{1}{l^2} g_{tt} \right) = \left( \frac{r_+}{l} \right)^4 \left( 1 + J^2 \right)^2 \delta(r)$$ (30a)

$$\sqrt{-g} \left( G_{t\phi} + \frac{1}{l^2} g_{t\phi} \right) = - \left( \frac{r_+}{l} \right)^4 \left( 1 + J^2 \right) J l \delta(r)$$ (30b)

$$\sqrt{-g} \left( G_{\phi\phi} + \frac{1}{l^2} g_{\phi\phi} \right) = \left( \frac{r_+}{l} \right)^4 (J l)^2 \delta(r)$$ (30c)

without any modification to the other components.

Similarly, the $P_a$ components of $F$ provide the torsion tensor and we find here that the torsion free condition is modified at the origin by

$$d e^0 + e^0_{\ ab} \omega^a \wedge e^b = - \frac{J r_+}{\sqrt{1 - J^2}} \delta(r) \ dr \wedge d\phi$$ (31a)

$$\quad d e^1 + e^1_{\ ab} \omega^a \wedge e^b = \frac{J r_+}{\sqrt{1 - J^2}} \delta(r) \ dr \wedge d\phi$$ (31b)

and no modification for the other component. The fact that the Wilson loops (27) belong always to the same class [under the conjugation (24)] indicates the absence of matter at the horizons $r = r_\pm$, extending the validity of Eqs. (30,31) to the whole space-time. The RHS’s correspond to the energy-momentum tensor of the matter, which is responsible for this particular geometry. This is in contrast with the (3+1)-dimensional Schwarschild metric configuration, which cannot be written as the solution of Einstein’s equations with a source localized at the origin. The difference is understood from the absence of gravitational interaction in (2+1) dimensions. We stress that Eqs. (30,31) are valid only for the choice of Dreibein (19). For example, a rotation of the Dreibein by $\exp \phi J_0$ would add to the RHS of Eq. (29) the term $\delta(r) J_0$. 

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It is interesting to comment on the Poincaré case $l \to \infty$ (i.e., a vanishing cosmological constant). In this limit and for $M > 0$, $J \neq 0$, the metric (5) becomes

$$ds^2 = \left( -\sqrt{M}dt + \frac{J}{2\sqrt{M}}d\phi \right)^2 + \frac{dr^2}{r^2} - M + \left( r^2 - \frac{J^2}{4M} \right) d\phi^2 \quad (32)$$

and is a solution of the corresponding limit equations (30), i.e., Einstein’s equations with a singularity at the origin. Near the source (for $r < \frac{J}{2\sqrt{M}}$), the space-time has closed timelike curves, $\phi$ being a periodic time coordinate. For $r > \frac{J}{2\sqrt{M}}$ the space-time is a helix expanding in the time coordinate $r$, since the variable $R \equiv -\sqrt{M}t + \frac{J}{2\sqrt{M}}\phi$ increases by an amount $\pi \frac{J}{\sqrt{M}}$ whenever $\phi \to \phi + 2\pi$. For $J = 0$ the metric becomes

$$ds^2 = -\frac{1}{M}dr^2 + Mdt^2 + r^2 d\phi^2 \quad (33)$$

which corresponds to a cylinder expanding in the time coordinate $r$.

The gauge formulation of gravity is useful in the analysis of classical solution. We have computed the group element associated with the BTZ black hole. We can choose it continuous but not differentiable where the metric does not exist. We have shown that the gauge invariants coincide with the geometric ADM invariants [5]. Using the Wilson loops, we were able to locate the source of matter at the origin appearing in the anti-de Sitter Einstein’s equations.

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References


