Stable Models of Elliptical Galaxies

Yan Guo
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University, Providence, RI 02912
and
Gerhard Rein
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Vienna, Austria

Abstract
We construct stable axially symmetric models of elliptical galaxies. The particle density on phase space for these models depends monotonically on the particle energy and on the third component of the angular momentum. They are obtained as minimizers of suitably defined energy-Casimir functionals, and this implies their nonlinear stability. Since our analysis proceeds from a rigorous but purely mathematical point of view it should be interesting to determine if any of our models match observational data in astrophysics. The main purpose of these notes is to initiate some exchange of information between the astrophysics and the mathematics communities.

1 Introduction
Consider a large ensemble of mass points (stars) which interact only by the gravitational field which they create collectively. Such a collisionless, self-gravitating gas is used to model galaxies or globular clusters, cf. [3, 5] and
the references there. The time evolution of the density \( f = f(t, x, v) \geq 0 \) of the stars in phase space is governed by the following nonlinear system of partial differential equations:

\[
\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,
\]

\[
\Delta U = 4\pi \rho, \quad \lim_{|x| \to \infty} U(t, x) = 0,
\]

\[
\rho(t, x) = \int f(t, x, v) dv.
\]

Here \( t \in \mathbb{R} \) denotes time, \( x, v \in \mathbb{R}^3 \) denote position and velocity respectively, \( \rho \) is the spatial mass density of the stars, and \( U \) the gravitational potential which the stars induce collectively. Collisional or relativistic effects are neglected. This system, which Sir J. H. Jeans introduced at the beginning of the last century for the above modeling purposes, is sometimes in the astrophysics literature referred to as the collisionless Boltzmann-Poisson system. Following the mathematics convention we call it the Vlasov-Poisson system.

In the present paper we are interested in the steady states of this system. Besides their very existence, questions of interest are the possible shapes of such steady states, in particular their symmetry type, and whether or not they are dynamically stable. We approach these questions from a mathematics point of view, in particular, only such information as can be extracted from the system stated above is to enter our arguments. Our “ideal” reader is an astrophysicist interested in these same questions. We find it deplorable that there is little communication between astrophysicists and mathematicians investigating these problems, and the present paper is an attempt to change this. Axially symmetric steady states of the the Vlasov-Poisson system were obtained in [22] as perturbations of spherically symmetric ones via the implicit function theorem. From an astrophysics point of view axisymmetric models of elliptical galaxies have been investigated in [18, 27, 28], but as to their stability no rigorous results are known to us. In the present paper we follow the variational approach developed in [7, 8, 9, 10, 11, 21, 23, 24]: We obtain axially symmetric steady states with vanishing and with non-vanishing velocity field as minimizers of appropriately defined energy-Casimir functionals. These steady states are no longer restricted to small perturbations of spherically symmetric ones, and, most importantly, they are nonlinearly stable. The question we would like to raise is: Do our mathematical construc-
tions yield suitable models for real-world galaxies, or, if not, how can the mathematical approach be modified to do so?

If $U_0 = U_0(x)$ is a stationary potential then any function $f_0 = \phi(E)$ of the particle energy

$$E = E(x,v) = \frac{1}{2}|v|^2 + U_0(x)$$

(1.1)

solves the Vlasov equation with the potential $U_0$, since $E$ is a conserved quantity along particle trajectories. So if $U_0$ is the potential induced by $f_0$ then this is a steady state of the Vlasov-Poisson system—at this point it must be emphasized that to obtain a self-consistent model by this approach, i.e., to make sure that $U_0$ is indeed the potential induced by $f_0$, is mathematically non-trivial. Any steady state obtained by this ansatz is necessarily spherically symmetric, as follows from a result of Gidas, Ni & Nirenberg [6].

In the present paper we make $f_0$ depend on an additional invariant of the particle trajectories. If $U_0$ is axially symmetric, i.e., invariant under all rotations about, say, the $x_3$-axis, then the corresponding component of angular momentum,

$$P = P(x,v) = x_1 v_2 - x_2 v_1$$

(1.2)

is such an additional invariant. To obtain steady states with a number density $f_0$ which depends on $E$ and $P$ we minimize the energy-Casimir functional

$$\frac{1}{2} \iint |v|^2 f(x,v) dv dx - \frac{1}{2} \iint \rho(x) \rho(y) \frac{1}{|x-y|} dx dy + \iint Q(f(x,v),P(x,v)) dv dx$$

subject to the constraint that $f$ is axially symmetric and has prescribed mass $M > 0$:

$$\iint f(x,v) dv dx = M.$$  

The corresponding Euler-Lagrange equation then shows that any minimizer is a steady state of the desired form. In order to conclude that this steady state is nonlinearly stable it is essential that the energy-Casimir functional is conserved along solutions. This is always true for the sum of the first two terms which are the kinetic and the potential energy of the system. However, the third term is conserved only if $P$ is constant along particle orbits which is true in general only if the solution is axially symmetric. Thus, the stability result which we derive is restricted to axially symmetric perturbations of the steady state. The fact that the third term in our functional is not conserved along general solutions makes the terminology “Casimir functional”
questionable, but we stick to it for lack of a better alternative. The situation is similar to the one of anisotropic, spherically symmetric steady states with $f_0$ depending on the particle energy and the modulus of the angular momentum, $|x \times v|$, where stability against spherically symmetric perturbations holds, cf. [10], but the general case is open. For isotropic steady states where $f_0$ depends only on the particle energy no such restriction is necessary, cf. [11, 24].

We want to stress the fact that in this article every conclusion is rigorously derived in the mathematical sense. Our method stems from our long term research project in which both the existence and the stability of galaxy configurations has been mathematically investigated. Since the Vlasov-Poisson system can be viewed as an infinite dimensional Hamiltonian system, even the celebrated Antonov stability theorems for polytropes are, strictly speaking, only valid in the linearized sense. This is like the case of center equilibria in a $2 \times 2$ Hamiltonian system: in general no dynamical stability assertion can be deduced from a linearized analysis, and some Liapunov type methods are needed. Recently, we were able to verify the validity of Antonov’s theorems in a dynamical, nonlinear sense [11]. Our variational method also yields stability of certain Camm models [10].

The paper proceeds as follows: In the next section we formulate the basic set up and state our main results. In order to put these into perspective we include a short discussion of stability concepts from a mathematics point of view. In the third section we discuss various examples and some further properties of the steady states which we obtain. For interested readers we collect the details of our mathematical proofs in an appendix.

We conclude the introduction with some references to the mathematical literature. In addition to our work mentioned above the stability of spherically symmetric steady states of the Vlasov-Poisson system in the present stellar dynamics case is also investigated in [1, 29, 30]. Global classical solutions to the initial value problem for the Vlasov-Poisson system were first established in [17], cf. also [26] and [20]. For the plasma physics case where the sign in the Poisson equation is reversed, the stability problem is better understood, and we refer to [4, 12, 13, 19]. A rather general condition which guarantees finite mass and compact support of steady states, but not their stability, is established in [25].
2 Main results

Let us first fix some notation. For a measurable function \( f = f(x,v) \geq 0 \) we define its induced spatial density

\[
\rho_f(x) := \int f(x,v) \, dv, \, x \in \mathbb{R}^3,
\]

and its induced gravitational potential

\[
U_f := -\rho_f \star \frac{1}{|\cdot|}.
\]

Next we define

\[
E_{\text{kin}}(f) := \frac{1}{2} \iint |v|^2 f(x,v) \, dv \, dx,
\]

\[
E_{\text{pot}}(f) := -\frac{1}{8\pi} \int |\nabla U_f(x)|^2 \, dx = -\frac{1}{2} \iint \frac{\rho_f(x) \rho_f(y)}{|x-y|} \, dx \, dy,
\]

\[
C(f) := \iint Q(f(x,v), P(x,v)) \, dv \, dx
\]

where \( Q \) is a given function satisfying certain assumptions specified below and \( P \) is defined in (1.2). We will minimize the energy-Casimir functional

\[
\mathcal{H}_C := E_{\text{kin}} + E_{\text{pot}} + C
\]

under a mass constraint, i. e., over the set

\[
\mathcal{F}_M := \{ f \in L^1(\mathbb{R}^6) \mid f \geq 0, \ i\iint f(x,v) \, dv \, dx = M, \ C(f) + E_{\text{kin}}(f) < \infty \},
\]

where \( M > 0 \) is prescribed. Eventually, we need to restrict ourselves to axially symmetric functions in this set:

\[
\mathcal{F}_M^S := \{ f \in \mathcal{F}_M \mid f(Ax,Av) = f(x,v), \ x,v \in \mathbb{R}^3,
\]

\[
A \text{ any rotation about the } x_3\text{-axis}\}.
\]

For the function \( Q \) determining the Casimir functional we make the following

**Assumptions on \( Q \):** \( Q, \ \partial_j Q, \ \partial^2_j Q \in C([0,\infty[ \times \mathbb{R}), \ Q \geq 0, \ Q(0,\cdot) = 0 = \partial_j Q(0,\cdot), \) and with constants \( C, C', F, F' > 0, \) and \( 0 < k, \ k' < 3/2, \)
(Q1) \( Q(f, P) \geq C f^{1+1/k}, f \geq F, P \in \mathbb{R}, \)
(Q2) \( \partial_t^2 Q(f, P) > 0, f > 0, P \in \mathbb{R}, \)
(Q3) \( Q(f, \cdot) \) is increasing on \(-\infty, 0[\) and decreasing on \(0, \infty[, f \geq 0,\)
(Q4) \( Q(f, 0) \leq C' f^{1+1/k'}, f \leq F'. \)

The first two assumptions are essential whereas the assumptions (Q3) and (Q4) can be modified in various ways, cf. the next section. Note that under the assumptions above \( \partial_t Q(\cdot, P) : [0, \infty[ \to [0, \infty[ \) is one-to-one and onto for any \( P \in \mathbb{R}. \) The steady states which we shall obtain will be of the form

\[
 f_0(x, v) = \phi(E_0 - E, P),
\]

where the particle energy \( E \) is defined in terms of the induced potential \( U_0 \) as in (1.1),

\[
 \phi(E, P) := \begin{cases} 
 (\partial_t Q(\cdot, P))^{-1}(E), & E \geq 0, P \in \mathbb{R}, \\
 0, & E < 0, P \in \mathbb{R},
\end{cases} \tag{2.1}
\]

and \( E_0 \leq 0 \) is some cut-off energy. We should point out that if \( Q \) does not depend on \( P \)—a case which is included in our assumptions—then no symmetry assumptions need to be made in the choice of the set \( F_M \) nor anywhere else. We now state our first main result:

**Theorem 1** For every \( M > 0 \) there is a minimizer \( f_0 \in F_M^S \) of the energy-Casimir functional \( \mathcal{H}_C. \) Let \( U_0 \) denote the potential induced by \( f_0. \) Then \( f_0 \) is a function of the particle energy \( E \) and angular momentum \( P \) as defined in (1.1) and (1.2),

\[
 f_0(x, v) = \phi(E_0 - E, P)
\]

where \( \phi \) is defined in terms of \( Q \) by (2.1). The parameter \( E_0 \) plays the role of a Lagrange multiplier and is given by

\[
 E_0 := \frac{1}{M} \iint [E + \partial_t Q(f_0, P)] f_0 dv dx.
\]

In particular, \( f_0 \) is an axially symmetric steady state of the Vlasov-Poisson system with total mass \( M. \)
The crucial part here is the existence assertion a detailed proof of which is given in the appendix. If one computes the Euler-Lagrange condition corresponding to our variational problem one obtains the assertions on the form of the minimizer. For the details of this tedious but straight forward argument we refer to \[10\] or \[11\].

**Remark 1.** If one is interested only in the existence of a steady state the usual approach is to prescribe some suitable function \(f_0(x,v) = \phi(E_0 - E,P)\), compute the corresponding spatial density \(\rho_0\) which becomes, via \(E\), a functional of the potential \(U_0\), and one is then left with the problem of proving that the semilinear Poisson equation

\[
\triangle U_0 = 4\pi \int \phi \left( E_0 - \frac{1}{2} |v|^2 + U_0, P(x,v) \right) dv
\]

has a suitable solution. The non-trivial problem here is to decide which solutions have finite total mass and compact support in space. The first requirement for \(\phi\) from the point of view of our theorem is that \(\phi\) is a strictly increasing function of \(E_0 - E\) so that a corresponding function \(Q\) and the Casimir functional can be defined. Of course, for the existence question alone this monotonicity condition which in astrophysics textbooks appears as a stability condition is not necessary. A mathematical approach which makes no such monotonicity assumption can be found in \[25\]. The point is that the minimizer which we obtain is not just any old steady state, but a nonlinearly stable one.

**Some remarks on stability concepts.** It may be useful to briefly review various concepts of stability before we state and discuss our stability theorem; for a more detailed such discussion we refer to \[14\]. Consider a nonlinear dynamical system which for short we write as

\[
\dot{x} = A(x),
\]

\(A\) being some nonlinear operator on the possibly infinite dimensional state space where solutions \(t \mapsto x(t)\) take their values. Let \(x_0\) be a steady state, i. e., \(A(x_0) = 0\).

- The “appropriate” stability approach is certainly one where we can deal with our problem directly, i. e., without simplifying or linearizing it in any way. The steady state is called *nonlinearly stable* or *Liapunov stable* if one can guarantee that sufficiently small perturbations of the
steady state launch solutions which are arbitrarily small perturbations
of the steady state for future times. More formally: One must find two
norms $\| \cdot \|, \| | \cdot | \|$ on the state space such that for any (small) number
$\epsilon > 0$ there always exists another (small) number $\delta > 0$ such that every
solution $x(t)$ with $\| x(0) - x_0 \| < \delta$ satisfies $\| x(t) - x_0 \| < \epsilon$ for all $t \geq 0$.
Some comments are in order:

- It is desirable but not always possible to use the same norms for
  measuring the initial perturbation and the one at time $t$; only on
  finite dimensional spaces are all norms equivalent.
- Sometimes one may not even obtain norms but more general tools
to measure the deviations with.
- From a physics point of view it is unsatisfactory if no rule is given
  on how to obtain the $\delta$ if $\epsilon$ is prescribed. From this point of view
a better result would be an estimate of the form

$$ \| | x(t) - x_0 | \| \leq C \| x(0) - x_0 \|, \quad t \geq 0 $$

possibly with some explicit constant $C$. Unfortunately, mathe-
maticians are not always (not often?) able to provide this.
- A global existence result which guarantees that solutions exist for
  all time $t \geq 0$ at least for initial data close to the steady state is a
necessary prerequisite and integral part for all the above.

* Since assessing nonlinear stability in the sense above is difficult, the
  problem is often approached via linearization. By Taylor expansion
one can compute the linearization of $A$ at $x_0$; formally $A_{\text{lin}} = D_x A(x_0)$.
There are now at least two sub-concepts of linearized stability: One is
to repeat the definition for Liapunov stability given above, but replace
$A$ by its linearization $A_{\text{lin}}$. After all, saying that $A$ is nonlinear does
not forbid $A$ happening to be linear. This is what we want to call
linearized stability. A somewhat different concept is spectral stability:
One considers solutions of the linearized equation of the form $e^{\lambda t}x$ and
tries to find out what the possible eigenvalues $\lambda$ are. If the real parts
of all these eigenvalues are strictly negative then one calls the steady
state spectrally stable. Again, some comments are in order:
In general there is no guarantee that the linearized problem actually has solutions of the form $e^{\lambda t}x$. If it does, then in order to draw any conclusions one must know that there are sufficiently many such solutions to get the general solution of the linearized problem by superposition (completeness of the eigenfunctions). And even if this goes well, there is simply no general result which will allow any conclusion on the behavior of the original, nonlinear system, unless the system is finite dimensional!

If one is dealing with a conservative system with some sort of Hamiltonian structure (such as the Vlasov-Poisson system), then the best to expect as far as spectral stability is concerned may be that all the eigenvalues are purely imaginary (due to inherent symmetries of the spectrum). In this case no stability follows for the nonlinear system, not even in the finite dimensional case.

Assume one can establish the existence of a growing mode, i.e., of a solution of the form $e^{\lambda t}x$ where $\lambda$ has positive real part. Then again there is no general result saying that $x_0$ is now also nonlinearly unstable, unless the system is finite dimensional. However, such a growing mode is a valuable first step toward proving a nonlinear instability result. How difficult it may be to get to a mathematically rigorous nonlinear instability result from there is illustrated by [12, 13]

The upshot from all this is: One should be very careful to draw conclusions about stability for nonlinear, infinite dimensional systems from linearization, in particular, for Hamiltonian ones.

All this said let us now return to our minimizer $f_0$ from Theorem 1. To investigate its dynamical stability we note first that if $\int f = M$,

$$
\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = d(f, f_0) - \frac{1}{8\pi} \|\nabla U_f - \nabla U_0\|^2_2,
$$

(2.3)

where $\|\cdot\|_2$ denotes the usual norm on $L^2(\mathbb{R}^3)$ and

$$
d(f, f_0) := \int \int \left[ Q(f, P) - Q(f_0, P) + (E - E_0)(f - f_0) \right] dv dx;
$$

we are allowed to subtract the term $E_0(f - f_0)$ from the integrand since its integral vanishes. Since $Q(\cdot, P)$ is convex, the integrand can be estimated
from below by

\[ \left[ \partial_t Q(f_0, P) + E - E_0 \right] (f - f_0). \]

According to Theorem 1, this quantity is zero where \( f_0 > 0 \), while it equals \( (E - E_0) f \geq 0 \) where \( f_0 = 0 \). Thus we see that

\[ d(f, f_0) \geq 0, \ f \in \mathcal{F}_M. \]

Now we note that the left hand side of (2.3) is constant along axially symmetric solutions. Moreover, and this is the crucial point, the term \( \| \nabla U_f - \nabla U_0 \|^2 \) which seems to have the wrong sign in (2.3) vanishes along minimizing sequences by Theorem A1 stated in the appendix.

Before we state our stability result we point out that if we shift a minimizer in the \( x_3 \) direction we obtain another minimizer. Moreover, we do in general not know whether the minimizers are unique up to spatial shifts. As discussed in the next section a minimizer must a-posteriori have the following additional symmetry property

\[ f_0(x_1, x_2, x_3^*, v_1, v_2, -v_3) = f_0(x_1, x_2, x_3^* + x_3, v_1, v_2, v_3), \ x, v \in \mathbb{R}^3, \]

for some \( x_3^* \in \mathbb{R} \). If we take without loss of generality \( x_3^* = 0 \) we refer to this as reflexion symmetry. This symmetry propagates along solutions of the time dependent problem, and, by restricting ourselves to data with this symmetry, we can at least ignore the non-uniqueness due to shifts:

**Theorem 2** For every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for any solution \( t \mapsto f(t) \) of the Vlasov-Poisson system with \( f(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M^S \) and reflexion symmetric,

\[ d(f(0), f_0) + \frac{1}{8\pi} \| \nabla U_{f(0)} - \nabla U_0 \|^2 < \delta \]

implies that for all \( t \geq 0 \),

\[ d(f(t), f_0) + \frac{1}{8\pi} \| \nabla U_{f(t)} - \nabla U_{f_0} \|^2 < \epsilon, \]

provided \( f_0 \) is unique or at least isolated in the reflexion symmetric subset of \( \mathcal{F}_M^S \).
Since after what was said above the proof is simple and instructive we include it here:

**Proof.** Assume the assertion were false. Then there exist $\epsilon > 0$, $t_n > 0$, and $f_n(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M^S$ reflexion symmetric and such that for all $n \in \mathbb{N}$,

$$d(f_n(0), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n(0)} - \nabla U_0\|_2^2 < \frac{1}{n}$$  \hspace{1cm} (2.4)

but

$$d(f_n(t_n), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n(t_n)} - \nabla U_0\|_2^2 \geq \epsilon.$$  \hspace{1cm} (2.5)

By (2.4) and (2.3),

$$\lim_{n \to \infty} \mathcal{H}_C(f_n(0)) = h_M^S,$$

where $h_M^S$ is the finite, negative minimum of $\mathcal{H}_C$ on $\mathcal{F}_M^S$, cf. Lemma A 3. Since $\mathcal{H}_C$ is conserved along classical, axially symmetric solutions as launched by $f_n(0)$, since mass is conserved, and since the assumed symmetry propagates,

$$\lim_{n \to \infty} \mathcal{H}_C(f_n(t_n)) = h_M^S \text{ and } f_n(t_n) \in \mathcal{F}_M^S, \; n \in \mathbb{N},$$

i.e., $(f_n(t_n))$ is a reflexion symmetric minimizing sequence for $\mathcal{H}_C$ in $\mathcal{F}_M^S$. Up to a subsequence we may therefore assume by Theorem A 1 that

$$\|\nabla U_{f_n(t_n)} - \nabla U_0\|_2^2 \to 0;$$  \hspace{1cm} (2.6)

note that due to the reflexion symmetry of the minimizing sequence and any minimizer the shifts along the $x_3$-axis in the statement of Theorem A 1 must be zero in the present situation. It is at this point that the uniqueness or isolation of the minimizer $f_0$ is used. Since $\lim_{n \to \infty} \mathcal{H}_C(f_n(t_n)) = h_M^S = \mathcal{H}_C(f_0)$ we conclude by (2.6) and (2.3) that

$$d(f_n(t_n), f_0) \to 0, \; n \to \infty,$$

and we arrive at a contradiction to (2.5). \hfill \Box

**Remark 2.** If the minimizer is not isolated, i.e., if arbitrarily close to it with respect to the “distance” used in the theorem there are other minimizers which do not result from a simple translation in space, then we obtain a stability result of the following form:
Let \( M \subset F \) denote the set of all minimizers of \( H \). Then for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for any solution \( t \mapsto f(t) \) of the Vlasov-Poisson system with \( f(0) \in C^1(\mathbb{R}^6) \cap F \),

\[
\inf_{f_0 \in M} \left[ d(f(0), f_0) + \frac{1}{8\pi} \| \nabla U_f(0) - \nabla U_0 \|^2 \right] < \delta
\]

implies that for all \( t \geq 0 \),

\[
\inf_{f_0 \in M} \left[ d(f(t), f_0) + \frac{1}{8\pi} \| \nabla U_f(t) - \nabla U_{f_0} \|^2 \right] < \epsilon.
\]

**Remark 3.** Although we only showed that \( d(f, f_0) \geq 0 \) for \( f \in F \), one may think of this term as a weighted \( L^2 \)-difference of \( f \) and \( f_0 \), and by a Taylor expansion a stronger and more explicit estimate can be obtained if \( \partial^2 f Q \) is bounded away from zero.

**Remark 4.** The restriction \( f(0) \in F \) for the perturbed initial data would be acceptable from a physics point of view: A perturbation of a given galaxy, say by the gravitational pull of some outside object, results in a perturbed state which is just a rearrangement of the original state, in particular, its mass remains unchanged. But in general symmetry properties of the steady state will be destroyed by such a perturbation. However, we need that \( f(0) \) is axially symmetric so that \( \mathcal{C} \) is conserved along the resulting axially symmetric solution \( f(t) \). So while this symmetry restriction for the perturbation seems necessary within the present mathematical framework it is quite undesirable from a physics point of view. If the steady state does not depend on \( P \) and thus is isotropic, then no symmetry restrictions are necessary anywhere in our results, and the stability is with respect to quite general perturbations.

**Remark 5.** Our stability result is of the nonlinear Liapunov type discussed above in general, but it suffers from the defect that given \( \epsilon \) it does not say how \( \delta \) needs to be chosen. The reason for this is that our proof is by contradiction and relies on a compactness result.

## 3 Examples and further properties

In this section we investigate some additional properties of the resulting steady states, keeping the discussion somewhat informal. First we present
some examples for functions $Q$ which satisfy our assumptions or possible variations of them:

**Examples.** A simple class of examples for functions $Q$ which satisfy the assumptions (Q1)–(Q4) is given by

$$Q(f, P) = f^{1+1/k} g(P), \ f \geq 0, \ P \in \mathbb{R},$$

where $0 < k < 3/2$ and $g$ is positive, continuous, bounded, bounded away from zero, increasing on $]-\infty, 0]$, and decreasing on $]0, \infty[$. Note that in this case,

$$f_0(x, v) = C(E_0 - E)^k g(P)^{-k}$$

on its support. Examining the proof of Lemma A 3 in the appendix, which is the only place where the assumptions (Q3) and (Q4) enter, one can see that these assumptions can for example be replaced by

(Q3') $Q(\lambda f, P) \geq \lambda^{1+1/k'} Q(f, P), \ f \geq 0, \ P \in \mathbb{R}$, with $1/2 \leq k' < 3/2$,

(Q4') $Q(f, P) \leq C' f^{1+1/k''}, \ f \leq F', \ |P| \geq P_0$, with constants $C' > 0, \ F' > 0, \ P_0 > 0$, and $0 < k'' < 3/2$.

Examples which satisfy the assumptions (Q1), (Q2), (Q3'), (Q4') but not the original ones are provided by (3.1) with $1/2 \leq k < 3/2$ and $g$ which has all the properties stated above except the monotonicity.

**Regularity and boundary condition.** Since by Theorem 1 $f_0$ is a function of the quantities $E$ and $P$ which are constant along particle trajectories we are justified to call $f_0$ a steady state provided $U_0$ is sufficiently regular to allow for the definition of particle trajectories to begin with. Now by construction $\triangle U_0 = 4\pi \rho_0$ on $\mathbb{R}^3$, at least in the sense of distributions, and from the very construction of $f_0$ certain integrability properties of $\rho_0$ follow, cf. the appendix. Applying the usual Sobolev space arguments one can eventually conclude that $U_0$ is twice continuously differentiable and $\lim_{|x| \to \infty} U_0(x) = 0$, cf. [11, Thm. 3] for the technical details.

**Finite mass and compact support.** By construction, the steady states have finite mass $M > 0$ which we prescribe by our mass constraint. To continue we note that by Theorem 1 and a change of variables,

$$\rho_0(x) = \int f_0(x, v) dv$$

$$= 2\pi \int_{U_0(x)}^{E_0} \int_{-\sqrt{2(E-U_0(x))}}^{\sqrt{2(E-U_0(x))}} \phi(E_0 - E, r(x)p) dp dE \quad (3.2)$$
where $U_0(x) < E_0$, and $\rho_0$ is zero else; here $r(x) := \sqrt{x_1^2 + x_2^2}$. Assume that $\lim_{|x| \to \infty} U_0(x) = 0$, see the discussion above. Then clearly the cut-off energy cannot be positive, i.e., $E_0 \leq 0$, since otherwise we get infinite mass “at spatial infinity”. Since $U_0 < 0$ everywhere, (3.2) shows that $\rho_0$ will have compact support if and only if $E_0 < 0$. An additional, sufficient condition which implies this is

$$f \partial_f Q(f, P) \leq 3Q(f, P), \quad f \geq 0, \quad P \in \mathbb{R},$$

(3.3)

which holds for example for $Q(f, P) = f^{1+1/k} g(P)$ with $1/2 \leq k < 3/2$ and some function $g$. To see this we rewrite the formula for $E_0$ from Theorem 1:

$$E_0 = \frac{1}{M} \left[ E_{\text{kin}}(f_0) + 2E_{\text{pot}}(f_0) + \iint f_0 \partial_f Q(f_0, P)\, dv\, dx \right].$$

Now we note that for solutions of the Vlasov-Poisson system,

$$\frac{d}{dt} \iint x \cdot vf = 2E_{\text{kin}}(f) + E_{\text{pot}}(f)$$

so that for a steady state the right hand side is zero. Thus by (3.3),

$$E_{\text{kin}}(f_0) + 2E_{\text{pot}}(f_0) + \iint f_0 \partial_f Q(f_0, P)\, dv\, dx \leq$$

$$E_{\text{kin}}(f_0) + 2E_{\text{pot}}(f_0) + 3 \iint Q(f_0, P)\, dv\, dx =$$

$$3E_{\text{kin}}(f_0) + 3E_{\text{pot}}(f_0) + 3C(f_0) = 3\mathcal{H}_C(f_0) < 0$$

by Lemma A 3 (a). An alternative condition which also guarantees compact support is the following:

$$\phi(E_0 - E, P) \leq C_1 (E_0 - E)^{k_1}, \quad E \to -\infty, \quad P \in \mathbb{R},$$

$$\phi(E_0 - E, P) \geq C_2 (E_0 - E)^{k_2}, \quad E \to E_0-, \quad P \in \mathbb{R}$$

(3.4)

for positive constants $C_1$, $C_2$ and $0 < k_1, k_2 < 3/2$. This assumption also implies that $E_0 < 0$, cf. [24, Thm. 3].

In the spherically symmetric case our approach provides an explicit bound on the radius of the steady state, cf. [9]. Note that in this case one can under appropriate assumptions decide whether $U_0$ crosses the cut-off energy level $E_0$ by direct examination of the semilinear Poisson equation (2.2), since this
equation reduces to an ordinary differential equation with respect to the
radial variable, cf. [25]. However, in the axially symmetric case this is no
longer true, and the corresponding analysis of the genuine partial differential
equation (2.2) would be much more difficult.

**Symmetry.** By construction the steady states which we obtain are axially symmetric, but so are the spherically symmetric ones whose existence
was known already. In order to make sure that we have found qualitatively
new steady states we must show that they are in general not spherically
symmetric. To see this we take $Q$ such that $\phi(E_0 - E, \cdot)$ is not constant on
any neighborhood of $P = 0$. We claim that neither $\rho_0$ nor $U_0$ are spherically
symmetric in this case. Indeed, if $\rho_0$ were spherically symmetric the same
were true for $U_0$. If $U_0$ were spherically symmetric we take some
$x \in \mathbb{R}^3$ with
$$r(x) = \sqrt{x_1^2 + x_2^2} \neq 0 \text{ small and take } A \in \text{SO}(3) \text{ such that } Ax = (0, 0, |x|),$$ hence
$$r(Ax) = 0 \neq r(x) \text{ but } U_0(Ax) = U_0(x).$$ Inserting this into the formula (3.2) for
$\rho_0$ we see that in general $\rho_0(Ax) \neq \rho_0(x)$ so $\rho_0$ is not spherically symmetric.

Indeed, this is a bit more than just saying that $f_0$ is not spherically
symmetric. To see this, take a spherically symmetric steady state $f_0$ with
induced density $\rho_0$, potential $U_0$, and particle energy $E = |v|^2/2 + U_0(x)$. Let
$g_0(x,v) = \psi(E,P)$ be any function which is odd in $P$, and such that $f_0 + g_0 \geq 0$.
Since $P$ is not invariant under general rotations neither is $f_0 + g_0$, but the
fact that $\psi$ is odd in $P$ implies that $\rho_{f_0 + g_0} = \rho_0$ and the same holds for the
potential. So this trivial construction gives an axially symmetric steady
state where the phase space density is not spherically symmetric but the
macroscopic quantities $\rho_0$ and $U_0$ are. Our steady states are in general not
of this trivial type.

Another symmetry issue refers to the dependence of the minimizer on the
variable $x_3$. Given a minimizer $f_0$ of which we do not yet know any
symmetry with respect to $x_3$ we can do a symmetric decreasing rearrangement
of $f_0$ with respect to $x_3$ while keeping all other variables fixed. Denote this
rearrangement by $f^*_0$; as to the rearrangement concept we refer the reader
to [16, Ch. 3]. Obviously, the rearrangement does not change the kinetic
energy, and neither does it change the Casimir functional, since $P$ does not
depend on $x_3$. By [16, Thms. 3.7, 3.9] it can at most decrease the potential
energy, with equality iff $f_0$ is already symmetric and decreasing in $x_3$ up to
a shift in $x_3$. But since $f_0$ minimizes the energy-Casimir functional it must
posses this symmetry, and without loss of generality we can assume that $f_0$
is reflexion symmetric in $x_3$. The corresponding symmetry in the variable $v_3$ follows from the form which $f_0$ must have due to Theorem 1.

**Stationary versus static solutions.** If instead of the Vlasov-Poisson system we consider a self-gravitating fluid as described by the Euler-Poisson system then every so-called static solution, i.e., every steady state with vanishing velocity field, is spherically symmetric, cf. [15]. This turns out to be false for the steady states of the Vlasov-Poisson system which we obtain. By definition the velocity field equals $j_0/\rho_0$ on the support of $\rho_0$ where the mass current density $j_0$ is given by

$$j_0(x) = \int v f_0(x,v) dv = 2\pi \int_{U_0(x)}^{E_0} \int_{U_0(x)}^{\sqrt{2(E-U_0(x))}} s\phi(E_0-E,r(x)p) dp dE \ e_t(x)$$

with $e_t(x):= (-x_2,x_1,0)/r(x)$; on the $x_3$-axis the velocity field vanishes. Obviously, if $\phi$ is even in $P$ then $j_0$ vanishes identically, so there exist static solutions which are not spherically symmetric among the ones we obtain. On the other hand, if, say, $\phi(E,P) > \phi(E,-P)$ for all $E \geq 0$ and $P > 0$ the velocity field is non-trivial and corresponds to an average rotation about the axis of symmetry in the counterclockwise direction.

**Less regular dependence on $P$.** When examining the appendix the reader may notice that at no place we really make use of the fact that all the functions under consideration can be taken to be axially symmetric. This is because as opposed to spherical symmetry, exploiting axial symmetry for example in the estimates for the potential is technically quite unpleasant. However, it is possible and even necessary if one wishes to study examples of the form

$$Q(f,P) = f^{1+1/k} P^{-2l/k}, \ f \geq 0, \ P \in \mathbb{R}$$

which for $l \neq 0$ do not satisfy the assumptions on $Q$ which we stated above. The corresponding steady states would take the form

$$f_0(x,v) = C(E_0-E)^k P^{2l}$$

which is analogous to the classical, spherically symmetric polytropes, except that the square of the third component of the angular momentum replaces the square of the modulus of the angular momentum. Clearly, (3.5) satisfies the crucial convexity condition (Q2), provided $k > 0$. When we examine the scaling arguments in Lemma A 3 we find that they go through, provided

$$l > -1, \ 0 < k < l + \frac{3}{2}.$$
The place where significant changes become necessary are the estimates in Lemma A 1 and Lemma A 6. Clearly, one can no longer control $\|f\|_{1+1/k}$ and $\|\rho_f\|_{1+1/n}$ in terms of $E_{\text{kin}}(f)$, $C(f)$, and $M$. Instead, one gains control of the weighted norms

$$\int \int P^{-l/k}(x,v) f^{1+1/k}(x,v) dv \, dx, \quad \int r^{-2l/n}(x) \rho_f^{1+1/n}(x) dx$$

where $r(x) = \sqrt{x_1^2 + x_2^2}$. We conjecture that the weight factor can be dealt with in the estimates for the potential energy etc. if one makes proper use of the axial symmetry of the potential.

4 Appendix

We now give the proof of the existence part in Theorem 1 and the additional results used in the proof of Theorem 2. First we collect some estimates for $\rho_f$ and $U_f$ induced by an element $f \in \mathcal{F}_M$. These estimates make no use of symmetry and rely on the assumption (Q1). Their main point is to see that $\mathcal{H}_C$ is bounded from below on $\mathcal{F}_M$ and to establish certain bounds along minimizing sequences of $\mathcal{H}_C$. Constants denoted by $C$ are positive, may depend only on $Q$ and $M$, and their value may change from line to line. By $\|\cdot\|_p$ we denote the usual $L^p$-norm of functions over $\mathbb{R}^3$ or $\mathbb{R}^6$ as the case may be.

Lemma A 1 Let $n := k + 3/2 < 3$. Then for any $f \in \mathcal{F}_M$ the following holds:

(a) \[
\int \int f^{1+1/k}(x,v) dv \, dx \leq C(1 + C(f))
\]

(b) \[
\int \rho_f^{1+1/n}(x) dx \leq C \left(1 + E_{\text{kin}}(f) + C(f)\right)
\]

(c) $U_f \in L^6(\mathbb{R}^3)$ with $\nabla U_f \in L^2(\mathbb{R}^3)$, the two forms of $E_{\text{pot}}(f)$ stated in Section 2 are equal, and

$$\int |\nabla U_f|^2 dx \leq C \|\rho_f\|_{6/5}^2 \leq C \left(1 + E_{\text{kin}}(f) + C(f)\right)^{n/3}.$$

Proof. Part (a) is a direct consequence of (Q1), splitting the $v$-integral in the definition of $\rho_f$ into small and large $v$’s and optimizing the split yields
(b), while the extended Young’s inequality and interpolation together with (b) imply (c). For details cf. [10] or [11].

The estimates above have an immediate but important consequence:

**Lemma A 2** The energy-Casimir functional $H_C$ is bounded from below on $F_M$, i.e.,

$$h_M := \inf_{f \in F_M} H_C(f) > -\infty,$$

and along any minimizing sequence of $H_C$ in $F_M$ the quantities $E_{\text{kin}}(f), C(f), \|f\|_{1+1/k},$ and $\|\rho_f\|_{1+1/n}$ are bounded.

**Proof.** Lemma A 1 yields the estimate

$$H_C(f) \geq E_{\text{kin}}(f) + C(f) - C(1 + E_{\text{kin}}(f) + C(f))^{n/3}, \ f \in F_M,$$

and since $n < 3$ the assertions follow.

Lemma A 1 and Lemma A 2 remain valid if we replace $F_M$ by its axially symmetric subset $F^S_M$. Of course the infimum of $H_C$ on this smaller set may be larger; we denote

$$h^S_M := \inf_{f \in F^S_M} H_C(f).$$

The assumptions (Q3) and (Q4) determine the behavior of $H_C$ under scaling transformations which we use to show that $h_M$ is negative and to relate the $h_M$’s for different values of $M$:

**Lemma A 3**  

(a) Let $M > 0$. Then $-\infty < h_M < 0$.

(b) For $0 < \bar{M} \leq M$,

$$h_M \geq (\bar{M}/M)^{5/3} h_M.$$  

Proof. Given any function $f$, we define a rescaled function $\bar{f}(x,v) = a f(bx,cv)$, where $a, b, c > 0$. Then

$$\int \int \bar{f} \, dv \, dx = a(b)\int \int f \, dv \, dx,$$

i.e. $f \in F_M$ iff $\bar{f} \in F_{\bar{M}}$ where $\bar{M} = a(b)\int \int f \, dv \, dx$, where $M = a(b)\int \int f \, dv \, dx$. Next

$$H_C(\bar{f}) = a b^{-3} c^{-5} E_{\text{kin}}(f) + a^2 b^{-5} c^{-6} E_{\text{pot}}(f)$$

$$+ (bc)^{-3} \int \int Q(a f(x,v),(bc)^{-1} P(x,v)) \, dv \, dx.$$  

(4.2)
To prove (a) we fix a function \( f \in \mathcal{F}_1 \) with \( f \leq F' \) and let \( a = M(bc)^3 \) so that \( \bar{f} \in \mathcal{F}_M \). Then by (Q3) and (Q4) and with positive constants \( C_1, C_2, C_3 \) which depend on \( f \),

\[
\mathcal{H}_C(\bar{f}) \leq M c^{-2} E_{\text{kin}}(f) + M^2 b E_{\text{pot}}(f) + M a^{-1} \int \int Q(a f(x,v),0) dv dx
\leq C_1 c^{-2} - C_2 b + C_3 a^{1/k'},
\]

provided \( a \leq 1 \) so that (Q4) can be applied; note that \( E_{\text{pot}}(f) < 0 \). We want the negative term to dominate as \( b \to 0 \), so we let \( c = b^{-\gamma/2} \) for some \( \gamma > 0 \). Then \( a = Mb^{3(1-\gamma/2)} \), and

\[
\mathcal{H}_C(\bar{f}) \leq C_1 b^\gamma - C_2 b + C_3 M^{1/k'} b^{3(1-\gamma/2)/k'}.
\]

Since \( 0 < k' < 3/2 \) we can fix \( \gamma \in [1,2[ \) such that \( 3(1-\gamma/2)/k' > 1 \). For \( b > 0 \) sufficiently small \( \mathcal{H}_C(\bar{f}) \) will then be negative and \( a = Mb^{3(1-\gamma/2)} < 1 \) as required. This proves part (a) of the lemma. To prove (b) we choose \( a = c = 1 \) and \( b = (\bar{M}/M)^{-1/3} \) so that the mapping \( \mathcal{F}_M \to \mathcal{F}_{\bar{M}}, f \mapsto \bar{f} \) is one-to-one and onto and \( b^{-1} \leq 1 \). By (4.2),

\[
\mathcal{H}_C(\bar{f}) = b^{-3} E_{\text{kin}}(f) + b^{-5} E_{\text{pot}}(f) + b^{-3} \int \int Q(f(x,v),b^{-1} P(x,v)) dv dx
\geq b^{-5} E_{\text{kin}}(f) + b^{-5} E_{\text{pot}}(f) + b^{-5} \int \int Q(f(x,v),P(x,v)) dv dx
\]

\[
= b^{-5} \mathcal{H}_C(f);
\]

we multiplied the two positive terms by \( b^{-2} \leq 1 \) and used the monotonicity of \( Q \) in the \( P \)-variable which we required in (Q3). By the choice of \( b \) and the definitions of \( h_M \) and \( h_{\bar{M}} \) the proof is complete. \( \square \)

It is again obvious that the assertions of the lemma remain valid if we restrict ourselves to the axially symmetric functions in \( \mathcal{F}_M \). The reader might also check that the scaling arguments work under the assumptions (Q3') and (Q4') from Section 3 as well.

Next we provide a splitting estimate which will be used to show that along a minimizing sequence the mass cannot escape to infinity; here and in the following we denote for \( 0 < R < S \leq \infty \),

\[
B_R := \{ x \in \mathbb{R}^3 | |x| \leq R \},
B_{R,S} := \{ x \in \mathbb{R}^3 | R \leq |x| < S \}.
\]

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Lemma A 4 Let \( f \in \mathcal{F}_M \). Then
\[
\sup_{a \in \mathbb{R}^3} \int_{a + B_R} f(x,v) dv dx \geq \frac{1}{RM} \left( -2E_{\text{pot}}(f) - \frac{M^2}{R} - \frac{C\|\rho_f\|_{1+1/n}}{R^{(5-n)/(n+1)}} \right), \quad R > 1.
\]

The proof follows from splitting the potential energy into three parts according to \(|x - y| \leq 1/R\), \(1/R < |x - y| < R\), and \(R \geq |x - y|\); for the details we refer to [11] or [23, Lemma 3]. The splitting estimate above has the following important consequence for minimizing sequences:

Lemma A 5 Let \((f_i) \subset \mathcal{F}_M\) be a minimizing sequence of \(\mathcal{H}_C\). Then there exist \(\delta_0 > 0\), \(R_0 > 0\), \(i_0 \in \mathbb{N}\), and a sequence of shift vectors \((a_i) \subset \mathbb{R}^3\) such that
\[
\int_{a_i + B_R} \int f_i(x,v) dv dx \geq \delta_0, \quad i \geq i_0, \quad R \geq R_0.
\]
If \((f_i) \subset \mathcal{F}_M^S\), i.e., the minimizing sequence consists of axially symmetric functions then one can choose \(a_i = (\bar{a}_i, z_i)\) with \(z_i \in \mathbb{R}\), i.e., only shifts along the axis of symmetry need to be admitted.

Proof. By Lemma A 2, \(\|\rho_{f_i}\|_{1+1/n}\) is bounded. By Lemma A 3 (a) we have
\[
E_{\text{pot}}(f_i) \leq \mathcal{H}_C(f_i) \leq \frac{1}{2}h_M < 0, \quad i \geq i_0,
\]
for a suitable \(i_0 \in \mathbb{N}\). Thus by Lemma A 4 there exist \(\delta_0 > 0\), \(R_0 > 0\), and a sequence of shift vectors \((a_i) \subset \mathbb{R}^3\) as required.

Now assume that the functions \(f_i\) are axially symmetric, and let \(a_i = (\bar{a}_i, z_i)\) with \(\bar{a}_i \in \mathbb{R}^2\) and \(z_i \in \mathbb{R}\). Suppose we can show that the sequence \((\bar{a}_i)\) is bounded. Then we can pick some \(\bar{R} > 0\) such that
\[
(\bar{a}_i,0) + B_{R_0} \subset B_{\bar{R}}, \quad i \in \mathbb{N}
\]
and thus
\[
a_i + B_{R_0} \subset (0,0,z_i) + B_{\bar{R}}, \quad i \in \mathbb{N},
\]
which implies our assertion in the symmetric case: For \(R \geq \bar{R}\),
\[
\int_{(0,0,z_i)+B_{\bar{R}}} \int f_i(x,v) dv dx \geq \int_{a_i + B_{R_0}} \int f_i(x,v) dv dx \geq \delta_0, \quad i \geq i_0.
\]
So it suffices to show that \((\bar{a}_i)\) is bounded. Assume the contrary and choose some integer \(N \in \mathbb{N}\) such that \(N \delta_0 > M\). If \(|\bar{a}_i|\) is sufficiently large then simple geometry tells us that there exist rotations \(A_1, \ldots, A_N\) about the \(x_3\)-axis such that the balls \(A_k a_i + B_{R_0}\), \(k = 1, \ldots, N\), are pairwise disjoint. By axial symmetry of \(f_i\),

\[
\int_{A_k a_i + B_{R_0}} f_i(x,v) dv dx = \int_{a_i + B_{R_0}} f_i(x,v) dv dx \geq \delta_0, \quad k = 1, \ldots, N,
\]

and since the balls on the left hand side are pairwise disjoint,

\[
\int \int f_i(x,v) dv dx \geq \sum_{k=1}^{N} \int_{A_k a_i + B_{R_0}} f_i(x,v) dv dx \geq N \delta_0 > M
\]

which violates the mass constraint and gives the desired contradiction. \(\square\)

We will also need to exploit the well known compactness properties of the solution operator of the Poisson equation:

**Lemma A 6** Let \((\rho_i) \subset L^{1+1/n}(\mathbb{R}^3)\) be bounded and \(\rho_i \rightharpoonup \rho_0\) weakly in \(L^{1+1/n}(\mathbb{R}^3)\).

(a) For any \(R > 0\),

\[
\nabla U_{1_B R \rho_i} \to \nabla U_{1_B R \rho_0} \text{ strongly in } L^2(\mathbb{R}^3).
\]

(b) If in addition \((\rho_i)\) is bounded in \(L^1(\mathbb{R}^3)\), \(\rho_0 \in L^1(\mathbb{R}^3)\), and for any \(\epsilon > 0\) there exists \(R > 0\) and \(i_0 \in \mathbb{N}\) such that

\[
\int_{|x| \geq R} |\rho_i(x)| dx < \epsilon, \quad i \geq i_0
\]

then

\[
\nabla U_{\rho_i} \to \nabla U_{\rho_0} \text{ strongly in } L^2(\mathbb{R}^3).
\]

For the details of the proof we refer to [11] or [23, Lemma 5]; it relies on the fact that over bounded sets the solution operator to the Poisson equation is compact in the appropriate spaces and the fact that in the context of the lemma we can make the contribution to the potential energy coming from outside sufficiently large balls small.

We are now ready to prove the existence of a minimizer, more precisely we prove:

\[
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\]
Theorem A 1 Let \((f_i) \subset \mathcal{F}_M^S\) be a minimizing sequence of \(\mathcal{H}_C\). Then there exists a sequence of shifts \((z_i) \subset \mathbb{R}\) and a subsequence, again denoted by \((f_i)\), such that for any \(\epsilon > 0\) there exists \(R > 0\) with

\[
\int_{(0,0,z_i)+B_R} f_i(x,v) dv dx \geq M - \epsilon, \quad i \in \mathbb{N},
\]

\[
Tf_i := f_i(\cdot + (0,0,z_i),\cdot) \rightharpoonup f_0 \text{ weakly in } L^{1+1/k}(\mathbb{R}^6), \quad i \to \infty,
\]

and

\[
\int_{B_R} f_0 \geq M - \epsilon.
\]

Finally,

\[
\nabla U_{Tf_i} \to \nabla U_0 \text{ strongly in } L^2(\mathbb{R}^3), \quad i \to \infty,
\]

and \(f_0 \in \mathcal{F}_M^S\) is a minimizer of \(\mathcal{H}_C\).

Proof. We split \(f \in \mathcal{F}_M^S\) into three different parts:

\[
f = 1_{B_{R_1} \times \mathbb{R}^3} f + 1_{B_{R_1} \times R_2 \times \mathbb{R}^3} f + 1_{B_{R_2} \times R_3 \times \mathbb{R}^3} f =: f_1 + f_2 + f_3;
\]

the parameters \(R_1 < R_2\) of the split are yet to be determined. The induced spacial densities are denoted by \(\rho_k\), their masses by \(M_k\), and the induced potentials by \(U_k, k = 1,2,3\). With

\[
I_{lm} := \int \int \frac{\rho_l(x) \rho_m(y)}{|x-y|}, \quad l,m = 1,2,3,
\]

we have

\[
\mathcal{H}_C(f) = \mathcal{H}_C(f_1) + \mathcal{H}_C(f_2) + \mathcal{H}_C(f_3) - I_{12} - I_{13} - I_{23}.
\]

If we choose \(R_2 > 2R_1\) then

\[
I_{13} \leq \frac{C}{R_2}.
\]

Next, we use the Cauchy-Schwarz inequality, the extended Young’s inequality, and interpolation to get

\[
I_{12} + I_{23} = \frac{1}{4\pi} \left| \int \nabla(U_1 + U_3) \cdot \nabla U_2 dx \right| \leq C \| \rho_1 + \rho_3 \|_{6/5} \| \nabla U_2 \|_2 \\
\leq C \| \rho \|_{1+1/n}^{(n+1)/6} \| \nabla U_2 \|_2.
\]
Using the estimates above and Lemma A 3 (b) we find
\[ h^S_M - \mathcal{H}_C(f) \leq \left( 1 - \left( \frac{M_1}{M} \right)^{5/3} - \left( \frac{M_2}{M} \right)^{5/3} - \left( \frac{M_3}{M} \right)^{5/3} \right) h^S_M \]
\[ + C \left( R_2^{-1} + \|\rho\|^{(n+1)/6}_{1+1/n} \|\nabla U_2\|_2 \right) \]
\[ \leq \frac{C}{M^2} (M_1 M_2 + M_1 M_3 + M_2 M_3) h^S_M \]
\[ + C \left( R_2^{-1} + \|\rho\|^{(n+1)/6}_{1+1/n} \|\nabla U_2\|_2 \right) \]
\[ \leq C h^S_M (M_1 M_3 + C \left( R_2^{-1} + \|\rho\|^{(n+1)/6}_{1+1/n} \|\nabla U_2\|_2 \right); \quad (4.3) \]

observe that by Lemma A 3 (a) \( h^S_M < 0 \) and that constants denoted by \( C \) are positive and depend on \( M \) and \( Q \), but not on \( R_1 \) or \( R_2 \). We want to use (4.3) to show that up to a subsequence and a shift \( M_3 \) becomes small along any minimizing sequence for \( i \) large provided the splitting parameters are suitably chosen.

So let \( (f_i) \subset \mathcal{F}^S_M \) be a minimizing sequence and define \( R_0, \delta_0, \) and \( (z_i) \) according to Lemma A 5. The shifted sequence \( (T f_i) \) is again minimizing so by Lemma A 2 we can pick a subsequence, again denoted by \( (f_i) \), with a weak limit as stated in the theorem. Now choose \( R_1 > R_0 \) so that by Lemma A 5, \( M_{i,1} \geq \delta_0 \) for \( i \) large. By (4.3),
\[ -C h^S_M \delta_0 M_{i,3} \leq \frac{C}{R_2} \|\nabla U_{0,2}\|_2 + C \|\nabla U_{i,2} - \nabla U_{0,2}\|_2 + \mathcal{H}_C(T f_i) - h^S_M \]

(4.4)
where \( U_{i,l} \) is the potential induced by \( f_{i,l} \) which in turn has mass \( M_{i,l} \), \( i \in \mathbb{N} \cup \{0\} \), and the index \( l = 1, 2, 3 \) refers to the splitting. Given any \( \epsilon > 0 \) we increase \( R_1 > R_0 \) such that the second term on the right hand side of (4.4) is small, say less than \( \epsilon/4 \). Next choose \( R_2 > 2R_1 \) such that the first term is small. Now that \( R_1 \) and \( R_2 \) are fixed, the third term in (4.4) converges to zero by Lemma A 6 (a). Since \( (T f_i) \) is minimizing the remainder in (4.4) follows suit. Therefore, for \( i \) sufficiently large,
\[ \int_{(0,0,z_i)+B_{R_2}} \int T f_i = M - M_{i,3} \geq M - (-C h^S_M \delta_0)^{-1} \epsilon. \]

(4.5)
Clearly, \( f_0 \geq 0 \) a.e., and \( f_0 \) is axially symmetric. By weak convergence we have that for any \( \epsilon > 0 \) there exists \( R > 0 \) such that
\[ M \geq \int_{B_R} f_0 \, dv \, dx \geq M - \epsilon \]

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which in particular implies that $f_0 \in L^1(\mathbb{R}^6)$ with $\int f_0 dv dx = M$. The functional $C$ is convex, so by Mazur’s Lemma and Fatou’s Lemma

$$C(f_0) \leq \limsup_{i \to \infty} C(Tf_i).$$

The strong convergence of the gravitational fields now follows by Lemma A 6 (b), and in particular,

$$\mathcal{H}_C(f_0) \leq \limsup_{i \to \infty} \mathcal{H}_C(Tf_i) = h^S_M$$

so that $f_0 \in \mathcal{F}_M^S$ is a minimizer of $\mathcal{H}_C$. $\square$

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