\[ \begin{align*}
\left( \frac{\partial}{\partial t} \psi(z) \right) & = \left( \int_{-\infty}^{\infty} m(-\xi) \psi(z+\xi) d\xi \right) = 0 \\
\left( \int_{-\infty}^{\infty} p(z) \psi(z+\xi) d\xi \right) & = 0 \\
\left( \int_{-\infty}^{\infty} m \psi(z+\xi) d\xi \right) & = 0
\end{align*} \]

\[ \psi(z) = \psi(0) e^{i \int_{-\infty}^{z} t(z') \, d\tau(z')} \]

Another interesting property of CSS states that if \( A \) is a constant and the local time evolution operator is \( \hat{U}(t) \), then

\[ \psi(t) = \psi(0) e^{i \int_{0}^{t} A(t') \, d\tau(t')} \]

TWO-MODE SPIN SQUEEZING

A two-mode squeezed state is a state of two bosonic fields that is minimized with respect to the variance \( \Delta s^2 \) of the number of photons in each mode. This state is characterized by a squeezing parameter \( \beta \) that describes the amount of squeezing.

\[ \hat{S} = \sqrt{N} \left( \hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{b} \right) \]

As in the usual case, the momentum and position operators also have the property of having minimal uncertainty, and the squeezing parameter \( \beta \) is given by

\[ \beta = \frac{\Delta s}{N} \]

In this state, the commutator of the position and momentum operators is

\[ [\hat{a}, \hat{a}^\dagger] = 1 \]

and the state is given by

\[ \psi(0) = \frac{1}{\sqrt{N}} \hat{a} \hat{b}^\dagger \psi(0) \]

TWO-MODE SPIN SQUEEZING

Neuper optimal two-mode spin squeezing via feedback
SU(2)→HW(2) contraction. That is, we make the substitutions [3, 13]

\[ S_x \rightarrow s - b^\dagger b, \]
\[ S_y \rightarrow \sqrt{\frac{s}{2}} (b + b^\dagger), \]
\[ S_z \rightarrow \sqrt{\frac{s}{2}} (b - b^\dagger), \]

where \( b^\dagger b \) is the number operator for photons subtracted from the x linear polarization mode (and added to the y linear polarization mode). We will assume that the probe beam is in a coherent spin state oriented in the x direction (i.e., x linearly polarized), and take the limit \( s \rightarrow \infty \) so that this contraction is exact.

Under this contraction, \( [\hat{S}_y, \hat{S}_z] \rightarrow i\hat{s} \). This means that, under the Heisenberg picture, the operator \( \hat{S}_y \) transforms as

\[ \hat{S}_y^\text{out} = \hat{S}_y^\text{in} + [\alpha \hat{S}_z, \hat{S}_y] \approx \hat{S}_y^\text{in} + \frac{\alpha n}{2} \hat{J}_z, \]

where \( n = 2s \) is the total number of photons in the pulse. Now we consider two samples, where we will use the notation \( \hat{J}_k^{(1)} \) and \( \hat{J}_k^{(2)} \), where \( k \in \{x, y, z\} \), for the spin operators for samples 1 and 2, respectively, and \( \hat{J}_k = \hat{J}_k^{(1)} \pm \hat{J}_k^{(2)} \). We may replace \( \hat{J}_z \) with \( \hat{J}_z^{(+)} \), the total z component of spin for the two samples, giving

\[ \hat{S}_y^\text{out} \approx \hat{S}_y^\text{in} + \frac{\alpha n}{2} \hat{J}_z^{(+)}. \]

Thus a measurement of \( \hat{S}_y^\text{out} \) gives a QND measurement of \( \hat{J}_z^{(+)} \). In considering detection, it is more convenient to consider the Stokes vector for the light integrated over time interval \( \delta t \). The transformation of this component is

\[ \delta \hat{S}_y^\text{out} \approx \delta \hat{S}_y^\text{in} + \frac{\alpha \delta n}{2} \hat{J}_z^{(+)}, \]

where \( \delta n \) is the total number of photons in time \( \delta t \). In Ref. [1], TMSS states are obtained by performing QND measurements on \( \hat{J}_y^{(+)} \) as well. They achieve this by applying a magnetic field in the x direction, in order to induce Larmor precession of the y and z components of the spin. This means that Eq. (8) becomes

\[ \delta \hat{S}_y^\text{out} (t) \approx \delta \hat{S}_y^\text{in} (t) + \frac{\alpha \delta n}{2} [\hat{J}_z^{(+)} \cos(\Omega t) + \hat{J}_y^{(+)} \sin(\Omega t)], \]

where \( \Omega \) is the frequency of the precession. In this case measurement of \( \delta \hat{S}_y^\text{out} \) alternately gives QND measurements of \( \hat{J}_z^{(+)} \) and \( \hat{J}_y^{(+)} \), thus giving reduced uncertainty in both \( \hat{J}_z^{(+)} \) and \( \hat{J}_y^{(+)} \).

For the TMSS states considered in Ref. [1] there is the requirement that \( \langle \hat{J}_x^{(1)} \rangle = -\langle \hat{J}_x^{(2)} \rangle \gg 1 \). Here we will instead take the convention that both \( \langle \hat{J}_x^{(1)} \rangle \) and \( \langle \hat{J}_x^{(2)} \rangle \) are large and close to the total spin \( j \), and replace \( J_y^{(+)} \) with \( J_y^{(-)} \). This is equivalent to a trivial rotation of coordinates for one of the modes and does not require a physical alteration to the experiment. If we now define

\[ J_y^{(\Omega)} (t) := J_y^{(+)} \cos(\Omega t) + J_y^{(-)} \sin(\Omega t), \]

then Eq. (9) becomes

\[ \delta \hat{S}_y^\text{out} (t) \approx \delta \hat{S}_y^\text{in} (t) + \frac{\alpha \delta n}{2} J_y^{(\Omega)} (t). \]

In order for this to be accurate, we require \( \delta n \) to be large, so that we may perform the SU(2)→HW(2) contraction. Under this contraction, \( \delta \hat{S}_y = \sqrt{\delta n/2X} \), where \( X \approx b + b^\dagger \). Therefore the transformation in the quadrature \( X \) is

\[ X^\text{out} (t) = X^\text{in} (t) + a \sqrt{\delta n} J_y^{(\Omega)} (t). \]

As the initial state is equivalent to the vacuum state [under the SU(2)→HW(2) contraction], it has a variance of 1 in its measured value. The same is true for \( X^\text{out} \), as the extra term only changes the mean.

Measurements are made on the output beam by splitting the beam into +45° and −45° linearly polarized components at a polarizing beam splitter and measuring the intensities at the two outputs. We then define the photocurrent as

\[ I_c (t) = \lim_{\delta t \rightarrow 0} \frac{\delta N_+ - \delta N_-}{\sqrt{\nu \delta t}}, \]

where \( \delta N_+ \) and \( \delta N_- \) are the photon counts at the +45° and −45° linearly polarized outputs and \( \nu \) is the photon flux \( \delta n/\delta t \). We may interpret \( I_c (t) \) as the measured value of \( X \) divided by \( \sqrt{\nu} \), so the photocurrent is given by

\[ I_c (t) = a \sqrt{\nu} J_y^{(\Omega)} (t) c + \xi (t), \]

where the subscript \( c \) indicates the average for the conditioned evolution, and \( \xi (t) \) is a real Gaussian white noise term such that \( \langle \xi (t) \xi (t') \rangle = \delta (t - t') \).

We take the limit of small time intervals \( \delta t \), though for finite photon flux this would mean that the SU(2)→HW(2) contraction is no longer valid, as \( \delta n \) would go to zero. In order for the limit to be rigorously correct, we need to take the limit of large \( \nu \) as well as small \( \delta t \). Nevertheless, this limit should be an accurate approximation for large photon flux. The conditioned master equation is then

\[ d\rho_c = \Gamma D[J_y^{(\Omega)}] \rho_c dt + \sqrt{\nu} dW(t) \mathcal{H}[J_z^{(\Omega)}] \rho_c, \]

where \( \Gamma = a^2 \nu /4 \), \( dW(t) = \xi (t) dt \) is an infinitesimal Wiener increment, \( D[\rho] \rho = r \rho r^\dagger + r^\dagger r \rho + \rho r^\dagger r )/2 \), and \( \mathcal{H}[\rho] \rho = \rho r + r^\dagger \rho - Tr[r + r^\dagger] \rho \rho \).
\section{Feedback}

The quadrature (12) is measured by detecting the photocurrent (13), thereby yielding TMSS states. The drawback is that this will yield nonzero values of $\langle J_y^{(+)} \rangle$ and $\langle J_y^{(-)} \rangle$, whereas for the optimal TMSS states [3], the expectation values are zero. In order to obtain states closer to the optimal TMSS states, we apply feedback to bring these expectation values closer to zero. For example, when $\cos(\Omega t) = 1$, we may apply a Hamiltonian proportional to $J_y^{(+)}$ in order to bring $\langle J_y^{(+)} \rangle$ towards zero, analogous to the case considered by TMW. When $\sin(\Omega t) = 1$, we wish to apply a Hamiltonian proportional to $J_y^{(-)}$ in order to bring $\langle J_y^{(-)} \rangle$ towards zero. For arbitrary $t$, we will apply the Hamiltonian

$$H_B = F(t) I_c(t),$$

where

$$F(t) = \frac{\lambda(t)}{\sqrt{T}} J_y^{(\Omega)}(t),$$

and

$$J_y^{(\Omega)}(t) = J_y^{(+)} \cos(\Omega t) - J_y^{(-)} \sin(\Omega t).$$

We choose this definition for $J_y^{(\Omega)}$ because $[J_y^{(\Omega)}, J_y^{(\Omega)}] = -i J_y^{(+)}$ [we will omit the (t) from this point for brevity].

As $\langle J_y^{(+)} \rangle$ is close to 2 for the states we are considering, this means that rotation about $J_y^{(\Omega)}$ will alter the expectation value of $J_z^{(+)}$. This Hamiltonian may be implemented by a radio frequency magnetic field [14]. This generates a Hamiltonian proportional to $J_y$, which becomes $J_y^{(\Omega)}$ due to the Larmor precession of the atomic spin.

Using this feedback, we obtain the unconditioned master equation [15]

$$\dot{\rho} = -i\hbar \{ c \frac{d}{dt} F(t) + F(t) c \}, \quad \rho + \{ c - i F(t) I_c \} \rho,$$

with

$$c = \sqrt{T} J_z^{(\Omega)}.$$  \hspace{1cm} (20)

In the single-mode case TMW pointed out that the term $c \frac{d}{dt} F(t) + F(t) c$ is proportional to $J_x J_y + J_y J_z$, which is the two-axis countertwisting Hamiltonian [16]. In a similar way we can derive a Hamiltonian for the two-mode case that produces TMSS states and is analogous to the two-axis countertwisting Hamiltonian. It is straightforward to show that

$$c \frac{d}{dt} F(t) + F(t) c = \lambda(t) \{ [(J_y^{(1)})^2 + (J_y^{(2)})^2 - (J_y^{(1)})^2 \}

- (J_y^{(2)})^2 \sin(2\Omega t) + (J_y^{(1)})^2 J_y^{(2)} + J_y^{(1)} J_y^{(2)} J_z^{(1)}

+ 2(J_y^{(1)} J_y^{(2)} + J_y^{(1)} J_z^{(2)}) \cos(2\Omega t)

+ 2(J_y^{(1)} J_y^{(2)} + J_y^{(1)} J_z^{(2)} \}. \hspace{1cm} (21)

In the limit of large $\Omega$, the contribution to the evolution from the terms containing $\sin(2\Omega t)$ and $\cos(2\Omega t)$ is negligible. Omitting these terms yields

$$c \frac{d}{dt} F(t) + F(t) c \approx 2\lambda(t) (J_y^{(1)} J_y^{(2)} + J_y^{(1)} J_y^{(2)}),$$

which indicates that it is possible to produce two-mode spin squeezing using the Hamiltonian

$$H = J_y^{(1)} J_y^{(2)} + J_y^{(1)} J_y^{(2)}.$$  \hspace{1cm} (22)

A. Simple feedback

Now we will consider the appropriate feedback strength, $\lambda(t)$. It is easily shown that the variation of $\langle J_z^{(1)} \rangle_c$ due to the conditioned evolution (15) is

$$\text{Tr}(J_z^{(1)} d\rho_c) = 2 \sqrt{\Gamma} W ((J_z^{(1)})^2)_c - (J_z^{(1)})^2_c.$$  \hspace{1cm} (23)

The variation in $\langle J_z^{(2)} \rangle_c$ due to the feedback is

$$i(\{ H_B, J_z^{(2)} \})_c = i \frac{\lambda}{\sqrt{T}} I_c (J_z^{(2)})_c,$$

$$= -\frac{\lambda}{\sqrt{T}} I_c (J_z^{(2)})_c.$$  \hspace{1cm} (24)

If $\langle J_z^{(2)} \rangle_c = 0$, then the appropriate feedback to keep this equal to zero is

$$\lambda(t) = \frac{2 \sqrt{\Gamma} (J_z^{(2)})^2_c}{\langle J_z^{(2)} \rangle_c}.$$  \hspace{1cm} (25)

If the unconditioned state is close to being pure, then we may also use this expression with the unconditioned averages without significant loss of accuracy:

$$\lambda(t) = \frac{2 \sqrt{\Gamma} (J_z^{(2)})^2_c}{\langle J_z^{(2)} \rangle_c}.$$  \hspace{1cm} (26)

Note that this result is comparable to that found for the case of feedback for single-mode spin squeezing by TMW:

$$\lambda(t) = \frac{2 \sqrt{\Gamma} (J_z^{(2)})^2_c}{\langle J_z^{(2)} \rangle_c}.$$  \hspace{1cm} (27)

This is due to the fact that the commutation relations for $J_x$, $J_y$, and $J_z$ are identical to those for $J_x^{(+)}$, $J_y^{(\Omega)}$, and $J_z^{(\Omega)}$.

B. Analytic feedback

Next we will consider the possibilities for using an analytic expression for the feedback strength $\lambda(t)$. In Ref. [9] the approximate relations

$$\langle J_z^{(2)} \rangle \approx \langle 4 \Omega t + 2/|J| \rangle^{-1},$$  \hspace{1cm} (29)

$$\langle J_z \rangle \approx J \exp(-\Omega/2),$$  \hspace{1cm} (30)
are used to derive the analytic feedback scheme
\[ \lambda(t) = \frac{\Gamma \exp(\Gamma t/2)}{1 + 2j \Gamma t}. \]  
(31)

In the two-mode case, it is possible to obtain the corresponding relation for \( \langle J_{z}^{(1)} \rangle \), but not for \( \langle (J_{z}^{(2)})^2 \rangle \).

In these derivations it is convenient to define the scaled variables \( v = \Gamma t \) and \( \Lambda(t) = \lambda(t)/\Gamma \). In terms of these variables the master equation (19) becomes
\[ \dot{\rho} = -\frac{i}{2} \left[ (J_{x}^{(1)})^2 \rho + D[J_{z}^{(1)}] \rho \right], \]  
(32)
where the dot indicates a derivative with respect to \( v \). Using these variables, the evolution is independent of \( \Gamma \). The value of \( \Gamma \) does not qualitatively affect the evolution and merely provides a scaling to the time and to \( \lambda(t) \).

It is straightforward to show that
\[ \frac{d}{dv} \langle J_{x}^{(1)} \rangle = \text{Tr}(J_{x}^{(1)} \dot{\rho}) \]
\[ = -\frac{1}{2} \langle (J_{x}^{(1)})^2 \rangle + 2\Lambda \langle (J_{z}^{(1)})^2 \rangle - \frac{1}{2} \Lambda^2 \langle J_{x}^{(1)} \rangle \]
\[ = -\frac{1}{2} \langle J_{x}^{(1)} \rangle + 2 \langle (J_{z}^{(1)})^2 \rangle \]
\[ \langle J_{x}^{(1)} \rangle \approx 2j \exp(-v/2), \]  
(33)
which is equivalent to that in the single-mode case. This technique fails to find the corresponding expression for \( \langle (J_{z}^{(1)})^2 \rangle \), but we have found numerically that a good approximation is given by
\[ \langle (J_{z}^{(1)})^2 \rangle \approx \frac{e^{-v/4}}{2v + 1/j}. \]  
(35)

Using these results in the expression for the feedback (27), we obtain
\[ \lambda(t) = \frac{\Gamma \exp(\Gamma t/2)}{1 + 2j \Gamma t}. \]  
(36)

C. Optimal feedback

The third alternative that we will consider is optimal feedback. In order to derive this, it is convenient to define the variables
\[ \zeta = \frac{\langle (J_{z}^{(1)})^2 + (J_{y}^{(1)})^2 \rangle}{2j}, \]
\[ \chi = \frac{\langle J_{x}^{(1)} \rangle}{2j}. \]  
(37)

The initial state used is that with individual coherent spin states in each arm, so both \( \zeta \) and \( \chi \) are equal to 1. As time progresses both \( \zeta \) and \( \chi \) decrease. In order for the feedback to be optimal, the value of \( \zeta \) should be as small as possible for each value of \( \chi \). Therefore, if \( \chi \) decreases by an amount \( \delta \chi \), then the amount by which \( \zeta \) decreases should be maximized. This means that the feedback that is optimum will be that which produces the maximum slope. Therefore, in order to determine the optimal feedback, we wish to solve
\[ \frac{d}{dv} \frac{\partial \zeta}{\partial \chi} = 0. \]  
(38)
This can be solved using
\[ \frac{d}{dv} \left[ \frac{d}{dv} \frac{\partial \zeta}{\partial \chi} \right] = 0. \]  
(39)
Using the master equation (19), it is straightforward to show that
\[ \text{Tr}[J_{x}^{(1)} \dot{\rho}] = -\frac{1}{2} \{ 1 + \Lambda^2 \langle (J_{z}^{(1)})^2 \rangle + 2 \Lambda \langle (J_{z}^{(1)})^2 \rangle \}. \]  
(40)

Determining the corresponding expression for the numerator in Eq. (39) is more difficult, and if it is performed directly it leads to an extremely complicated expression. However, there is a more convenient way of determining this expression. Note that
\[ \langle (J_{z}^{(1)})^2 \rangle = \langle (J_{x}^{(1)})^2 + (J_{y}^{(1)})^2 \rangle \]
\[ \langle (J_{z}^{(1)})^2 \rangle \approx \frac{1}{2} \langle (J_{x}^{(1)})^2 + (J_{y}^{(1)})^2 \rangle. \]  
(41)

Numerically this approximation was found to be very accurate. Using this result we find that
\[ \frac{1}{2} \text{Tr}[\langle (J_{x}^{(1)})^2 + (J_{y}^{(1)})^2 \rangle \dot{\rho}] \approx \text{Tr}[\langle (J_{z}^{(1)})^2 \rangle \dot{\rho}] \]
\[ = \Lambda^2 \langle (J_{x}^{(1)})^2 + (J_{y}^{(1)})^2 \rangle - \Lambda \langle (J_{x}^{(1)})^2 \rangle \langle (J_{z}^{(1)})^2 \rangle \]
\[ = \frac{1}{2} \langle J_{x}^{(1)} \rangle \cos \Omega t \sin \Omega t \]  
(43)

Thus we find that Eq. (39) becomes
\[ \frac{d}{dv} \left[ \frac{\Lambda^2 d - \Lambda e}{-f - \Lambda^2 f + \Lambda g} \right] = 0, \]  
(44)
where
\[ d = \langle (J_{x}^{(1)})^2 \rangle - \langle (J_{z}^{(1)})^2 \rangle, \]
\[ e = \langle (J_{x}^{(1)})^2 \rangle \langle (J_{z}^{(1)})^2 \rangle, \]
\[ f = \frac{1}{2} \langle J_{x}^{(1)} \rangle, \]
\[ g = 2 \langle (J_{z}^{(1)})^2 \rangle. \]  
(45)

Solving this gives
\[ \Lambda = \frac{-fd \pm \sqrt{(fd)^2 + ef(ef - dg)}}{fe - dg}. \]  
(46)
Numerically it is found that the correct solution that maximizes the slope is that with the positive sign.

Note that this derivation of the optimum feedback does not rely on any commutation relations other than the usual $su(2)$ commutation relations. It therefore is also applicable to the case of feedback for single-mode spin squeezing, with $J_x^{(1)}$, $J_y^{(1)}$, and $J_z^{(1)}$ replaced by $J_x$, $J_y$, and $J_z$.

IV. NUMERICAL RESULTS

A. Unconditioned master equation

The unconditioned master equation (32) was solved using a simple, finite step method with step sizes of $\delta \tau = 1/1000$. It was found that better results were obtained as $\Omega$ was increased. In order to estimate the best result in the limit of large $\Omega$, $\Omega$ was assigned the maximum value possible with this step size, $\tau/2\delta \tau$. The initial condition used was that where the individual modes were in independent coherent spin states oriented along the $x$ axis.

At each time step the quantities $\zeta$, $\chi$, and the purity

$$P = \text{Tr}[\rho^2]$$

were calculated. The results for the simple feedback scheme of Eq. (27) and a spin of $j = 5$ are shown in Fig. 1. Initially the system behaves as we would expect, with the sum of the variances $\zeta$ decreasing, until a time of approximately $\tau = 3$. Then the variances dramatically increase, and the purity drops almost to zero. After this, however, the system stabilizes. Note also that the value of $\chi$ drops regularly as time increases until the change at $\tau = 3$.

Recall that the system must be entangled if $\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \rangle < \langle J_x^{(1)} \rangle^2$ [3], which is equivalent to $\zeta < \chi$. Therefore a state with $\zeta < \chi$ can be described as TMSS. Evidently the feedback produces quite strong spin squeezing. In order to determine how close to optimal the states produced by this feedback are, the value of $\zeta$ was plotted against $\chi$ and compared with the plot for optimal TMSS states in Fig. 2. As time progresses the state travels from the upper right corner towards the lower left corner of the figure.

The state starts out very close to optimum and does not deviate significantly from optimum until $\zeta < 0.4$. Eventually the value of $\zeta$ increases dramatically, and the system spirals towards its equilibrium state at $(0.5, 0.5)$. The results for $j = 5$ are typical, and similar results are obtained for other spins $j > 1/2$.

For the purposes of teleportation, the important issue is how close it is possible to get to the optimal TMSS states considered for teleportation in Ref. [3]. This state is indicated by the cross in Fig. 2. As can be seen, the states obtained by feedback closely approach this optimal state. Similar results are obtained for higher spin, but for smaller spin the states obtained by feedback are further from the TMSS states considered for teleportation.

The plots of $\zeta$ versus $\chi$ for analytic feedback and optimal feedback are also shown in Fig. 2. The results for analytic feedback are very similar to those for simple feedback, with $\chi$ dramatically increasing at later times. In contrast, under optimal feedback the value of $\zeta$ monotonically decreases to an asymptotic value for $\chi = 0$. For small times the three feedback schemes produce very similar results, and the states obtained using optimal feed-
back do not pass significantly closer to the optimal state for teleportation.

The last alternative that we consider in this section is the Hamiltonian (23), which is analogous to the two-axis countertwisting Hamiltonian in the single-mode case. The results for this Hamiltonian are also shown in Fig. 2. For this case there is significant spin squeezing, but the squeezing is generally poorer than for feedback. Nevertheless, at later times the value of $\zeta$ does not rise dramatically (until $\chi$ falls below zero), as opposed to the results for the simple or analytic feedback.

**B. Conditioned master equation**

Next we consider the results for the conditioned master equation. There are two examples where it is necessary to consider the conditioned master equation. In the case of single feedback, at the time that $\zeta$ increases dramatically the purity of the state has greatly decreased. This means that the feedback based on the unconditioned averages, Eq. (27), is a poor approximation of the feedback based on conditioned averages, Eq. (26), and will not accurately keep the average $\langle J_z^{(\gamma)} \rangle_c$ equal to zero. For this reason we may obtain better results using the conditioned master equation and the feedback (26).

The second case that we consider here is that without any feedback. In this case there is no reduction in the variances for the unconditioned equation. If we consider the conditioned equation, however, there is a reduction in the variances, though the means of $\langle J_z^{(+)} \rangle_c$ and $\langle J_z^{(-)} \rangle_c$ are nonzero. For this reason we will consider the conditioned master equation for this case, with $\zeta$ defined by

$$\zeta = \frac{(\langle J_z^{(+)} \rangle_c - \langle J_z^{(+)\gamma} \rangle_c)^2 + (\langle J_z^{(-)} \rangle_c - \langle J_z^{(-\gamma)} \rangle_c)^2}{2j}. \quad (48)$$

To minimize the random differences between the two cases the same random numbers were used for each. The results for these two cases are plotted in Fig. 3.

The results for the feedback (26) are extremely close to the line for optimal states, with only temporary excursions from it. In particular, the value of $\zeta$ does not increase dramatically at large times. This indicates that improved results may be obtained using feedback based on the conditioned state.

On the other hand, implementing this feedback may be far more difficult. In the forms of feedback considered for the unconditioned equation, the feedback was proportional to the measurement record, with a proportionality constant $\lambda$ that is a function of time only. This feedback may be performed efficiently by calculating $\lambda(t)$ beforehand and programming it into a field-programmable gate array [17]. In contrast, the value of $\lambda$ given by Eq. (26) is dependent on the measurement record, and therefore would need to be calculated during the measurement, introducing a significant time delay.

**V. COMPARISON WITH SINGLE-MODE CASE**

There is clearly a great similarity between the single-mode and two-mode cases. The operators $J_x^{(+)}$, $J_y^{(\gamma)}$, and $J_z^{(\gamma)}$ obey exactly the same commutation relations as $J_x$, $J_y$, and $J_z$ in the single-mode case. This means that, for example, the optimal feedback also applies to the single-mode case (as was mentioned in Sec. III C). As we may replace the sum of the variances $\langle (J_z^{(+)} - \langle J_z^{(+)\gamma} \rangle_c)^2 \rangle$ with $2\langle (J_z^{(\gamma)} \rangle_c^2 \rangle$, it might appear that this case is identical to the single-mode case. Nevertheless, there is a subtle difference due to the fact that the operator $J_z^{(\gamma)}$ is time dependent.

For the case of spin 1/2, there is complete equivalence between the single-mode case (for spin $j = 1/2$) and the two-mode case. We will take the Hilbert space for the single-mode case to be that for two spin 1/2 particles, so that it is the same as for the two-mode case. The optimal TMSS states for $j = 1/2$ are also optimal single-mode squeezed states in terms of the total spin. That is,
they minimize $\langle (J_x^+)^2 \rangle$ for a given $\langle J_x^+ \rangle$. The converse is also true: the states optimized for single-mode spin squeezing are automatically optimized for two-mode spin squeezing.

In addition, both the single- and two-mode countertwisting Hamiltonians produce optimal spin squeezed states. Not only this, but the feedback as given by Eq. (28) in the single-mode case, or Eq. (27) in the two-mode case, and the optimal feedback described in Sec. III C, give optimal spin squeezed states. The only feedback that does not give optimal states is the analytic feedback for the single- or two-mode case. These results are depicted in Fig. 4. In this figure the variables $\zeta$ and $\chi$ are defined for the single-mode case as

$$\zeta = 2\langle J_x^+ \rangle/j, \quad \chi = \langle J_x \rangle/j. \quad (49)$$

To explain these results analytically, first note that, for spin 1/2,

$$J_z^+ J_y^+ + J_y^+ J_z^+ = 2(J_z^{(1)} J_y^{(2)} + J_y^{(1)} J_z^{(2)}). \quad (50)$$

This means that the countertwisting Hamiltonians for the one- and two-mode cases are identical, and therefore produce identical states.

In addition we find that $\langle (J_x^+)^2 \rangle - \langle (J_y^-)^2 \rangle$ commutes with these Hamiltonians. This means that for the states produced by countertwisting, the values of $\langle (J_x^+)^2 \rangle$ and $\langle (J_y^-)^2 \rangle$ will be identical. This means that, if the states produced are optimal single-mode spin squeezed states (i.e., $\langle (J_x^+)^2 \rangle$ is minimized), then they must also be optimal TMSS states.

To show that optimal single-mode spin squeezed states are produced by the countertwisting Hamiltonian, note that for total spin 1 we have the differential equations

$$\frac{d}{dt} \langle J_x^+ \rangle = 2\langle J_x^+ J_y^- \rangle, \quad (51)$$

$$\frac{d}{dt} \langle J_y^+ \rangle = -\langle J_x \rangle, \quad (52)$$

$$\frac{d}{dt} \langle J_y^- \rangle = \langle J_x \rangle. \quad (53)$$

The solution of these equations is

$$\langle J_x \rangle^2 = 4\langle J_x^+ \rangle^2 - \langle J_y^- \rangle^2. \quad (54)$$

This is the relation for optimal single-mode spin squeezed states as given by Sørensen and Mølmer. This shows that the states produced by the one- and two-mode countertwisting Hamiltonians and the one- and two-mode optimal states are all identical.

For the case of feedback, it is sufficient to show that the feedbacks of Eqs. (28) and (27) give optimal states, as this will imply that the optimal feedback gives optimal spin squeezed states. The derivation of this in case is lengthy, and is given in Appendix B. It is interesting to note that in this case the feedback case is given by Eq. (B7), and differs dramatically from the analytic feedback used for higher spin.

The equivalence between the single- and two-mode cases does not hold for any total spin higher than 1. As shown in Fig. 5, there are significant differences between the optimal single- and two-mode spin squeezed states for total spin above 1. As can be seen, for a total spin of 2 or more the values of $\zeta$ are significantly less for the single-mode case than for the two-mode case. This reflects the fact that in the two-mode case we wish to minimize $\langle (J_y^-)^2 \rangle$ as well as $\langle (J_x^+)^2 \rangle$.

As there are such strong similarities between the single-mode and two-mode cases, it is reasonable to apply the squeezing parameter as considered in Refs. [9, 18, 19] to the two-mode case. In the single-mode case the squeezing parameter was defined by

$$\xi_s^2 = 2\langle J_y^- \rangle^2 \langle J_x \rangle^2. \quad (55)$$

With the definitions of $\zeta$ and $\chi$ for the single-mode case above, this can be expressed as $\xi_s^2 = \zeta^2/\chi^4$. It would seem reasonable to use the same definition for the two-mode case.

For the two-mode case we find that the minimum value of $\xi_s^2$, namely $\xi_{s \text{min}}$, varies with spin $j$ as shown in Fig. 6(a). It is interesting the values of $\xi_{s \text{min}}$ have been multiplied by $j + 1$ in order to more easily compare the results for different spins. Very similar results are obtained for the simple feedback of Eq. (27) and the optimal feedback of Eq. (46). Using the analytic feedback of Eq. (36) gives results that are similar to, but slightly above, those for the other two feedback schemes. All of these feedback schemes give results that are significantly above the result for the optimal states. Using the countertwisting
Hamiltonian (23) gives results that are higher than those for feedback.

Similar results are obtained for the single-mode case [see Fig. 6(b)]. The results for feedback are noticeably above those for optimal states, and the results for two-axis countertwisting are significantly above those for feedback. To summarize, the scaling constants for each of the cases are as given in Table I. In this table we can see the similarities between the single-mode and two-mode cases: the scaling constants obtained by feedback are significantly above those for optimal states, and the scaling constants for two-axis countertwisting (or its two-mode equivalent) are significantly above those for feedback. In both cases there are only small differences between the results for the different types of feedback. The only qualitative difference between the scaling constants for the single- and two-mode cases are that in the single-mode case the analytic feedback gives slightly better results than the simple feedback of Eq. (27). In contrast, in the two-mode case the scaling constants are indistinguishable.

VI. EXPERIMENTAL PROSPECTS

There are some complications in applying this theory to the experiment of Ref. [1]. In this experiment, $\alpha \approx 5 \times 10^{-12}$, and $\nu \approx 2 \times 10^{10} \text{s}^{-1}$. This means that $\Gamma \approx 1.4 \times 10^{-8} \text{s}^{-1}$, which implies that the time required for maximal squeezing is on the order of $10^3 \text{s}$ (over 20 yr). Therefore a far more intense beam or a stronger interaction would be required to obtain the two-mode spin squeezing described here.

It must be emphasized that this long time is required for the measurements, not the feedback. Feedback may be applied to the states obtained in Ref. [1] with a negligible increase in the time required for the experiment. Nevertheless, for the conditions of this experiment, although the feedback will bring the means of $J_{z}^{(+)}$ and $J_{y}^{(+)}$ towards zero, it will not significantly reduce the variances. It is only when the QND measurement can be performed strongly enough (i.e., with a strong enough interaction over a long enough time) to obtain spin squeezing close to maximum that an improvement in the variance is obtained by using feedback.

Another problem is spontaneous losses due to absorption of QND probe light. The loss rate due to this is $N \gamma g^2 \pi /4 \Delta^2$, where $N = 10^3$ is the total number of atoms and $g$ is the one-photon Rabi frequency. As in the case of single-mode spin squeezing [9], this loss will be very large over the time period $1/\Gamma$ for free space. Both this problem and the problem of the long interaction time may

![Graph](image-url)
TABLE I: The scaling constants for the minimum squeezing parameter \( \xi^2_{\text{min}} \), for both single- and two-mode spin squeezing, for optimal states, simple feedback, analytic feedback, optimal feedback, and two-axis countertwisting (or its equivalent in the two-mode case).

<table>
<thead>
<tr>
<th></th>
<th>Optimal</th>
<th>Optimal feedback</th>
<th>Analytic feedback</th>
<th>Simple feedback</th>
<th>Countertwisting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single</td>
<td>1</td>
<td>1.6368</td>
<td>1.6793</td>
<td>1.6777</td>
<td>1.9562</td>
</tr>
<tr>
<td>Two mode</td>
<td>3/4</td>
<td>1.0584</td>
<td>1.0692</td>
<td>1.0692</td>
<td>1.292</td>
</tr>
</tbody>
</table>

be overcome by performing the experiment in a cavity in the strong-coupling regime.

Two other common experimental problems are inefficient detectors and time delays. As for the case of single-mode spin squeezing, inefficient detectors do not affect the scaling. The system has the potential to be far more sensitive to time delays than in the single-mode case; however. The problem is that, as the spin component that is being measured is rotating rapidly, the feedback may be correcting for measurements of a different spin component.

In order to correct the right spin component, the rotation of the spin component that is measured should have completed an integral number of rotations during the time delay. Experimentally, this means that the magnetic field should be adjusted such that the frequency \( \Omega \) is an integral multiple of \( 2\pi/\tau \), where \( \tau \) is the time delay. Provided that this is done, time delays should not be more of a problem than in the single-mode case.

VII. CONCLUSIONS

Two-mode spin squeezed states are important states to produce for quantum teleportation. Here we have shown that it is possible to produce states very close to the optimal TMS states by adapting the feedback for single-mode spin squeezing considered by TMW. These states are not conditioned on the measurement record, in contrast to the conditional two-mode spin squeezing discussed in Ref. [1].

Using the simple feedback scheme (27) it is possible to obtain states that are quite close to optimal TMS states for small times, but strongly diverge from these states at later times. In particular, for spins above about 5 it is possible to obtain states very close to the TMS states considered for teleportation in Ref. [3]. An analytic feedback scheme (36) also gives similar results. This feedback scheme is more practical experimentally, as the appropriate feedback strength to be used is easily calculated.

We have derived the optimal feedback that produces the maximum possible spin squeezing. This feedback is also applicable to the case of feedback for single-mode spin squeezing. This feedback gives states even closer to the optimal TMS states.

We have also derived a Hamiltonian for the two-mode case that is equivalent to the two-axis countertwisting Hamiltonian introduced in Ref. [16]. This Hamiltonian produces spin squeezing, but not as strongly as the measurements with feedback.

In the case of spin 1/2 both the feedback (except for the analytic feedback) and the countertwisting Hamiltonian produce optimal TMS states. These states are equivalent to optimal single-mode spin squeezed states if the two modes are considered as a single spin-1 system. In the single-mode case optimal spin squeezed states are produced both by feedback and by the countertwisting Hamiltonian.

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APPENDIX A: RELATION OF VARIABLES TO EXPERIMENT

Here we give further explanation of the variables introduced in Eq. (1). The coupling constant \( a \) is given by

\[
a = \frac{\sigma}{A(1 + 1/2) \Delta \alpha_c},
\]

where \( \sigma \) is the resonant absorption cross section, \( A \) is the area of the transverse cross section of the light beam, \( I \) is the nuclear spin, \( \gamma \) is the spontaneous emission rate of the upper atomic level, \( \Delta \) is the detuning, and \( \alpha_c \) is the dynamic vector polarizability. We have omitted the bounds on the integral (1) for greater generality, so we may apply this expression to multiple samples. Explicit bounds are unnecessary, as there is no contribution to the integral from regions where there are no atoms.

The continuous spin operators for the sample are defined as

\[
J_k(z, t) = \lim_{\delta z \to 0} \frac{1}{\delta z} \sum_{\mu} \frac{1}{2} \sigma^\mu_k,
\]

where \( k \in \{x, y, z\} \) and \( \sigma^\mu_k \) is the Pauli operator for particle \( \mu \). The sum is over all particles between \( z \) and \( z + \delta z \) over the cross section of the sample. The operator for the entire sample is obtained by integrating over \( z \).
The field is described by the continuous-mode annihilation operators $a_k(z,t)$, where $k = x$ and $y$ for the $x$ polarized and $y$ polarized modes, respectively. The instantaneous Stokes parameters are

$$S_x(z, t) = \frac{1}{2} [a_x^\dagger(z, t) a_x(z, t) - a_y^\dagger(z, t) a_y(z, t)],$$
$$S_y(z, t) = \frac{1}{2} [a_x^\dagger(z, t) a_y(z, t) + a_y^\dagger(z, t) a_x(z, t)],$$
$$S_z(z, t) = -\frac{i}{2} [a_x^\dagger(z, t) a_y(z, t) - a_y^\dagger(z, t) a_x(z, t)].$$  \hspace{1cm} \text{(A3)}

The Stokes vector for the entire pulse at position $z$ is given by $\mathbf{S}(z) = \int S(z, t) dt$.

**APPENDIX B: FEEDBACK FOR TOTAL SPIN 1**

Here we show that the feedback of either Eqs. (28) or (27) gives optimal spin squeezed states for a total spin of 1. To see this, note first that the first term in Eq. (19) is just the same as that produced by the countertwisting Hamiltonian, and so will produce optimal states. In order to show that the feedback produces optimal states, it remains to be shown that the second term, $D[\varphi(t)] \rho$, does not alter the evolution. Specifically, in the single-mode case

$$\text{Tr} \{J_z D[J_x - i\lambda J_y] \rho\} = -\frac{1}{2} (1 + \lambda^2) \langle J_x \rangle + \lambda \langle J_x^2 + J_y^2 \rangle.$$  \hspace{1cm} \text{(B1)}

If the state $\rho$ is an optimal state, then $\langle J_x^2 + J_y^2 \rangle = 1$, so this simplifies to

$$\text{Tr} \{J_z D[J_x - i\lambda J_y] \rho\} = -\frac{1}{2} (1 + \lambda^2) \langle J_x \rangle + \lambda.$$  \hspace{1cm} \text{(B2)}

Similarly we can show that

$$\text{Tr} \{J_z^2 D[J_x - i\lambda J_y] \rho\} = -\frac{\lambda}{2} \langle J_x \rangle + \lambda^2 \langle J_x^2 - J_y^2 \rangle.$$  \hspace{1cm} \text{(B3)}

For optimal states, $\langle J_x^2 \rangle = 1$, so this becomes

$$\text{Tr} \{J_z^2 D[J_x - i\lambda J_y] \rho\} = -\frac{\lambda}{2} \langle J_x \rangle + \lambda^2 (1 - \langle J_y^2 \rangle).$$  \hspace{1cm} \text{(B4)}

It is simple to show that both Eqs. (B2) and (B4) will be zero if Eq. (54) is satisfied, and the feedback is given by Eq. (28).

Therefore, if the state is already in an optimal state, and the feedback as given by Eq. (28) is used, then the state will continue to be in an optimal state. On the other hand, if some other feedback is used, then optimal states will not be obtained. For example, the analytic feedback considered by LMW does not give optimal states. In order to determine analytic feedback that will give optimal states, note that the differential equations for $\langle J_x \rangle$ and $\langle J_z^2 \rangle$ are

$$\frac{d}{dv} \langle J_x \rangle = \lambda(2 \langle J_x^2 \rangle - 1),$$
$$\frac{d}{dv} \langle J_z^2 \rangle = -\lambda \langle J_y^2 \rangle.$$  \hspace{1cm} \text{(B5)}

Solving these using the feedback (28) and Eq. (54) gives

$$\langle J_x \rangle = \sqrt{2e^{-v} - e^{-2v}},$$
$$\langle J_z^2 \rangle = \frac{1}{2} e^{-v}.$$  \hspace{1cm} \text{(B6)}

This implies that the value of $\lambda$ should change with time as

$$\lambda(v) = \frac{1}{\sqrt{2e^{-v} - 1}}.$$  \hspace{1cm} \text{(B7)}

This is quite different from the analytic expression applied for larger spins.

The case for feedback for two-mode spin squeezing is analogous to this, except that $J_x$, $J_y$, and $J_z$ are replaced with $J_x^{(1)}$, $J_y^{(1)}$, and $J_z^{(1)}$. This therefore shows that optimal states are obtained in the one- and two-mode cases using the feedback given by Eqs. (28) and (27), respectively.

(1940).


